NEW CONDITIONS ON SOLUTIONS FOR PERIODIC SCHRÖDINGER EQUATIONS WITH SPECTRUM ZERO

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Abstract. This paper is concerned with the following Schrödinger equation:

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases}$$

where the potential V and f are periodic with respect to x and 0 is a boundary point of the spectrum $\sigma(-\triangle + V)$. By a new technique for showing the boundedness of Cerami sequences, we are able to obtain the existence of nontrivial solutions with mild assumptions on f.

1. INTRODUCTION

In this paper, we consider the semilinear Schrödinger equation:

$$(1.1) \qquad \begin{cases} -\triangle u + V(x)u = f(x,u), & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a potential and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonlinear coupling which is superlinear as $|u| \to \infty$. Note that, if $V(x)$ is periodic in x, then the operator $-\triangle + V$ has purely continuous spectrum $\sigma(-\triangle + V)$ which is bounded below and consists of closed disjoint intervals ([17, Theorem XIII.100]). As we know, the nonlinear Schrödinger equation with periodic potential and nonlinearities has been widely investigated in the literature over the past several decades for both its importance in applications and mathematical interest. There are many results on the existence and

multiplicity of solutions for problem (1.1) depending on the location of 0 in $\sigma(A)$, see, e.g. [1, 2, 3, 4, 9, 10, 11, 13, 14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 26, 29, 30, 31, 32] and the references therein.

For the case of $0 < \sigma(A)$, Coti-zelati and Rabinowitz proved in [32] the existence of infinitely many solutions with $f \in C^2$ and the classic Ambrosetti-Rabinowitz super-quadratic condition. In [11], under a general superlinear assumption and monotone condition on f, Li, Wang and Zeng obtained the existence of ground state solutions by concentration compactness argument. We also refer the reader to [4, 12, 15, 19, 23, 27] and the reference therein where the condition (AR) was replaced by more general superlinear conditions.

For the case that 0 lies in a spectral gap of $\sigma(A)$, the interest in the study of various qualitative properties of the solutions has steadily increased in recent years. In [7], relying on a degree theory and a linking-type argument developed there, Kryszeuski and Szulkin obtained a nontrival solution under condition (AR) and infinitely many geometrically distinct nontrival solutions with additional locally Lipschitzian assumption on f (see (A8) in [7]). The stronger results to be those of Szulkin and Weth [22], following the approach of Pankov [14], they proved the existence of ground state solutions of Nehai-Pankov type with the Nehari type assumption, i.e.,

(Ne) $t \to \frac{f(x,t)}{|t|}$ is strictly increasing on $(-\infty, 0) \cup (0, \infty)$.

In [30], Yang obtained a least energy solution by using a different method (based on the approach of [20]). Liu [9] improved the result of Szulkin and Weth [22] by relaxing (Ne) to a weaker version, i.e.,

(WN) $t \to \frac{f(x,t)}{|t|}$ is non-decreasing on $(-\infty, 0) \cup (0, \infty)$.

In recent paper of the author [24], (WN) was weaken to the following more generic condition (S4) and the least energy solution was established there.

(S4)   There exists $\theta_0 \in (0, 1)$ such that

$$\frac{1 - \theta^2}{2} tf(x,t)$$
$$\geq \int_{\theta t}^{t} f(x,s)\mathrm{d}s = F(x,t) - F(x,\theta t), \quad \forall \, \theta \in [0, \theta_0], \; (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

To our best knowledge, there are only several papers deal with the case that 0 is a boundary point of the spectrum $\sigma(A)$. In [2], Bartsch and Ding obtained a nontrival solution with condition (AR). Later, this result was improved by Willem and Zou in [29] by using an improved generalized weak link theorem. In [31], Yang et al. proved the existence of a nontrivial solution for (1.1) with (Ne) and the following condition (F), which seems to be necessary to obtain the existence of one weak solution of (1.1) in [2, 29, 31].

(F)   There exist constants $c_0 > 0$, $2 < \mu < 2^*$ such that

$$F(x,t) \geq c_0|t|^\mu, \quad \forall\, (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

However, (F) is a severe restriction, since it strictly controls the growth of $f(x,t)$ as $|t| \to \infty$. There are much functions which are superlinear at both zero and infinity, but do not satisfy the condition (F). For example $f(x,t) = at|t|^{\alpha-2}\ln(1 + |t|^{1/N})$ with $a > 0$ and $\alpha \in (2, 2^* - 1/N)$. In recent paper [25], Tang weakened (F) to a milder condition and obtained a least energy solution with (S4). A related result can be found in [13]. In author's recent paper [16], infinitely many large energy solutions was obtained with a weaker condition than (WN).

Motivated by above works, in the present paper, we will continue to study the existence of nontrivial solutions with mild assumptions on nonlinearity. More precisely, we introduce the following assumptions:

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_i, $i = 1, 2, \ldots, N$;

(V2) $0 \in \sigma(A)$ and there exists $\beta > 0$ such that $(0, \beta] \cap \sigma(A) = \emptyset$;

(S1) $f \in C(\mathbb{R}^{N+1}, \mathbb{R})$ is 1-periodic in x_i, $i = 1, 2, \ldots, N$, and there exist constants $b_1, b_2 > 0$, $2 < \mu \leq p < 2^*$ such that

(1.2) $b_1 \min\{|t|^\mu, |t|^2\} \leq tf(x,t), \quad |f(x,t)| \leq b_2(1+|t|^{p-1}), \quad \forall\, (x,t) \in \mathbb{R}^N \times \mathbb{R},$

where $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := +\infty$ if $N = 1$ or 2;

(S2) $f(x,t) = o(|t|)$ as $|t| \to 0$ uniformly in x;

(S3) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^2} = \infty$, a.e. $x \in \mathbb{R}^N$;

Proof of the main result are based on variational methods applied to the following functional,

(1.3) $\quad \Phi(u) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)|u|^2\right) \mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x, \quad \forall\, u \in H^1(\mathbb{R}^N).$

The hypotheses on $f(x,u)$ imply that $\Phi \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and

(1.4) $\langle\Phi'(u), v\rangle = \displaystyle\int_{\mathbb{R}^N} (\nabla u\nabla v + V(x)uv)\, \mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x, \quad \forall\, u, v \in H^1(\mathbb{R}^N),$

and a standard argument shows that the critical points of Φ are weak solutions of (1.1). Under assumption (V1), $A := -\triangle + V$ is a self-adjoint operator, acting on $H := L^2(\mathbb{R}^N)$ with domain $D(A) = H^2(\mathbb{R}^N, \mathbb{R})$. Setting $H^- := P_0 H$ and $H^+ := (id - P_0)H$, where $(P_\lambda)_{\lambda \in \mathbb{R}} : H \to H$ denote the spectral family of A, then we have the orthogonal decomposition $H = H^- + H^+$. Let $E = D(|A|^{\frac{1}{2}})$ be equipped with the inner product

(1.5) $\qquad\qquad (u,v) = (|A|^{\frac{1}{2}}u, |A|^{\frac{1}{2}}v)_{L^2}, \quad \forall\, u,\ v \in E$

and norm $\|u\| = \||A|^{\frac{1}{2}}u|_2$, where $(\cdot, \cdot)_{L^2}$ denote the inner product of $L^2(\mathbb{R}^N)$. Then $H^1(\mathbb{R}^N) \subset E$ and we have the decomposition

$$E = E^- \oplus E^+, \quad u = u^+ + u^-$$

where $E^\pm = E \cap H^\pm$ orthogonal with respect to both $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)$ and $u^\pm \in E^\pm$.

Define another norm on E

(1.6) $$\|u\|_\mu := (\|u\|^2 + |u|_\mu^2)^{\frac{1}{2}}, \quad \forall \, u \in \, E,$$

where $|\cdot|_\mu$ is the usual $L^\mu(\mathbb{R}^N)$ norm. Let E_μ^- be the completion of H^- with respect to $\|\cdot\|_\mu$, then

$$E_\mu = E_\mu^- \oplus E^+$$

is the completion of $H^1(\mathbb{R}^N)$ with respect to $\|\cdot\|_\mu$, moreover, E_μ is a reflective Banach space such that $H^1(\mathbb{R}^N) \subset E_\mu \subset E$ (see [2]).

Now, we are ready to state the main results of this paper.

Theorem 1.1. *Suppose that* (V1), (V2) *and* (S1)-(S4) *are satisfied, then problem* (1.1) *possesses a solution* $u \in E_\mu$ *such that* $\Phi(u) \geq \kappa_0$, *where* κ_0 *is a positive constant.*

At the end of this introduction, we state another result dealing with the case that 0 is a left end point of $\sigma(A)$, i.e. we replace (V2) by

(V2′) $0 \in \sigma(A)$ and there exists $\beta > 0$ such that $[-\beta, 0) \cap \sigma(A) = \emptyset$.

Theorem 1.2. *Suppose that* (V1), (V2) *hold and* $-f$ *satisfies* (S1)-(S4), *then problem* (1.1) *possesses a solution* $u \in E_\mu$ *such that* $\Phi(u) \leq -\kappa_0$, *where* κ_0 *is a positive constant.*

The proof of theorem 1.2 is analogous to theorem 1.1 by working with $-\Phi$ instead of Φ.

Remark 1.3. Condition (V2) *implies that V cannot be constant. The new condition* (S4) *firstly introduced by Tang* [24] *is weaker and unifies condition* (WN), (Ne), (AR) *and the following weaker version of* (AR):

(WAR) *there exists a* $\mu > 2$ *such that*

$$0 \leq \mu F(x, u) \leq u f(x, u), \quad (x, u) \in \mathbb{R}^N \times \mathbb{R}.$$

We point out that the assumption "strictly increasing" in (Ne) is very crucial in the argument of Szulkin and Weth [22] and the idea of monotone trick was firstly introduced by [19]. Later, it is developed by Jeanjean [6] for Landesman-Lazer type problems in $\mathbb{R}^N$ and Zou et al. [20, 29] for strongly indefinite problem. We emphasize

that in [31], Yang et al. used a generalized linking theorem established in [20] and considered a family of perturbed functionals

$$\Phi_\lambda(u) = \frac{1}{2}\|u^+\|^2 - \lambda \left(\frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x \right), \quad \lambda \in [1,2].$$

To our knowledge, this approach is not very satisfactory, because working with a family of perturbed functionals makes things unnecessary complicated. In the present paper, we no longer use monotonicity trick since without the condition (WN) or (Ne). The main ingredient in our approach is the observation that: although energy functional may possess unbounded Palais-Smale sequences, we will prove all Cerami sequences for the energy functional are bounded and establish the existence of nontrivial solutions with a new super-quadratic condition. Before proceeding to the proof of Theorem 1.1, we give two nonlinear examples to illustrate the assumptions.

Example 1.4. $f(x,t) = h(x)t\ln(1+|t|^{1/N})$, where $h \in C(\mathbb{R}^N, \mathbb{R}^+)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$.

One can easily verify that the above functions f satisfy (S1)-(S4), but not (F).

Example 1.5. $f(x,u) = h(x)\eta(u)$, where $h(x) \in C(\mathbb{R}^N, \mathbb{R}^+)$ is 1-periodic in x_i, $i = 1, 2, \ldots, N$; $\eta(-u) = -\eta(u)$ for $u \geq 0$, and

$$\eta(u) = \begin{cases} 0, & 0 \leq u < 1/2; \\ 2(u - \frac{1}{2}), & 1/2 \leq u < 1; \\ 3u^2 - \frac{15}{2}u^{3/2} + \frac{11}{2}u, & u \geq 1. \end{cases}$$

It is not difficult to verify that the above function f satisfies (S4) with $\theta_0 = 1/10$. Nevertheless, it satisfies neither (WN) nor (WAR).

This paper is organized as follows. In Section 2, some preliminary results are presented. The proof of main results is given in the last Section.

2. PRELIMINARIES

Throughout this paper, we denote by $|\cdot|_s$ the usual $L^s(\mathbb{R}^N)$ norm for $s \in [1, \infty)$. For any $s \in [2, 2^*]$, by Soblev embedding theorem, there exists an embedding constant $\gamma_s \in (0, \infty)$ such that

(2.1) $$|u|_s \leq \gamma_s\|u\|_{H^1}, \quad \forall u \in H^1(\mathbb{R}^N).$$

Since the spectrum of A restricted on H^+ is contained in $(\beta, +\infty)$, the norm $\|\cdot\|$ is equivalent to the $H^1(\mathbb{R}^N)$ norm on E^+. By the definition of E_μ, we know all norms $\|\cdot\|$, $\|\cdot\|_{H^1}$, $\|\cdot\|_\mu$ are equivalent on E^+, i.e.,

(2.2) $$\|u\|_\mu = \|u\| \sim \|u\|_{H^1}, \quad \forall u \in E^+.$$

But it is not true on $H^1(\mathbb{R}^N) \cap H^-$ because of $0 \in \sigma(A)$ as a right end point of $\sigma(A)$, thus the norm $\|\cdot\|$ is weaker than $H^1(\mathbb{R}^N)$ norm and $H^1(\mathbb{R}^N) \cap H^-$ is not complete with respect to $\|\cdot\|$. Moreover, we can not look for solutions of (1.1) in the completion E of $H^1(\mathbb{R}^N)$ under norm $\|\cdot\|$, because $\int_{\mathbb{R}^N} F(x,u)\mathrm{d}x$ is not well defined due to our assumption on $f(x,u)$.

To solve this problem, we set

$$(2.3) \quad E_m^- := E^- \cap P_{-\frac{1}{m}} H \subset H^- \subset E^-, \quad E_m := E_m^- \oplus E^+ \subset E, \quad \forall\, m \in \mathbb{N}^*.$$

Since the spectrum of A restricted on E_m is bounded away from 0, the norm $\|\cdot\|$ is equivalent to the $H^1(\mathbb{R}^N)$ norm on E_m, i.e., there exist positive constants c_1, c_2 such that

$$(2.4) \qquad\qquad\qquad c_1\|u\| \leq \|u\|_{H^1} \leq c_2\|u\|, \quad \forall\, u \in E_m.$$

Denote orthogonal projection as follows:

$$(2.5) \qquad\qquad\qquad Q_m := P_{-\frac{1}{m}} + (id - P_0): \ E \to E_m.$$

Then for any $u \in H^1(\mathbb{R}^N)$,

$$(2.6) \qquad Q_m u \to u \text{ as } m \to \infty, \text{ with respect to } \|\cdot\| \text{ and } |\cdot|_s, \ 2 \leq s < 2^*.$$

By (1.3), (1.4) and (1.6), one has

$$(2.7) \qquad\qquad \Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \Psi(u), \quad \forall\, u \in H^1(\mathbb{R}^N),$$

and

$$(2.8) \quad \langle\Phi'(u), v\rangle = (u^+, v^+) - (u^-, v^-) - \int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x, \quad \forall\, u, v \in H^1(\mathbb{R}^N),$$

where and in the sequel $\Psi(u) := \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x$. Set

$$\Phi_m = \Phi|_{E_m}, \ \Psi_m = \Psi|_{E_m}, \ \forall\, m \in \mathbb{N},$$

then Ψ_m is well defined in E_m and $\Phi_m, \Psi_m \in C^1(E_m, \mathbb{R})$. Moreover

$$(2.9) \qquad\qquad\qquad \langle\Psi'_m(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x,$$

and

$$(2.10) \qquad \langle\Phi'_m(u), v\rangle = (u^+, v^+) - (u^-, v^-) - \langle\Psi'_m(u), v\rangle, \qquad \forall\, u, v \in E_m.$$

The following generalized linking theorem plays an important role in proving our main results.

Let X be a Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$. For a functional $\varphi \in C^1(X, \mathbb{R})$, φ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in X one has $\varphi(u) \leq \liminf_{n \to \infty} \varphi(u_n)$, and φ' is said to be weakly sequentially continuous if $\lim_{n \to \infty} \langle \varphi'(u_n), v \rangle = \langle \varphi'(u), v \rangle$ for each $v \in X$.

Lemma 2.1. ([5, Theorem 4.5], [10, Theorem 2.1]). *Let X be a Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$, and let $\varphi \in C^1(X, \mathbb{R})$ of the form*

$$\varphi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \psi(u), \quad u = u^+ + u^- \in X^+ \perp X^-.$$

Suppose that the following assumptions are satisfied:

(I1) *$\psi \in C^1(X, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;*

(I2) *ψ' is weakly sequentially continuous;*

(I3) *there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that*

$$\kappa := \inf \varphi(S_\rho) > \sup \varphi(\partial Q),$$

where

$$S_\rho = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{se + v : v \in X^-, s \geq 0, \|se + v\| \leq r\}.$$

Then for some $c \geq \kappa$, there exists a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0.$$

Such a sequence is called a Cerami sequence on the level c, or a $(C)_c$ sequence.

Lemma 2.2. *Suppose that (S1) and (S2) are satisfied. Then for any $m \in \mathbb{N}$, Ψ_m is nonnegative, weakly sequentially lower semi-continuous, and Ψ'_m is weakly continuous.*

By (2.4) and using Sobolev's imbedding theorem, one can checks easily the above lemma, so we omit the proof.

Lemma 2.3. *Suppose that (V1), (V2), (S1) and (S2) are satisfied. Then for any $m \in \mathbb{N}$, there is a $\rho > 0$ such that $\kappa := \inf \Phi(S_\rho^+) = \inf \Phi_m(S_\rho^+) > 0$, where $S_\rho^+ = \partial B_\rho \cap E^+$.*

Lemma 2.3 can be proved in the same way as [22, Lemma 2.4].

Lemma 2.4. *Suppose that (V1), (V2) and (S1)-(S3) are satisfied. Let $e \in E^+$ with $\|e\| = 1$. Then for any $m \in \mathbb{N}$, there is a $r_0 > 0$ such that $\sup \Phi_m(\partial Q) \leq 0$,*

where

$$Q = \{se + w : w \in E_m^-, \ s \geq 0, \ \|se + w\| \leq r_0\}.$$

Proof. (S1) yields that $F(x,t) \geq 0$, $\forall \ (x,t) \in \mathbb{R}^{N+1}$, so we have $\Phi_m(u) \leq 0$, $\forall \ u \in E_m^-$. Next, it is sufficient to show that $\Phi_m(u) \to -\infty$ and $u \in E_m^- \oplus \mathbb{R}e$, $\|u\| \to \infty$, $\forall \ m \in \mathbb{N}$. Arguing indirectly, assume that for some sequence $\{w_n + s_n e\} \subset E_m^- \oplus \mathbb{R}e$ with $\|w_n + s_n e\| \to \infty$, there exist $m \in \mathbb{N}$ and $M > 0$ such that $\Phi_m(w_n + s_n e) \geq -M$, $\forall \ n \in \mathbb{N}$. Set $v_n = (w_n + s_n e)/\|w_n + s_n e\| = v_n^- + t_n e$, then $\|v_n^- + t_n e\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E_m, then $v_n \to v$ a.e. on $\mathbb{R}^n$, $v_n^- \rightharpoonup v^-$ in E_m^-, $t_n \to \bar{t}$, and

$$(2.11) \quad \frac{-M}{\|w_n + s_n e\|^2} \leq \frac{\Phi_m(w_n + s_n e)}{\|w_n + s_n e\|^2} = \frac{t_n^2}{2} - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} \mathrm{d}x.$$

If $\bar{t} = 0$, then it follows from (S1) and (2.11) that

$$0 \leq \frac{1}{2}\|v_n^-\|^2 + \int_{\mathbb{R}^N} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} \mathrm{d}x \leq \frac{t_n^2}{2} + \frac{-M}{\|w_n + s_n e\|^2} \to 0,$$

which yields $\|v_n^-\| \to 0$, and so $1 = \|v_n\| \to 0$, a contradiction.

If $\bar{t} \neq 0$, then $v \neq 0$, it follows from (S3), (2.11) and Fatou's lemma that

$$
\begin{aligned}
0 &\leq \limsup_{n \to \infty} \left[\frac{t_n^2}{2} - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} \mathrm{d}x \right] \\
&= \limsup_{n \to \infty} \left[\frac{t_n^2}{2} - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x, w_n + s_n e)}{(w_n + s_n e)^2} v_n^2 \mathrm{d}x \right] \\
&\leq \frac{1}{2} \lim_{n \to \infty} \frac{t_n^2}{2} - \liminf_{n \to \infty} \int_{\mathbb{R}^N} \frac{F(x, w_n + s_n e)}{(w_n + s_n e)^2} v_n^2 \mathrm{d}x \\
&\leq \frac{\bar{t}^2}{2} - \int_{\mathbb{R}^N} \liminf_{n \to \infty} \frac{F(x, w_n + s_n e)}{(w_n + s_n e)^2} v_n^2 \mathrm{d}x = -\infty,
\end{aligned}
$$

a contradiction. ∎

Applying Lemmas 2.1-2.4, we obtain the following lemma.

Lemma 2.5. *Suppose that* (V1), (V2) *and* (S1)-(S3) *are satisfied. Then there exist a constant $c_* \geq \kappa$ and a sequence $\{u_n\} \subset E_m$ satisfying*

$$(2.12) \quad \Phi_m(u_n) \to c_*, \quad \|\Phi_m'(u_n)\|(1 + \|u_n\|) \to 0, \quad \forall \ m \in \mathbb{N}.$$

The following lemma is very crucial to demonstrate the boundedness of $(C)_c$ sequence.

Lemma 2.6. ([24, Lemma 2.3]). *Suppose that* (V1), (V2), (S1), (S2) *and* (S4) *are satisfied. Then*

$$\Phi(u) \geq \Phi(tu^+) + \frac{t^2\|u^-\|^2}{2} + \frac{1-t^2}{2}\langle\Phi'(u), u\rangle + t^2\langle\Phi'(u), u^-\rangle$$

$$(2.13) \qquad -t^2\int_{t|u^+|>\theta_0|u|} f(x, u)u^+\mathrm{d}x, \qquad \forall\, t \geq 0,\ u \in H^1(\mathbb{R}^N).$$

Lemma 2.7. ([25, Lemma 3.2]). *Suppose that* (V1) *and* (V2) *are satisfied. Then the following conclusions hold.*

(i). $E_\mu \hookrightarrow L^s(\mathbb{R}^N)$ *for* $\mu \leq s \leq 2^*$;

(ii). $E_\mu \hookrightarrow H^1_{loc}(\mathbb{R}^N)$ *and* $E_\mu \hookrightarrow\hookrightarrow L^s_{loc}(\mathbb{R}^N)$ *for* $2 \leq s < 2^*$;

(iii). *For* $\mu \leq s \leq 2^*$, *there exists a constant* $C_s > 0$ *such that*

$$(2.14) \quad |u|^s_s \leq C_s\left[\|u\|^s + \left(\int_\Omega |u|^\mu\mathrm{d}x\right)^{s/\mu} + \left(\int_{\Omega^c} |u|^2\mathrm{d}x\right)^{s/2}\right], \quad \forall\, u \in E^-_\mu,$$

where $\Omega \subset \mathbb{R}^N$ *is any measurable set,* $\Omega^c = \mathbb{R}^N \setminus \Omega$.

Lemma 2.8. *Suppose that* (V1), (V2) *and* (S1)-(S4) *are satisfied. Then for any sequence* $\{u_n\} \subset E_m$ *satisfying*

$$(2.15) \qquad \Phi_m(u_n) \to c \geq 0, \quad \langle\Phi'_m(u_n), u^\pm_n\rangle \to 0, \quad \forall\, m \in \mathbb{N},$$

is bounded in E.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E. By Lemma 2.7, we have $v_n \rightharpoonup v$ in $L^s_{loc}(\mathbb{R}^N)$, $2 \leq s < 2^*$ and $v_n \to v$ a.e. on $\mathbb{R}^N$. If

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)} |v^+_n|^2\mathrm{d}x = 0,$$

then by Lions's concentration compactness principle ([8] or [28, Lemma 1.21]), $v^+_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. By (S1) and (S2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$(2.16) \qquad |f(x, u)| \leq \varepsilon|t| + C_\varepsilon|t|^{p-1}, \quad \forall\, (x, t) \in \mathbb{R}^N \times \mathbb{R}$$

and

$$(2.17) \qquad |F(x, t)| \leq \frac{1}{2}\varepsilon|t|^2 + \frac{1}{p}C_\varepsilon|t|^p \leq \varepsilon|t|^2 + C_\varepsilon|t|^p, \quad \forall\, (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

By (2.1), (2.2) and (2.4), for any $r > 0$,

$$
\text{(2.18)} \quad
\begin{aligned}
\int_{\mathbb{R}^N} F(x, rv_n^+) \mathrm{d}x &\leq \varepsilon \int_{R^N} |rv_n^+|^2 \mathrm{d}x + C_\varepsilon \int_{R^N} |rv_n^+|^p \mathrm{d}x \\
&\leq \varepsilon c_2^2 \gamma_2^2 r^2 \|v_n^+\|^2 + o(1) \to 0,
\end{aligned}
$$

and

$$
\text{(2.19)} \quad
\begin{aligned}
&\frac{r^2}{\|u_n\|} \int_{r|v_n^+| > \theta_0|u_n|} |f(x, u_n)| |v_n^+| \mathrm{d}x \\
&\leq \frac{r^2}{\|u_n\|} \int_{r|v_n^+| > \theta_0|u_n|} \left[\varepsilon |u_n| + C_\varepsilon |u_n|^{p-1} \right] |v_n^+| \mathrm{d}x \\
&\leq \frac{r^2}{\|u_n\|} \int_{r|v_n^+| > \theta_0|u_n|} \left[\varepsilon r \theta_0^{-1} |v_n^+|^2 + C_\varepsilon r^{p-1} \theta_0^{1-p} |v_n^+|^p \right] \mathrm{d}x \\
&\leq \frac{r^2}{\|u_n\|} \int_{\mathbb{R}^N} \left[\varepsilon r \theta_0^{-1} |v_n^+|^2 + C_\varepsilon r^{p-1} \theta_0^{1-p} |v_n^+|^p \right] \mathrm{d}x \\
&= \frac{\varepsilon r^3 \theta_0^{-1} c_2^2 \gamma_2^2 \|v_n^+\|^2 + C_\varepsilon r^{p+1} \theta_0^{1-p} |v_n^+|_p^p}{\|u_n\|} \to 0,
\end{aligned}
$$

since ε is chosen arbitrarily. Let $t_n = r/\|u_n\|$. By (2.12), (2.18), (2.19) and Lemma 2.6, one has

$$
\text{(2.20)} \quad
\begin{aligned}
c + o(1) &= \Phi_m(u_n) \\
&\geq \Phi_m(t_n u_n^+) + \frac{t_n^2 \|u_n^-\|^2}{2} + \frac{1 - t_n^2}{2} \langle \Phi_m'(u_n), u_n \rangle + t_n^2 \langle \Phi_m'(u_n), u_n^- \rangle \\
&\quad - t_n^2 \int_{t_n|u_n^+| > \theta_0|u_n|} f(x, u_n) u_n^+ \mathrm{d}x \\
&= \Phi_m(rv_n^+) + \frac{r^2 \|v_n^-\|^2}{2} + \left(\frac{1}{2} - \frac{r^2}{2\|u_n\|^2} \right) \langle \Phi_m'(u_n), u_n \rangle \\
&\quad + \frac{r^2}{\|u_n\|^2} \langle \Phi_m'(u_n), u_n^- \rangle - \frac{r^2}{\|u_n\|} \int_{r|v_n^+| > \theta_0|u_n|} f(x, u_n) v_n^+ \mathrm{d}x \\
&= \frac{r^2}{2} (\|v_n^+\|^2 + \|v_n^-\|^2) - \int_{\mathbb{R}^N} F(x, rv_n^+) \mathrm{d}x + o(1) \\
&\quad - \frac{r^2}{\|u_n\|} \int_{r|v_n^+| > \theta_0|u_n|} f(x, u_n) v_n^+ \mathrm{d}x \\
&= \frac{r^2}{2} + o(1),
\end{aligned}
$$

which leads to a contradiction if we take r big enough. Thus $\delta > 0$.

Going to a subsequence if necessary, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |v_n^+|^2 \mathrm{d}x > \frac{\delta}{2}$. Let $w_n(x) = v_n(x + k_n)$. Then

$$(2.21) \qquad \int_{B_{1+\sqrt{N}}(0)} |w_n^+|^2 \mathrm{d}x > \frac{\delta}{2}.$$

Since $V(x)$ is periodic, we have $\|w_n\| = \|v_n\| = 1$. Passing to a subsequence, we have $w_n \rightharpoonup w$ in E_m. By (2.4), $w_n \to w$ in $L_{loc}^s(\mathbb{R}^N)$, $2 \le s < 2^*$ and $w_n \to w$ a.e. on $\mathbb{R}^N$. Obviously, (2.21) implies that $w^+ \ne 0$ and $w \ne 0$. Now we define $\tilde{u}_n(x) = u_n(x + k_n)$, then $\tilde{u}_n/\|u_n\| = w_n \to w$ a.e. on $\mathbb{R}^N$.

For $x \in \{y \in \mathbb{R}^N : w(y) \ne 0\}$, we have $\lim_{n\to\infty} \tilde{u}_n(x) = \infty$. Hence, it follows from (2.7), (2.15), (S3) and Fatou's lemma that

$$0 = \lim_{n\to\infty} \frac{c + o(1)}{\|u_n\|^2} = \lim_{n\to\infty} \frac{\Phi_m(u_n)}{\|u_n\|^2}$$

$$= \lim_{n\to\infty} \left[\frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) - \int_{\mathbb{R}^N} \frac{F(x, \tilde{u}_n)}{\tilde{u}_n^2} w_n^2 \mathrm{d}x \right]$$

$$\le \frac{1}{2} - \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x, \tilde{u}_n)}{\tilde{u}_n^2} w_n^2 \mathrm{d}x$$

$$\le \frac{1}{2} - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x, \tilde{u}_n)}{\tilde{u}_n^2} w_n^2 \mathrm{d}x = -\infty.$$

This contradiction shows that $\{\|u_n\|\}_n$ is bounded. ∎

Lemma 2.9. *Suppose that* (V1), (V2) *and* (S1)-(S4) *are satisfied. Then there exists* $v^m \in E_m \setminus \{0\}$ *such that*

$$(2.22) \qquad \Phi_m(v^m) \le c_*, \qquad \Phi_m'(v^m) = 0, \quad \forall\, m \in \mathbb{N}.$$

Proof. For any fixed $m \in \mathbb{N}$, Lemma 2.5 implies the existence of a sequence $\{u_n\} \subset E_m$ satisfying (2.12). By Lemma 2.8, $\{u_n\}$ is bounded in E. If

$$\delta := \limsup_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2 \mathrm{d}x = 0.$$

Then by (2.4) and Lions's concentration compactness principle [8] or [28, Lemma 1.21], $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. By (2.16) and (2.17), it is not difficult to verify that

$$(2.23) \qquad \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2} f(x, u_n)u_n - F(x, u_n) \right] \mathrm{d}x = 0.$$

By (2.7), (2.8), (2.12) and (2.23), one can get that

$$(2.24) \qquad c_* = \Phi_m(u_n) - \frac{1}{2}\langle \Phi'_m(u_n), u_n \rangle + o(1)$$

$$(2.25) \qquad = \int_{\mathbb{R}^N} \left[\frac{1}{2} f(x, u_n) u_n - F(x, u_n) \right] dx + o(1) = o(1),$$

which is a contradiction. Thus $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |u_n|^2 dx > \frac{\delta}{2}$. Define $v_n = u_n(x + k_n)$ so that

$$(2.26) \qquad \int_{B_{1+\sqrt{N}}(0)} |v_n|^2 dx > \frac{\delta}{2}.$$

Since $V(x)$ and $f(x, u)$ are periodic in x, we have $\|v_n\| = \|u_n\|$ and

$$(2.27) \qquad \Phi_m(v_n) \to c_*, \quad \|\Phi'_m(v_n)\|(1 + \|v_n\|) \to 0.$$

Passing to a subsequence, we have $v_n \rightharpoonup v^m$ in E_m. By (2.4), we have $v_n \to v^m$ in $L^s_{loc}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $v_n \to v^m$ a.e. on $\mathbb{R}^N$. Hence it follows from (2.26) and (2.27) that $\Phi'_m(v^m) = 0$ and $v^m \ne 0$ in E_m. By (S4), we have

$$(2.28) \qquad \frac{1}{2} f(x, t) t \ge F(x, t), \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Then by (2.27), (2.28) and Fatou's lemma, one has

$$\begin{aligned}
c_* &= \lim_{n\to\infty} \left[\Phi_m(v_n) - \frac{1}{2}\langle \Phi'_m(v_n), v_n \rangle \right] \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2} f(x, v_n) v_n - F(x, v_n) \right] dx \\
&\ge \int_{\mathbb{R}^N} \lim_{n\to\infty} \left[\frac{1}{2} f(x, v_n) v_n - F(x, v_n) \right] dx \\
&= \int_{\mathbb{R}^N} \left[\frac{1}{2} f(x, v^m) v^m - F(x, v^m) \right] dx \\
&= \Phi_m(v^m) - \frac{1}{2}\langle \Phi'_m(v^m), v^m \rangle = \Phi_m(v^m).
\end{aligned}$$

$$(2.29)$$

This shows (2.22) holds.   ∎

3. Proof of Main Results

Lemma 3.1. *Suppose that* (V1), (V2) *and* (S1)-(S4) *are satisfied. Then the sequence* $\{v_m\}$ *obtained in Lemma* 2.9 *is bounded in* E_μ.

Proof.   By lemma 2.9, there exists a sequence still denote by $\{v_m\}_m$ and $c_0 \in [0, c_*]$ such that

$$(3.1) \qquad \Phi'_m(v_m) = 0, \qquad \Phi_m(v_m) \to c_0, \quad m \to \infty.$$

Hence $\{v_m\}_m$ is a $(PS)^*_{c_0}$ sequence for Φ. By a similar fashion as in the the proof of Lemma 2.8, we can prove that $\{v_m\}_m$ is bounded in E. We assert that $c_0 \geq \kappa_0 > 0$ for some positive constant κ_0. In fact, note that (2.1), (2.4), (2.8), (2.16) and (3.1) imply that

$$(3.2) \qquad \begin{aligned} \|v_m\|^2 &= \langle \Phi'_m(v_m), v_m^+ - v_m^- \rangle + \int_{\mathbb{R}^N} f(x, v_m)(v_m^+ - v_m^-)\mathrm{d}x \\ &\leq \varepsilon \gamma_2^2 c_2^2 \|v_m\|^2 + C_\varepsilon \gamma_p^p c_2^p \|v_m\|^p, \end{aligned}$$

which implies that

$$(3.3) \qquad \|v_m\| \geq \left(\frac{1 - \varepsilon \gamma_2^2 c_2^2}{C_\varepsilon \gamma_p^p c_2^p} \right)^{1/(p-2)} > 0, \text{ for } \varepsilon \text{ small enough.}$$

Let

$$(3.4) \qquad \varepsilon_0 = \frac{\theta_0}{2(2 + \theta_0)\gamma_2^2 c_2^2}, \qquad \beta_0 = \left(\frac{1}{\theta_0^{p-1}} + \frac{1}{p} \right) C_{\varepsilon_0} \gamma_p^p c_2^p.$$

Then (3.3) and (3.4) imply that

$$(3.5) \qquad 2p\beta_0 \|v_m\|^{p-2} > 2C_{\varepsilon_0} \gamma_p^p c_2^p \|v_m\|^{p-2} \geq 1.$$

Let $t_m = \frac{1}{\|v_m\|}(2p\beta_0)^{-1/(p-2)}$. Then (3.5) implies that $0 < t_m < 1$. It follows from (2.1), (2.4), (2.13), (2.16), (2.17), (3.1) and (3.4) that

$$\begin{aligned} &\Phi_m(v_m) \\ &\geq \frac{t_m^2 \|v_m\|^2}{2} - t_m^2 \int_{t_m|v_m^+|>\theta_0|v_m|} f(x, v_m)v_m^+ \mathrm{d}x - \int_{\mathbb{R}^N} F(x, t_m v_m^+)\mathrm{d}x \\ (3.6) \quad &\geq \frac{t_m^2 \|v_m\|^2}{2} - \left(\frac{t_m}{\theta_0} + \frac{1}{2} \right) \varepsilon_0 t_m^2 |v_m^+|_2^2 - \left(\frac{t_m}{\theta_0^{p-1}} + \frac{1}{p} \right) C_{\varepsilon_0} t_m^p |v_m^+|_p^p \\ &\geq \frac{t_m^2 \|v_m\|^2}{2} - \left(\frac{1}{\theta_0} + \frac{1}{2} \right) \varepsilon_0 \gamma_2^2 c_2^2 t_m^2 \|v_m\|^2 - \left(\frac{1}{\theta_0^{p-1}} + \frac{1}{p} \right) C_{\varepsilon_0} \gamma_p^p c_2^p t_m^p \|v_m\|^p \\ &= \frac{t_m^2 \|v_m\|^2}{4} - \beta_0 t_m^p \|v_m\|^p = \left(\frac{1}{2} - \frac{1}{p} \right) 2^{\frac{-p}{p-2}} (\beta_0 p)^{\frac{-2}{p-2}} := \kappa_0. \end{aligned}$$

This shows that $c_0 \geq \kappa_0 > 0$.

By (S1), (2.8) and (3.1), we have

(3.7)
$$\|v_m^+\|^2 - \|v_m^-\|^2 = \int_{\mathbb{R}^N} f(x, v_m)v_m \mathrm{d}x$$
$$\geq b_1 \left(\int_{|v_m|<1} |v_m|^\mu \mathrm{d}x + \int_{|v_m|\geq 1} |v_m|^2 \mathrm{d}x \right).$$

(2.1), (2.4), (2.14) and (3.7) imply the existence of a constant $C > 0$ such that

(3.8)
$$|v_m|_\mu^\mu$$
$$\leq 2^{\mu-1} \left(|v_m^+|_\mu^\mu + |v_m^-|_\mu^\mu \right)$$
$$\leq 2^{\mu-1}(1+C_\mu) \left[|v_m^+|_\mu^\mu + \|v_m^-\|^\mu + \int_{|v_m|<1} |v_m^-|^\mu \mathrm{d}x + \left(\int_{|v_m|\geq 1} |v_m^-|^2 \mathrm{d}x \right)^{\mu/2} \right]$$
$$\leq 2^{\mu-1}(1+C_\mu) \left[\left(1+\gamma_\mu^\mu c_2^\mu\right) \|v_m^+\|^\mu + 2^{\mu-1} \left(\int_{|v_m|<1} |v_m^+|^\mu \mathrm{d}x + \int_{|v_m|<1} |v_m|^\mu \mathrm{d}x \right) \right.$$
$$\left. + 2^{\mu/2} \left(\int_{|v_m|\geq 1} |v_m^+|^2 \mathrm{d}x + \int_{|v_m|\geq 1} |v_m|^2 \mathrm{d}x \right) \right]^{\mu/2}$$
$$\leq C \left(\|v_m^+\|^\mu + \|v_m^+\|^2 \right).$$

Since $\{\|v_m\|\}_m$ is bounded, by (1.6) and (3.8), $\{|v_m|_\mu\}_m$ is bounded and so $\{v_m\}$ is bounded in E_μ. ∎

Lemma 3.2. ([2, Corollary 2.3]). *Suppose that* (V1), (V2), (S1) *and* (S2) *are satisfied. If* $u \subset E$ *is a weak solution of the Schrödinger equations*

(3.9)
$$-\triangle u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N,$$

i.e.

(3.10)
$$\int_{\mathbb{R}^N} (\nabla u \nabla \psi + V(x)u\psi)\mathrm{d}x = \int_{\mathbb{R}^N} f(x, u)\psi \mathrm{d}x, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N),$$

then $u_n \to 0$ *as* $|x| \to \infty$.

Proof of Theorem 1.1. Since $\{v_m\}_m$ is bounded in E_μ, going to a subsequence if necessary, we may assume $v_m \rightharpoonup v$ in E_μ. By Lemma 2.7, $v_m \to v$ in $L_{loc}^s(\mathbb{R}^N)$, $2 \leq s < 2^*$, $v_m \to v$ a.e. on $\mathbb{R}^N$. (3.3) implies that $\{v_m\}_m$ is non-vanishing and passing to a $\mathbb{Z}^N$-translation if necessary, we may assume that $v \neq 0$. For any

$\phi \in C_0^\infty(\mathbb{R}^N)$, by (2.16), Lemma 2.7 and Hölder's inequality, we have

$$|\int_{\mathbb{R}^N} f(x, v_m)(id - Q_m)\phi \mathrm{d}x|$$

(3.11)
$$\leq \varepsilon \int_{\mathbb{R}^N} |v_m||(id - Q_m)\phi| \mathrm{d}x$$

$$+C_\varepsilon \int_{\mathbb{R}^N} |v_m|^{p-1}|(id - Q_m)\phi| \mathrm{d}x \to 0, \quad m \to \infty.$$

Observe that

$$(Av_m, \phi)_{L^2} = (Av_m, Q_m\phi)_{L^2}$$

(3.12)
$$= (\Phi'_m(v_m), Q_m\phi)_{L^2} + \int_{\mathbb{R}^N} f(x, v_m)\phi \mathrm{d}x$$

$$- \int_{\mathbb{R}^N} f(x, v_m)(id - Q_m)\phi \mathrm{d}x.$$

Then, by (3.1) and (3.11), taking limit $m \to \infty$ in (3.12), we get

$$(Av, \phi)_{L^2} = \int_{\mathbb{R}^N} f(x, v)\phi \mathrm{d}x.$$

This implies, by Lemma 3.2, that v is a nontrivial solution of problem (1.1). By a similar argument as (3.6) with v_m replaced by v, we can prove that $\Phi(v) \geq \kappa_0$. ∎

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