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GLOBAL BEHAVIORS FOR A CLASS OF MULTI-GROUP SIRS EPIDEMIC MODELS WITH NONLINEAR INCIDENCE RATE

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Abstract. In this paper, we study a class of multi-group SIRS epidemic models with nonlinear incidence rate which have cross patch infection between different groups. The basic reproduction number $\mathcal{R}_0$ is calculated. By using the method of Lyapunov functions, LaSalle's invariance principle, the theory of the nonnegative matrices and the theory of the persistence of dynamical systems, it is proved that if $\mathcal{R}_0 \leq 1$ then the disease-free equilibrium is globally asymptotically stable, and if $\mathcal{R}_0 > 1$ then the disease in the model is uniform persistent. Furthermore, when $\mathcal{R}_0 > 1$, by constructing new Lyapunov functions we establish the sufficient conditions of the global asymptotic stability for the endemic equilibrium.

1. INTRODUCTION

In the theoretical study of epidemic dynamical models, in recent years the multi-group epidemic models have been proposed to describe the spread of many infectious diseases in heterogeneous populations, such as mumps, gonorrhea, measles, West-Nile virus and HIV/AIDS (See, for example, [1-11]). A heterogeneous host population can be divided into several homogeneous groups according to modes of transmission, contact patterns, or geographic distributions, so that within-group and inter-group interactions could be modeled separately.

In [1], Guo, Li and Shuai have first succeeded to establish the completely global dynamics for a multi-group SIR model, by making use of the theory of non-negative matrices, Lyapunov functions and a subtle grouping technique in estimating the derivatives of Lyapunov functions guided by graph theory in 2006. Then, many researchers

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on different kinds of multi-group epidemic models, commonly follow this research to study the global stability of equilibria of models.

For multi-group epidemic models, the difficult problem is to obtain the global stability of endemic equilibrium. Many researches on the global stability of the endemic equilibrium for multi-group epidemic models are not perfect. Particularly, in [2] Muroya, Enatsu and Kuniya studied the following multi-group SIRS epidemic model with varying population sizes

$$(1)\qquad \begin{cases} \dfrac{dS_k}{dt} = & b_k - \mu_{k1}S_k - S_k\left(\displaystyle\sum_{j=1}^{n}\beta_{kj}I_j\right) + \delta_k R_k, \\ \dfrac{dI_k}{dt} = & S_k\left(\displaystyle\sum_{j=1}^{n}\beta_{kj}I_j\right) - (\mu_{k2}+\gamma_k)I_k, \\ \dfrac{dR_k}{dt} = & \gamma_k I_k - (\mu_{k3}+\delta_k)R_k. \end{cases}$$

By using the Lyapunov function techniques, the authors established that, under the conditions $\mu_{k1} \le \min\{\mu_{k2}, \mu_{k3}\}\,(k = 1, 2, \ldots, n)$ and matrix $B = (\beta_{kj})_{n\times n}$ is irreducible, if basic reproduction number $\mathcal{R}_0 \le 1$, then disease-free equilibrium $\mathbf{E}^0 = (S_1^0, 0, 0, S_2^0, 0, 0, \ldots, S_n^0, 0, 0)$ is globally asymptotically stable, and if $\mathcal{R}_0 > 1$, then model (1) is permanent and there exists an endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, R_1^*, S_2^*, I_2^*, R_2^*, \ldots, S_n^*, I_n^*, R_n^*)$. Furthermore, if $\mathcal{R}_0 > 1$ and $\mu_{k1}S_k^* - \delta_k R_k^* \ge 0\,(k = 1, 2, \ldots, n)$, then E^* also is globally asymptotically stable. In [3], Korobrinikov studied the following multi-group SIR and SEIR models:

$$(2)\qquad \begin{cases} \dfrac{dS}{dt} = & \lambda - \displaystyle\sum_{j=1}^{n}\beta_j S I_j - \mu S, \\ \dfrac{dI_i}{dt} = & p_i\displaystyle\sum_{j=1}^{n}\beta_j S I_j - \delta_i I_i, \\ \dfrac{dR_i}{dt} = & \omega_i I_i - \mu R_i, \quad i = 1, 2, \ldots, n. \end{cases}$$

and

$$(3)\qquad \begin{cases} \dfrac{dS}{dt} = & \lambda - \displaystyle\sum_{j=1}^{n}\beta_j S I_j - \mu S, \\ \dfrac{dE}{dt} = & \displaystyle\sum_{j=1}^{n}\beta_j S I_j + \sum_{j=1}^{n} r_j I_j - \sigma E, \\ \dfrac{dI_i}{dt} = & \gamma_i E - \delta_i I_i, \\ \dfrac{dR_i}{dt} = & \omega_i I_i - \mu R_i, \quad i = 1, 2, \ldots, n. \end{cases}$$

The author obtained the necessary and sufficient conditions for the global stability of equilibria. That is, the disease-free equilibrium of models (2) and (3) is globally asymptotically stable if and only if the basic reproduction number $\mathcal{R}_0 \leq 1$, and the endemic equilibrium of models (2) and (3) is globally asymptotically stable if and only if $\mathcal{R}_0 > 1$.

Motivated by above works, as an extension of above model (1) we consider the following multi-group SIRS epidemic model with nonlinear incidence rate:

$$(4) \qquad \begin{cases} \dfrac{dS_k}{dt} = & b_k - \displaystyle\sum_{j=1}^{n} \beta_{kj} f_k(S_k) g_j(I_j) - \mu_{k1} S_k + \delta_k R_k, \\ \dfrac{dI_k}{dt} = & \displaystyle\sum_{j=1}^{n} \beta_{kj} f_k(S_k) g_j(I_j) - (\mu_{k2} + \gamma_k) I_k, \\ \dfrac{dR_k}{dt} = & \gamma_k I_k - (\mu_{k3} + \delta_k) R_k, \quad k = 1, 2, \ldots, n. \end{cases}$$

We will study the dynamical behaviors of model (4). The basic reproduction number $\mathcal{R}_0$ is calculated. Under basic assumption $(\mathbf{H})$ (see Section 2), it is proved that only when $\mathcal{R}_0 \leq 1$ then the disease-free equilibrium is globally asymptotically stable by using the method of Lyapunov functions, LaSalle's invariance principle (See [13]) and the properties of nonnegative irreducible matrices (See [12,18]), and only when $\mathcal{R}_0 > 1$ then the disease in the model is uniform persistent by using the theory of persistence of dynamical systems (See [14]). Furthermore, when $\mathcal{R}_0 > 1$, we will establish new sufficient conditions of the global asymptotic stability of the endemic equilibrium under assumption $(\mathbf{H}^*)$ (see Theorem 2) by constructing new Lyapunov functions.

The organization of this paper is as follows. In the second section we give a model description, and further obtain a result on the positivity and boundedness of solutions of model (4). In the third section we discuss the existence and global asymptotic stability of the disease-free equilibrium, and the uniform persistence of the disease for model (4). In the fourth section we will study the global asymptotic stability of the endemic equilibrium of model (4). Lastly, in the fifth section we will give a conclusion.

2. Preliminaries

In model (4), for each $k = 1, 2, \ldots, n$, $S_k(t)$, $I_k(t)$ and $R_k(t)$ denote the numbers of susceptible, infected and recovered individuals in k-th group at time t, respectively. b_k is the recruitment rate of the total population, μ_{ki} $(i = 1, 2, 3)$ are the natural death rates of susceptible, infected and recovered individuals in k-th group, and death rate μ_{k2} also includes the disease-related death rate of the infected individuals in k-th group. δ_k is the rate at which recovered individuals in k-th group lose immunity and return to the corresponding susceptible class. γ_k is the recovery rate of the infected individuals in k-th group. In model (4), we assume that each two groups are connected by the direct transport such as airplanes or trains, etc. Therefore, for model (4), the only input is the recruitment. Moreover, not only for infective individuals I_k in k-th group, the disease is transmitted to the susceptible individuals S_k by the incidence rate $\beta_{kk} f_k(S_k) g_k(I_k)$ with a transmission rate β_{kk}, but also we consider cross path infection between different groups such that for each I_j ($j \neq k,\ j = 1, 2, \ldots, n$), who travel from other j-th group into k-th group, the disease is transmitted by the incidence rate $\beta_{kj} f_k(S_k) g_j(I_j)$ with a transmission rate β_{kj}. Furthermore, we assume that δ_k, β_{kj} are nonnegative constants, and $b_k, \mu_{k1}, \mu_{k2}, \mu_{k3}$ and γ_k are positive constants.

The initial condition for model (4) is given in the following form

$$S_k(0) > 0,\ I_k(0) \geq 0,\ R_k(0) \geq 0,\ k = 1, 2, \ldots, n. \tag{5}$$

We also have some special forms of nonlinear incidence rate in model (4), such as $f_k(S_k) = S_k$, $f_k(S_k) = \frac{S_k}{1+\lambda_k S_k}$, $g_j(I_j) = I_j$ and $g_j(I_j) = \frac{I_j}{1+\alpha_j I_j}$. In this paper, we always assume that

(H) $f_k(S_k)$ and $g_k(I_k)$ satisfy the local Lipschitz condition and are strictly monotone increasing on $S_k \in [0, \infty)$ and $I_k \in [0, \infty)$, respectively, $g_k(0) = 0$, $\frac{I_k}{g_k(I_k)}$ is also strictly monotone increasing on $I_k \in (0, \infty)$, $g_k'(0)$ exists with $g_k'(0) > 0$ for $k = 1, 2, \ldots, n$.

Remark 1. If function $g_k(I_k)$ ($k = 1, 2, \ldots, n$) satisfies that second order derivative $g_k''(I_k)$ exists and $g_k''(I_k) \leq 0$ for all $I_k \in [0, \infty)$, then we can easily prove that $\frac{I_k}{g_k(I_k)}$ is monotone increasing on $I_k \in (0, \infty)$.

By the biological meanings of natural death rates $\mu_{k1}, \mu_{k2}, \mu_{k3}$ of susceptible, infected and recovered individuals, we may assume that

$$\mu_{k1} \leq \min\{\mu_{k2}, \mu_{k3}\}, \quad k = 1, 2, \ldots, n. \tag{6}$$

Moreover, for simplicity in this paper, we also assume that

$$n \times n \text{ matrix} \quad \mathbf{B} = (\beta_{kj})_{n\times n} \quad \text{is irreducible.} \tag{7}$$

That is, an infected individual in the first group can cause infection to a susceptible individual in the second group through an infection path.

Let $S_k^0 = \frac{b_k}{\mu_{k1}}$, $k = 1, 2, \ldots, n$, and $\mathbf{S}^0 = (S_1^0, S_2^0, \ldots, S_n^0)^T$. The matrix $\tilde{\mathbf{M}}(\mathbf{S}^0)$ is defined by

$$\tilde{\mathbf{M}}(\mathbf{S}^0) = \left(\frac{\beta_{kj} f_k(S_k^0) g_j'(0)}{\mu_{k2} + \gamma_k} \right)_{n\times n}. \tag{8}$$

Further let

$$\tilde{\mathcal{R}}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)), \tag{9}$$

where $\rho(\tilde{\mathbf{M}}(\mathbf{S}^0))$ denotes the spectral radius of matrix $\tilde{\mathbf{M}}(\mathbf{S}^0)$.

Firstly, on the positivity and ultimate boundedness of solutions of model (4) with initial condition (5), we have the following results.

Lemma 1. *For any solution $(S_k(t), I_k(t), R_k(t), 1 \le k \le n)$ of model (4) with initial condition (5), we have*

(1) *if $S_k(0) > 0, I_k(0) > 0, R_k(0) > 0, 1 \le k \le n$, then $S_k(t) > 0, I_k(t) > 0$ and $R_k(t) > 0$ for $1 \le k \le n$ and $t \ge 0$;*

(2) *if condition (6) holds, then for each $k = 1, 2, \ldots, n$*

$$\begin{aligned} &\limsup_{t\to+\infty} N_k(t) \le S_k^0, \quad \limsup_{t\to+\infty} S_k(t) \le S_k^0, \\ &\limsup_{t\to+\infty} I_k(t) \le S_k^0, \quad \limsup_{t\to+\infty} R_k(t) \le S_k^0. \end{aligned} \tag{10}$$

Proof. Firstly, from assumption (**H**) for any initial condition (5) model (4) has a unique solution $(S_k(t), I_k(t), R_k(t), 1 \le k \le n)$. Let $m(t) = \min\{S_k(t), I_k(t), R_k(t), 1 \le k \le n\}$, then $m(0) > 0$ by $S_k(0) > 0, I_k(0) > 0$ and $R_k(0) > 0$ for any $k = 1, 2, \ldots, n$. By the continuity of solutions of model (4), if there exists a positive t_1 such that $m(t_1) = 0$, then we can assume that $m(t) > 0$ for all $0 \le t < t_1$.

If there exists a positive integer $k_1 \in \{1, 2, \ldots, n\}$ such that $m(t_1) = S_{k_1}(t_1) = 0$, then by the derivative definition, we have $\frac{dS_{k_1}(t_1)}{dt} \le 0$. But, by the first equations of model (4),

$$\frac{dS_{k_1}(t_1)}{dt} = b_{k_1} + \delta_{k_1} R_{k_1}(t_1) \ge b_{k_1} > 0,$$

which leads to a contradiction.

If there exists a positive integer $k_1 \in \{1, 2, \ldots, n\}$ such that $m(t_1) = I_{k_1}(t_1) = 0$, by the second equations of model (4), for any $t \in [0, t_1)$,

$$\frac{dI_{k_1}}{dt} = \sum_{j=1}^{n} \beta_{k_1 j} f_{k_1}(S_{k_1}) g_j(I_j) - (\mu_{k_1 2} + \gamma_{k_1}) I_{k_1} \ge -(\mu_{k_1 2} + \gamma_{k_1}) I_{k_1}.$$

Hence, we obtain

$$I_{k_1}(t) \ge I_{k_1}(0) e^{-(\mu_{k_1 2} + \gamma_{k_1})t}$$

for any $t \in [0, t_1)$, then by the continuity of solutions, we have

$$I_{k_1}(t_1) \ge I_{k_1}(0) e^{-(\mu_{k_1 2} + \gamma_{k_1})t_1} > 0,$$

which leads to a contradiction.

Similarly, if there exists a positive integer $k_1 \in \{1, 2, \ldots, n\}$ such that $m(t_1) = R_{k_1}(t_1) = 0$, then we also can obtain a contradiction.

Therefore, $m(t) > 0$ for all $t \ge 0$, which implies $S_k(t) > 0$, $I_k(t) > 0$ and $R_k(t) > 0$ for all $t \ge 0$ and $k = 1, 2, \ldots, n$.

Let $N_k(t) = S_k(t) + I_k(t) + R_k(t)$, $1 \le k \le n$, then from model (4) we have

$$\frac{dN_k(t)}{dt} = b_k - \mu_{k1} S_k(t) - \mu_{k2} I_k(t) - \mu_{k3} R_k(t).$$

This shows that in model (4), total population size $N_k(t)$ is variable along with time. By (6), we have

$$\frac{dN_k(t)}{dt} \le b_k - \mu_{k1} N_k(t), \ \ k = 1, 2, \ldots, n,$$

from this we obtain (10). This completes the proof. ■

3. Extinction and Permanence

In order to clearly study the persistence of the disease for model (4), we firstly introduce some notations and results on the uniformly persistence of dynamical systems given by Kuang in [14].

Let X be a complete metric space with metric d, and let X^0 be an open set of X such that $X = X^0 \cup \partial X^0$, where ∂X^0 is the boundary of X^0. We assume that ∂X^0 is nonempty. Suppose that $T(t) : X \to X$, $t \ge 0$ is a C^0-semigroup on X, that is, $T(0) = I$ an identity, $T(t+s) = T(t)T(s)$ for any $t, s \ge 0$ and $T(t)x$ is continuous in $(t, x) \in [0, \infty) \times X$. $T(t)$ is said to be point dissipative in X if there is a bounded set $B \subset X$ such that, for any $x \in X$, there is a $t_0 = t_0(x, B) > 0$ such that $T(t)x \in B$ for all $t \ge t_0$. For any $x \in X$, $\gamma^+(x) = \{T(t)x : t \ge 0\}$ is said to be the positive orbit through x, and its ω-limit set $\omega(x)$ is defined by

$$\omega(x) = \{y \in X : \text{there is a sequence } t_n \to \infty \ (n \to \infty) \text{ such that } \lim_{n\to\infty} T(t_n)x = y\}.$$

A set B in X is said to be invariant if $T(t)B \subset B$ for all $t \ge 0$, where $T(t)B = \{T(t)x : x \in B\}$. Let $B \subset X$ be invariant, then we easily prove that there is a negative orbit $\gamma^-(x)$ defined for all $t \le 0$ through each point $x \in B$ that belongs to B, and we can further define the α-limit set $\alpha(x)$ of $\gamma^-(x)$ in a similar manner. A nonempty invariant subset M of X is called an isolated invariant set if it is the maximal invariant set of a neighborhood of itself. The stable set $W^s(A)$ of a compact invariant set A is defined by

$$W^s(A) = \{x \in X : \omega(x) \ne \emptyset, \ \omega(x) \subset A\}$$

and its unstable set $W^u(A)$ is defined by

$$W^u(A) = \{x \in X : \alpha(x) \ne \emptyset, \ \alpha(x) \subset A\}.$$

Further, we assume that $T(t)$ for any $t \ge 0$ satisfies

$$T(t) : X^0 \to X^0, \quad T(t) : \partial X^0 \to \partial X^0.$$

This shows that X^0 and ∂X^0 are the invariant sets in X. Let A and B be two isolated invariant sets, A is said to be chained to B, written as $A \to B$, if there exists a point $x \in X$, and $x \notin A \cup B$, such that $x \in W^u(A) \cap W^s(B)$. A finite sequence $M_1, M_2, \ldots, M_k$ of isolated invariant sets is called a chain if $M_1 \to M_2 \to \cdots \to M_k$, and if $M_k = M_1$ the chain is called a cycle.

Lastly, we have that $T(t)$ is said to be uniformly persistent if there is a constant $\eta > 0$ such that, for any $x \in X^0$,

$$\liminf_{t\to\infty} d(T(t)x, \partial X^0) \geq \eta.$$

Now, on the uniform persistence of $T(t)$ we have the following result which is given in [14]. See Theorem 2.4 in [14, Chapter 8].

Lemma 2. *Suppose that $T(t)$ satisfies*

(i) *$T(t)$ is compact for $t \geq 0$ and point dissipative in X;*

(ii) *there exists a finite sequence $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ of compact and isolated invariant sets such that*

   (a) *$M_i \cap M_j = \emptyset$ for any $i, j = 1, 2, \ldots, k$ and $i \neq j$;*
   (b) *$\Omega(\partial X^0) \triangleq \cup_{x\in\partial X^0}\omega(x) \subset \cup_{i=1}^k M_i$;*
   (c) *no subset of $\mathcal{M}$ forms a cycle in ∂X^0;*
   (d) *$W^s(M_i) \cap X^0 = \emptyset$ for each $i = 1, 2, \ldots, k$.*

Then $T(t)$ is uniformly persistent.

It is easy to see that disease-free equilibrium $\mathbf{E}^0 = (S_1^0, 0, 0, S_2^0, 0, 0, \ldots, S_n^0, 0, 0)$ of model (4) always exist. On the global stability of $\mathbf{E}^0$ and the uniformly persistence of model (4), we have the following theorem. The technique used in the proof is similar to those of Guo et al. in [1] and Muroya et al. in [2].

Theorem 1. *Assume that conditions (6) and (7) hold.*

(i) *If $\tilde{\mathcal{R}}_0 \leq 1$, then disease-free equilibrium $\mathbf{E}^0$ is the unique equilibrium of model (4) and it is globally asymptotically stable in $\mathbf{\Gamma}$.*

(ii) *If $\tilde{\mathcal{R}}_0 > 1$, then $\mathbf{E}^0$ is unstable and model (4) is uniformly persistent in $\mathbf{\Gamma}^0$, where $\mathbf{\Gamma}^0$ is the interior of the feasible region $\mathbf{\Gamma}$, and $\mathbf{\Gamma} = \mathbf{\Gamma}_1 \times \mathbf{\Gamma}_2 \times \cdots \times \mathbf{\Gamma}_n$ with*

$$\mathbf{\Gamma}_k = \{(S_k, I_k, R_k) \in \mathbf{R}_+^3 : S_k + I_k + R_k \leq S_k^0\}, \quad k = 1, 2, \ldots, n.$$

Proof. (i) Let $\mathbf{S} = (S_1, S_2, \ldots, S_n)^T$, $\mathbf{I} = (I_1, I_2, \ldots, I_n)^T$, and put

$$\mathbf{M}(\mathbf{S}) = \Big(\frac{\beta_{kj} f_k(S_k) g_j(I_j)}{(\mu_{k2} + \gamma_k) I_j}\Big)_{n\times n}.$$

Since in $\mathbf{\Gamma}$, it holds that $0 \leq S_k \leq S_k^0$, $g_k(I_k) \leq g_k'(0)I_k$ for $k = 1, 2, \ldots, n$, we have $0 \leq \mathbf{M}(\mathbf{S}) \leq \tilde{\mathbf{M}}(\mathbf{S^0})$. Since $\mathbf{B}$ is irreducible, we obtain that $\tilde{\mathbf{M}}(\mathbf{S}^0)$ and $\mathbf{M}(\mathbf{S})$ are also irreducible. Therefore, $\rho(\mathbf{M}(\mathbf{S})) < \rho(\tilde{\mathbf{M}}(\mathbf{S^0}))$, provided $\mathbf{S} \neq \mathbf{S^0}$. (See, for example, Lemma 2.3 in [12]).

If $\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S^0})) \leq 1$, then for $\mathbf{S} \neq \mathbf{S^0}$, we have $\rho(\mathbf{M}(\mathbf{S})) < \rho(\tilde{\mathbf{M}}(\mathbf{S^0})) \leq 1$, and the equation

$$\mathbf{M}(\mathbf{S})\mathbf{I} = \mathbf{I}$$

has only trivial solution $\mathbf{I} = \mathbf{0}$. This shows that $\mathbf{E^0}$ is the unique equilibrium of model (4).

Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the positive left eigenvector of $\tilde{\mathbf{M}}(\mathbf{S^0})$ corresponding to the spectral radius $\rho(\tilde{\mathbf{M}}(\mathbf{S}^0))$, that is,

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)\rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) = (\alpha_1, \alpha_2, \ldots, \alpha_n)\tilde{\mathbf{M}}(\mathbf{S}^0).$$

Define the Lyapunov function as follows

$$L = \sum_{k=1}^{n} \frac{\alpha_k}{\mu_{k2} + \gamma_k} I_k.$$

Calculating the time derivative of L along any solution of model (4), when $\tilde{\mathcal{R}}_0 \leq 1$, we obtain

$$\begin{aligned}
\frac{dL}{dt} &= \sum_{k=1}^{n} \frac{\alpha_k}{\mu_{k2} + \gamma_k} \left[\sum_{j=1}^{n} \beta_{kj} f_k(S_k) g_j(I_j) - (\mu_{k2} + \gamma_k) I_k \right] \\
&= \sum_{k=1}^{n} \alpha_k \left[\sum_{j=1}^{n} \frac{\beta_{kj} f_k(S_k) g_j(I_j)}{(\mu_{k2} + \gamma_k)} - I_k \right] \\
&= (\alpha_1, \alpha_2, \ldots, \alpha_n)[\mathbf{M}(\mathbf{S})\mathbf{I} - \mathbf{I}] \\
&\leq \sum_{k=1}^{n} \alpha_k \left[\sum_{j=1}^{n} \frac{\beta_{kj} f_k(S_k^0) g_j'(0) I_j}{(\mu_{k2} + \gamma_k)} - I_k \right] \\
&= (\alpha_1, \alpha_2, \ldots, \alpha_n)[\tilde{\mathbf{M}}(\mathbf{S}^0)\mathbf{I} - \mathbf{I}] \\
&= [\rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) - 1](\alpha_1, \alpha_2, \ldots, \alpha_n)\mathbf{I} \\
&\leq 0.
\end{aligned}$$

If $\tilde{\mathcal{R}}_0 < 1$, then $\frac{dL}{dt} = 0$ if and only if $\mathbf{I} = 0$. If $\tilde{\mathcal{R}}_0 = 1$, then $\frac{dL}{dt} = 0$ implies

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)\mathbf{M}(\mathbf{S})\mathbf{I} = (\alpha_1, \alpha_2, \ldots, \alpha_n)\mathbf{I}. \tag{11}$$

If $\mathbf{S} \neq \mathbf{S^0}$, then we have

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)\mathbf{M}(\mathbf{S}) < (\alpha_1, \alpha_2, \ldots, \alpha_n)\tilde{\mathbf{M}}(\mathbf{S}^0) = (\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Hence, (11) has only trivial solution $\mathbf{I} = \mathbf{0}$. Therefore, $\frac{dL}{dt} = 0$ implies that $\mathbf{I} = \mathbf{0}$ or $\mathbf{S} = \mathbf{S^0}$. It can be verified that the maximal invariant subset of the set

$$\{(S_1, I_1, R_1, S_2, I_2, R_2, \ldots, S_n, I_n, R_n) \in \mathbf{\Gamma} : \frac{dL}{dt} = 0\}$$

is the singleton $\{\mathbf{E^0}\}$. By the LaSalle's invariance principle (See Theorem 6.4 in [13, Chapter 2]), we obtain that $\mathbf{E^0}$ is globally asymptotically stable.

(ii) If $\tilde{\mathcal{R}}_0 > 1$ and $\mathbf{I} \neq \mathbf{0}$, we know that

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)\tilde{\mathbf{M}}(\mathbf{S}^0) - (\alpha_1, \alpha_2, \ldots, \alpha_n) = [\rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) - 1](\alpha_1, \alpha_2, \ldots, \alpha_n) > 0$$

and hence

$$\frac{dL}{dt} = (\alpha_1, \alpha_2, \ldots, \alpha_n)[\mathbf{M}(\mathbf{S})\mathbf{I} - \mathbf{I}] > 0$$

in a neighborhood of $\mathbf{E^0}$ in $\mathbf{\Gamma}^0$ by the continuity of functions $f_k(S_k)$ and $g_k(I_k)$. This implies that $\mathbf{E^0}$ is unstable.

Now, we prove the uniform persistence of model (4) if $\tilde{\mathcal{R}}_0 > 1$ by using Lemma 2. Here, the technique given by Li et al. in [15] is developed. Let

$$\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2 \times \cdots \times \mathbf{X}_n, \quad \mathbf{X}^0 = \mathbf{X}_1^0 \times \mathbf{X}_2^0 \times \cdots \times \mathbf{X}_n^0$$

and

$$\partial\mathbf{X}^0 = \partial\mathbf{X}_1^0 \times \partial\mathbf{X}_2^0 \times \cdots \times \partial\mathbf{X}_n^0,$$

where for each $k = 1, 2, \ldots, n$

$$\mathbf{X}_k = \{(S_k, I_k, R_k) : S_k > 0, I_k \geq 0, R_k \geq 0\},$$

$$\mathbf{X}_k^0 = \{(S_k, I_k, R_k) : S_k > 0, I_k > 0, R_k \geq 0\}$$

and $\partial\mathbf{X}_k^0 = \{(S_k, I_k, R_k) : S_k > 0, I_k = 0, R_k \geq 0\}$. We have $\mathbf{X} = \mathbf{X}^0 \cup \partial\mathbf{X}^0$. Let $Y(t) = (S_k(t), I_k(t), R_k(t), 1 \leq k \leq n)$ be the solution of model (4) with initial value $Y(0) = Y_0$ in $\mathbf{X}$, then $Y(t) \in \mathbf{X}$ for all $t \geq 0$. For any $t \geq 0$ we define a map $T(t) : \mathbf{X} \to \mathbf{X}$ as follows

$$T(t)Y_0 = Y(t).$$

It is clear that $T(t)$ is a C^0-semigroup, that is, From Lemma 1, we can easily prove that $T(t)$ is compact for $t \geq 0$ and point dissipative in X, and $\mathbf{X}^0$ and $\partial\mathbf{X}^0$ are the positively invariable sets for $T(t)$. In $\partial\mathbf{X}^0$, we have $\mathbf{I} = 0$, and hence,

$$\begin{aligned} \frac{dS_k}{dt} &= b_k - \mu_{k1}S_k + \delta_k R_k, \\ \frac{dR_k}{dt} &= -(\mu_{k3} + \delta_k)R_k. \end{aligned} \tag{12}$$

It is clear that system (12) has a globally asymptotically stable equilibrium $(S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$. This shows that the disease-free equilibrium $\mathbf{E}^0$ in $\partial\mathbf{X}^0$ is a global attractor of $T(t)$, which implies that $\Omega(\partial\mathbf{X}^0) = \{\mathbf{E}^0\}$ and $\mathcal{M} = \{M_1\}$ with $M_1 = \{\mathbf{E}^0\}$. Therefore, the conditions (i), (ii)(a), (ii)(b) and (ii)(c) in Lemma 2 are satisfied.

Now, we prove the condition $(ii)(d)$ of Lemma 2. From $\tilde{\mathcal{R}}_0 > 1$, there is a small enough constant $\varepsilon_0 > 0$ such that $\rho(\tilde{\mathbf{M}}(\mathbf{S}^0, \varepsilon_0)) > 1$, where

$$\tilde{\mathbf{M}}(\mathbf{S}^0, \varepsilon_0) = \left(\frac{\beta_{kj} f_k(S_k^0 - \varepsilon_0) g_j(\varepsilon_0)}{(\mu_{k2} + \gamma_k)\varepsilon_0}\right)_{n\times n}.$$

If $W^s(\mathbf{E}^0) \cap \mathbf{X}^0 \neq \emptyset$, then there is a solution $(S_k(t), I_k(t), R_k(t), 1 \le k \le n)$ of model (4) with the initial value in $\mathbf{X}^0$ such that $(S_k(t), I_k(t), R_k(t)) \to (S_k^0, 0, 0)$ for each $k = 1, 2, \ldots, n$ as $t \to \infty$, then there is a $T > 0$ such that $S_k(t) > S_k^0 - \varepsilon_0$ and $I_k(t) < \varepsilon_0$ for all $t \ge T$ and $k = 1, 2, \ldots, n$. Thus, from assumption $(\mathbf{H})$ and the second equation of model (4) we have

$$\frac{dI_k(t)}{dt} \ge \sum_{j=1}^{n} \beta_{kj} f_k(S_k^0 - \varepsilon_0)\frac{g_j(\varepsilon_0)}{\varepsilon_0} I_j(t) - (d_2 + \gamma) I_k(t), \quad k = 1, 2, \ldots, n \tag{13}$$

for all $t \ge T$. Since $\mathbf{B}$ is irreducible, we obtain that $\tilde{\mathbf{M}}(\mathbf{S}^0, \varepsilon_0)$ is also irreducible. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the positive left eigenvector of $\tilde{\mathbf{M}}(\mathbf{S^0}, \varepsilon_0)$ corresponding to the spectral radius $\rho(\tilde{\mathbf{M}}(\mathbf{S}^0, \varepsilon_0))$, that is,

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)\rho(\tilde{\mathbf{M}}(\mathbf{S}^0, \varepsilon_0)) = (\alpha_1, \alpha_2, \ldots, \alpha_n)\tilde{\mathbf{M}}(\mathbf{S}^0, \varepsilon_0).$$

Define the Lyapunov function as follows

$$L(t) = \sum_{k=1}^{n} \frac{\alpha_k}{\mu_{k2} + \gamma_k} I_k(t).$$

Calculating the time derivative of $L(t)$, from (13) we obtain

$$\begin{aligned}
\frac{dL(t)}{dt} &\ge \sum_{k=1}^{n} \frac{\alpha_k}{\mu_{k2} + \gamma_k}\left[\sum_{j=1}^{n} \beta_{kj} f_k(S_k^0 - \varepsilon_0)\frac{g_j(\varepsilon_0)}{\varepsilon_0} I_j(t) - (\mu_{k2} + \gamma_k) I_k(t)\right] \\
&= \sum_{k=1}^{n} \alpha_k \left[\sum_{j=1}^{n} \frac{\beta_{kj} f_k(S_k^0) g_j(\varepsilon_0)}{(\mu_{k2} + \gamma_k)\varepsilon_0} I_j(t) - I_k(t)\right] \\
&= (\alpha_1, \alpha_2, \ldots, \alpha_n)[\tilde{\mathbf{M}}(\mathbf{S}^0, \varepsilon_0)\mathbf{I}(t) - \mathbf{I}(t)] \\
&= [\rho(\tilde{\mathbf{M}}(\mathbf{S}^0, \varepsilon_0)) - 1](\alpha_1, \alpha_2, \ldots, \alpha_n)\mathbf{I}(t) \\
&> 0
\end{aligned}$$

for all $t \ge T$. This implies $L(t) \ge L(T) > 0$ for all $t \ge T$, which leads to a contradiction with $\lim_{t\to\infty} L(t) = 0$. Therefore, $W^s(\mathbf{E}^0) \cap \mathbf{X}^0 = \emptyset$. Thus, from Lemma 2 we have that $T(t)$ is uniformly persistent. Consequently, model (4) is uniformly persistent. This completes the proof. ■

The ultimate boundedness of solutions in $\mathbf{\Gamma}^0$, together with the uniform persistence of model (4), implies the existence of a positive equilibrium of model (4) (See, Theorem D.3 in [16] or Theorem 2.8.6 in [17]). Therefore, we have the following corollary.

Corollary 1. *Assume that conditions (6) and (7) hold. If $\tilde{\mathcal{R}}_0 > 1$, then model (4) has at least one endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, R_1^*, S_2^*, I_2^*, R_2^*, \ldots, S_n^*, I_n^*, R_n^*)$.*

4. Global Stability for Endemic Equilibrium E^*

In this section, we assume $\tilde{\mathcal{R}}_0 > 1$, and we will prove that the endemic equilibrium of model (4) is globally asymptotically stable in $\mathbf{\Gamma}^0$. The method that we use here is to construct the appropriate Lyapunov functions and use Lemma 2.1 given in [1]. By Corollary 1, there exists an endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, R_1^*, S_2^*, I_2^*, R_2^*, \ldots, S_n^*, I_n^*, R_n^*) \in \mathbf{\Gamma}^0$ which satisfies

$$\tag{14} \begin{cases} b_k - \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) - \mu_{k1} S_k^* + \delta_k R_k^* = 0, \\ \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) - (\mu_{k2} + \gamma_k) I_k^* = 0, \\ \gamma_k I_k^* - (\mu_{k3} + \delta_k) R_k^* = 0, \quad k = 1, 2, \ldots, n. \end{cases}$$

Consider the auxiliary function as follows.

$$U_1 = \sum_{k=1}^{n} v_k \Big\{ \int_{S_k^*}^{S_k} \Big(1 - \frac{f_k(S_k^*)}{f_k(u)}\Big) du + \int_{I_k^*}^{I_k} \Big(1 - \frac{g_k(I_k^*)}{g_k(u)}\Big) du \Big\},$$

where constants $v_i\,(i = 1, 2, \cdots, n)$ are positive which will be determined in the following Lemma 4. We give the following symbols

$$\tag{15} \begin{aligned} & x_k = \frac{S_k}{S_k^*},\ y_k = \frac{I_k}{I_k^*},\ z_k = \frac{R_k}{R_k^*}, \\ & f_k(x_k) = \frac{f_k(S_k)}{f_k(S_k^*)},\ g_k(y_k) = \frac{g_k(I_k)}{g_k(I_k^*)},\ k = 1, 2, \ldots, n. \end{aligned}$$

and $h(x) = x - 1 - \ln x$.

Lemma 3. *Assume that condition (7) holds. If $\tilde{\mathcal{R}}_0 > 1$, then*

$$
\begin{aligned}
&\frac{dU_1(t)}{dt}\\
&=\sum_{k=1}^{n} v_k\Big\{-\mu_{k1}S_k^*\Big(1-\frac{1}{f_k(x_k)}\Big)(x_k-1)+\delta_k R_k^*\Big(1-\frac{1}{f_k(x_k)}\Big)(z_k-1)\Big\}\\
&\quad+\sum_{k=1}^{n} v_k\Big[1-\frac{1}{g_k(y_k)}\Big]\Big[g_k(y_k)-y_k\Big]\Big\{\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\Big\}\\
&\quad-\sum_{k=1}^{n} v_k\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\left[h\left(\frac{1}{f_k(x_k)}\right)+h\left(\frac{f_k(x_k)g_j(y_j)}{g_k(y_k)}\right)\right]\\
&\quad+\sum_{k=1}^{n}\Big\{\sum_{j=1}^{n} v_j\beta_{jk}f_j(S_j^*)g_k(I_k^*)-v_k(\mu_{k2}+\gamma_k)I_k^*\Big\}h\big(g_k(y_k)\big).
\end{aligned}
\tag{16}
$$

Proof. By model (4), and (14) and (15), we have

$$
\begin{aligned}
\frac{dS_k}{dt} &= b_k-\sum_{j=1}^{n}\beta_{kj}f_k(S_k)g_j(I_j)-\mu_{k1}S_k+\delta_k R_k\\
&\quad-\left[b_k-\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)-\mu_{k1}S_k^*+\delta_k R_k^*\right]\\
&=-\sum_{j=1}^{n}\beta_{kj}[f_k(S_k)g_j(I_j)-f_k(S_k^*)g_j(I_j^*)]-\mu_{k1}(S_k-S_k^*)+\delta_k(R_k-R_k^*)\\
&=-\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)[f_k(x_k)g_j(y_j)-1]-\mu_{k1}S_k^*(x_k-1)+\delta_k R_k^*(z_k-1),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{dI_k}{dt} &= \sum_{j=1}^{n}\beta_{kj}f_k(S_k)g_j(I_j)-(\mu_{k2}+\gamma_k)I_k\\
&\quad-\left[\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)-(\mu_{k2}+\gamma_k)I_k^*\right]\\
&=\sum_{j=1}^{n}\beta_{kj}[f_k(S_k)g_j(I_j)-f_k(S_k^*)g_j(I_j^*)]-(\mu_{k2}+\gamma_k)[I_k-I_k^*]\\
&=\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)[f_k(x_k)g_j(y_j)-1]-(\mu_{k2}+\gamma_k)I_k^*[y_k-1]\\
&=\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)[f_k(x_k)g_j(y_j)-y_k].
\end{aligned}
$$

Differentiating $U_1(t)$ and by the second equations of (14), we can obtain

$$
\begin{aligned}
&\frac{dU_1(t)}{dt}\\
&=\sum_{k=1}^n v_k\Big\{\Big[1-\frac{f_k(S_k^*)}{f_k(S_k)}\Big]\frac{dS_k}{dt}+\Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\frac{dI_k}{dt}\Big\}\\
&=\sum_{k=1}^n v_k\Big[1-\frac{f_k(S_k^*)}{f_k(S_k)}\Big]\Big\{-\sum_{j=1}^n\beta_{kj}[f_k(S_k)g_j(I_j)-f_k(S_k^*)g_j(I_j^*)]\\
&\quad-\mu_{k1}(S_k-S_k^*)+\delta_k(R_k-R_k^*)\Big\}\\
&\quad+\sum_{k=1}^n v_k\Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big\{\sum_{j=1}^n\beta_{kj}f_k(S_k)g_j(I_j)-(\mu_{k2}+\gamma_k)I_k\Big\}\\
&=\sum_{k=1}^n v_k\Big\{-\mu_{k1}S_k^*\Big(1-\frac{1}{f_k(x_k)}\Big)(x_k-1)+\delta_kR_k^*\Big(1-\frac{1}{f_k(x_k)}\Big)(z_k-1)\Big\}\\
&\quad-\sum_{k=1}^n v_k\Big[1-\frac{f_k(S_k^*)}{f_k(S_k)}\Big]\Big\{\sum_{j=1}^n\beta_{kj}[f_k(S_k)g_j(I_j)-f_k(S_k^*)g_j(I_j^*)]\Big\}\\
&\quad+\sum_{k=1}^n v_k\Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big\{\sum_{j=1}^n\beta_{kj}f_k(S_k)g_j(I_j)-\frac{I_k}{I_k^*}\sum_{j=1}^n\beta_{kj}f_k(S_k^*)g_j(I_j^*)\Big\}.
\end{aligned}\tag{17}
$$

For the last term of (17), we have

$$
\begin{aligned}
&\sum_{k=1}^n v_k\Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big\{\sum_{j=1}^n\beta_{kj}f_k(S_k)g_j(I_j)-\frac{I_k}{I_k^*}\sum_{j=1}^n\beta_{kj}f_k(S_k^*)g_j(I_j^*)\Big\}\\
&=\sum_{k=1}^n v_k\Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big\{\sum_{j=1}^n\beta_{kj}f_k(S_k)g_j(I_j)-\frac{I_k}{I_k^*}\sum_{j=1}^n\beta_{kj}f_k(S_k^*)g_j(I_j^*)\\
&\quad+\frac{g_k(I_k)}{g_k(I_k^*)}\sum_{j=1}^n\beta_{kj}f_k(S_k^*)g_j(I_j^*)-\frac{g_k(I_k)}{g_k(I_k^*)}\sum_{j=1}^n\beta_{kj}f_k(S_k^*)g_j(I_j^*)\Big\}\\
&=\sum_{k=1}^n v_k\Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big\{\Big[\frac{g_k(I_k)}{g_k(I_k^*)}-\frac{I_k}{I_k^*}\Big]\sum_{j=1}^n\beta_{kj}f_k(S_k^*)g_j(I_j^*)\\
&\quad+\sum_{j=1}^n\beta_{kj}\Big[f_k(S_k)g_j(I_j)-\frac{g_k(I_k)}{g_k(I_k^*)}f_k(S_k^*)g_j(I_j^*)\Big]\Big\}\\
&=\sum_{k=1}^n v_k\Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big[\frac{g_k(I_k)}{g_k(I_k^*)}-\frac{I_k}{I_k^*}\Big]\Big\{\sum_{j=1}^n\beta_{kj}f_k(S_k^*)g_j(I_j^*)\Big\}\\
&\quad+\sum_{k=1}^n v_k\Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big\{\sum_{j=1}^n\beta_{kj}\Big[f_k(S_k)g_j(I_j)-\frac{g_k(I_k)}{g_k(I_k^*)}f_k(S_k^*)g_j(I_j^*)\Big]\Big\}.
\end{aligned}\tag{18}
$$

It is obvious that

$$
\Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big[\frac{g_k(I_k)}{g_k(I_k^*)}-\frac{I_k}{I_k^*}\Big]=\frac{I_k}{g_k(I_k^*)}\Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big[\frac{g_k(I_k)}{I_k}-\frac{g_k(I_k^*)}{I_k^*}\Big]\le 0.
$$

Hence, from the first term of (18), we obtain

$$
\begin{aligned}
&\sum_{k=1}^{n} v_k \Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big[\frac{g_k(I_k)}{g_k(I_k^*)}-\frac{I_k}{I_k^*}\Big]\Big\{\sum_{j=1}^{n}\beta_{kj} f_k(S_k^*)g_j(I_j^*)\Big\}\\
&=\sum_{k=1}^{n} v_k \Big[1-\frac{1}{g_k(y_k)}\Big]\Big[g_k(y_k)-y_k\Big]\Big\{\sum_{j=1}^{n}\beta_{kj} f_k(S_k^*)g_j(I_j^*)\Big\}\\
&\le 0.
\end{aligned}
$$

From the second term of (17) and the second term of (18), we further have

$$
\begin{aligned}
&\sum_{k=1}^{n} v_k \Big[1-\frac{g_k(I_k^*)}{g_k(I_k)}\Big]\Big\{\sum_{j=1}^{n}\beta_{kj}\Big[f_k(S_k)g_j(I_j)-\frac{g_k(I_k)}{g_k(I_k^*)}f_k(S_k^*)g_j(I_j^*)\Big]\Big\}\\
&\quad-\sum_{k=1}^{n} v_k \Big[1-\frac{f_k(S_k^*)}{f_k(S_k)}\Big]\Big\{\sum_{j=1}^{n}\beta_{kj}[f_k(S_k)g_j(I_j)-f_k(S_k^*)g_j(I_j^*)]\Big\}\\
&=\sum_{k=1}^{n} v_k \Big[1-\frac{1}{g_k(y_k)}\Big]\Big\{\sum_{j=1}^{n}\beta_{kj} f_k(S_k^*)g_j(I_j^*)[f_k(x_k)g_j(y_j)-g_k(y_k)]\Big\}\\
&\quad-\sum_{k=1}^{n} v_k \Big[1-\frac{1}{f_k(x_k)}\Big]\Big\{\sum_{j=1}^{n}\beta_{kj} f_k(S_k^*)g_j(I_j^*)[f_k(x_k)g_j(y_j)-1]\Big\}\\
&=\sum_{k=1}^{n} v_k \sum_{j=1}^{n}\beta_{kj} f_k(S_k^*)g_j(I_j^*)\Big\{[f_k(x_k)g_j(y_j)-g_k(y_k)]\Big[1-\frac{1}{g_k(y_k)}\Big]\\
&\quad-[f_k(x_k)g_j(y_j)-1]\Big[1-\frac{1}{f_k(x_k)}\Big]\Big\}\\
&=\sum_{k=1}^{n} v_k \sum_{j=1}^{n}\beta_{kj} f_k(S_k^*)g_j(I_j^*)\Big[2-\frac{f_k(x_k)g_j(y_j)}{g_k(y_k)}-g_k(y_k)+g_j(y_j)-\frac{1}{f_k(x_k)}\Big]\\
&=\sum_{k=1}^{n} v_k \sum_{j=1}^{n}\beta_{kj} f_k(S_k^*)g_j(I_j^*)\Big[h\big(g_j(y_j)\big)-h\big(g_k(y_k)\big)\\
&\quad-h\left(\frac{1}{f_k(x_k)}\right)-h\left(\frac{f_k(x_k)g_j(y_j)}{g_k(y_k)}\right)\Big]\\
&=\sum_{k=1}^{n} v_k \sum_{j=1}^{n}\beta_{kj} f_k(S_k^*)g_j(I_j^*)\Big[h\big(g_j(y_j)\big)-h\big(g_k(y_k)\big)\Big]\\
&\quad-\sum_{k=1}^{n} v_k \sum_{j=1}^{n}\beta_{kj} f_k(S_k^*)g_j(I_j^*)\left[h\left(\frac{1}{f_k(x_k)}\right)+h\left(\frac{f_k(x_k)g_j(y_j)}{g_k(y_k)}\right)\right].
\end{aligned}
\tag{19}
$$

For the first term of (19), we have

$$
\begin{aligned}
&\sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) \Big[h\big(g_j(y_j)\big) - h\big(g_k(y_k)\big)\Big] \\
&= \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) h\big(g_j(y_j)\big) \\
&\quad - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) h\big(g_k(y_k)\big) \\
&= \sum_{j=1}^{n} v_j \sum_{k=1}^{n} \beta_{jk} f_j(S_j^*) g_k(I_k^*) h\big(g_k(y_k)\big) - \sum_{k=1}^{n} v_k(\mu_{k2}+\gamma_k) I_k^* h\big(g_k(y_k)\big) \\
&= \sum_{k=1}^{n} \Big\{ \sum_{j=1}^{n} v_j \beta_{jk} f_j(S_j^*) g_k(I_k^*) - v_k(\mu_{k2}+\gamma_k) I_k^* \Big\} h\big(g_k(y_k)\big).
\end{aligned} \tag{20}
$$

Thus, from (17)-(20) we finally obtain

$$
\begin{aligned}
&\frac{dU_1(t)}{dt} \\
&= \sum_{k=1}^{n} v_k \Big\{ -\mu_{k1} S_k^* \Big(1 - \frac{1}{f_k(x_k)}\Big)(x_k - 1) + \delta_k R_k^* \Big(1 - \frac{1}{f_k(x_k)}\Big)(z_k - 1) \Big\} \\
&\quad + \sum_{k=1}^{n} v_k \Big[1 - \frac{1}{g_k(y_k)}\Big] \Big[g_k(y_k) - y_k\Big] \Big\{ \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) \Big\} \\
&\quad - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) \left[h\left(\frac{1}{f_k(x_k)}\right) + h\left(\frac{f_k(x_k) g_j(y_j)}{g_k(y_k)}\right)\right] \\
&\quad + \sum_{k=1}^{n} \Big\{ \sum_{j=1}^{n} v_j \beta_{jk} f_j(S_j^*) g_k(I_k^*) - v_k(\mu_{k2}+\gamma_k) I_k^* \Big\} h\big(g_k(y_k)\big).
\end{aligned} \tag{21}
$$

Therefore, (16) is obtained. This completes the proof. ■

Now, we consider the following equation

$$
\sum_{k=1}^{n} \Big\{ \sum_{j=1}^{n} v_j \beta_{jk} f_j(S_j^*) g_k(I_k^*) - v_k(\mu_{k2}+\gamma_k) I_k^* \Big\} h\big(g_k(y_k)\big) = 0.
$$

We give the following lemma which comes from the well-known techniques in Guo et al. [1].

Lemma 4. *Assume that condition (7) holds. Let*

$$
\tilde{\beta}_{kj} = \beta_{kj} f_k(S_k^*) g_j(I_j^*), \quad k, j = 1, 2, \ldots, n.
$$

and

$$\tilde{\mathbf{B}} = \begin{bmatrix} \sum_{j\neq 1}\tilde{\beta}_{1j} & -\tilde{\beta}_{21} & \cdots & -\tilde{\beta}_{n1} \\ -\tilde{\beta}_{12} & \sum_{j\neq 2}\tilde{\beta}_{2j} & \cdots & -\tilde{\beta}_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ -\tilde{\beta}_{1n} & -\tilde{\beta}_{2n} & \cdots & \sum_{j\neq n}\tilde{\beta}_{nj} \end{bmatrix}.$$

Further, let $\mathbf{x} = (v_1, v_2, \ldots, v_n)^T$*, where* v_k $(k = 1, 2, \ldots, n)$ *denotes the cofactor of the k-th diagonal entry of* $\tilde{\mathbf{B}}$*. Then,* $v_k > 0$ *and*

$$\sum_{j=1}^{n} v_j \beta_{jk} f_j(S_j^*) g_k(I_k^*) = v_k(\mu_{k2} + \gamma_k) I_k^*, \quad k = 1, 2, \ldots, n. \tag{22}$$

Proof. Since $\mathbf{B}$ is irreducible, we have that matrices $(\tilde{\beta}_{kj})_{n\times n}$ and $\tilde{\mathbf{B}}$ are also irreducible. From Lemma 2.1 in [1], we know that $\mathbf{x} = (v_1, v_2, \ldots, v_n)^T$ is the solution of the following linear equation

$$\tilde{\mathbf{B}}\mathbf{x} = \mathbf{0}, \tag{23}$$

and we have $v_i > 0$ for $i = 1, 2, \ldots, n$. Then, from (23), we obtain

$$\begin{bmatrix} \tilde{\beta}_{11} & \tilde{\beta}_{21} & \cdots & \tilde{\beta}_{n1} \\ \tilde{\beta}_{12} & \tilde{\beta}_{22} & \cdots & \tilde{\beta}_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{\beta}_{1n} & \tilde{\beta}_{2n} & \cdots & \tilde{\beta}_{n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \left(\sum_{j=1}^{n}\tilde{\beta}_{1j}\right)v_1 \\ \left(\sum_{j=1}^{n}\tilde{\beta}_{2j}\right)v_2 \\ \vdots \\ \left(\sum_{j=1}^{n}\tilde{\beta}_{nj}\right)v_n \end{bmatrix}.$$

Therefore, we have

$$\sum_{j=1}^{n} v_j \tilde{\beta}_{jk} = v_k \sum_{j=1}^{n} \tilde{\beta}_{kj}, \quad k = 1, 2, \ldots, n.$$

From this, we finally obtain

$$\sum_{j=1}^{n} v_j \beta_{jk} f_j(S_j^*) g_k(I_k^*) = v_k \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) = v_k(\mu_{k2} + \gamma_k) I_k^*,\ k = 1, 2, \ldots, n.$$

This completes the proof. ■

Let $N_k = S_k + I_k + R_k$, $N_k^* = S_k^* + I_k^* + R_k^*$, $\tilde{N}_k = S_k + I_k + \tilde{c}_k R_k$, $\tilde{N}_k^* = S_k^* + I_k^* + \tilde{c}_k R_k^*$, $n_k = \frac{N_k}{N_k^*}$ and $\tilde{n}_k = \frac{\tilde{N}_k}{\tilde{N}_k^*}$. Furthermore, for each $k = 1, 2, \ldots, n$, we define the constants $\tilde{c}_k$ and $\tilde{\varepsilon}_k$ as follows

$$\tilde{c}_k = \frac{\mu_{k2} - \mu_{k1}}{\gamma_k} + 1, \quad \tilde{\varepsilon}_k = \tilde{c}_k(\mu_{k3} - \mu_{k1} + \delta_k) - \delta_k.$$

Lemma 5. *For each $k = 1, 2, \ldots, n$, we have the following two cases:*
(1) *Assume $\mu_{k1} = \mu_{k2} = \mu_{k3} = \mu_k$, and W_k is defined by*

$$W_k = \frac{(R_k - R_k^*)^2}{2} + \frac{\gamma_k^2}{4\mu_k(\mu_k + \gamma_k + \delta_k)} \frac{(N_k - N_k^*)^2}{2}, \tag{24}$$

then it holds that

$$\begin{aligned} \frac{dW_k}{dt} &= -(\mu_k + \gamma_k + \delta_k)\Big(R_k^*(z_k - 1) - \frac{\gamma_k N_k^*}{2(\mu_k + \gamma_k + \delta_k)}(n_k - 1)\Big)^2 \\ &\quad -\gamma_k R_k^* S_k^*(z_k - 1)(x_k - 1). \end{aligned} \tag{25}$$

(2) *Otherwise, and W_k is defined by*

$$W_k = \frac{(R_k - R_k^*)^2}{2} + \frac{\gamma_k}{\tilde{\varepsilon}_k} \frac{(\tilde{N}_k - \tilde{N}_k^*)^2}{2}, \tag{26}$$

then it holds that

$$\begin{aligned} \frac{dW_k}{dt} &= -(\gamma_k \tilde{c}_k + \mu_{k3} + \delta_k)(R_k^*)^2(z_k - 1)^2 - \frac{\gamma_k \mu_{k1}}{\tilde{\varepsilon}_k}(\tilde{N}_k^*)^2(\tilde{n}_k - 1)^2 \\ &\quad -\gamma_k R_k^* S_k^*(z_k - 1)(x_k - 1). \end{aligned} \tag{27}$$

Proof. Directly from model (4), we have

$$\begin{aligned} \frac{dR_k}{dt} &= \gamma_k I_k - (\mu_{k3} + \delta_k)R_k - [\gamma_k I_k^* - (\mu_{k3} + \delta_k)R_k^*] \\ &= \gamma_k(I_k - I_k^*) - (\mu_{k3} + \delta_k)(R_k - R_k^*) \\ &= \gamma_k[(N_k - N_k^*) + (S_k - S_k^*) + (R_k - R_k^*)] - (\mu_{k3} + \delta_k)(R_k - R_k^*) \\ &= \gamma_k[(\tilde{N}_k - \tilde{N}_k^*) - (S_k - S_k^*) - \tilde{c}_k(R_k - R_k^*)] - (\mu_{k3} + \delta_k)(R_k - R_k^*) \\ &= \gamma_k(\tilde{N}_k - \tilde{N}_k^*) - \gamma_k(S_k - S_k^*) - (\gamma_k \tilde{c}_k + \mu_{k3} + \delta_k)(R_k - R_k^*). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{(R_k - R_k^*)^2}{2}\right) \\ &= (R_k - R_k^*)\frac{dR_k}{dt} \\ &= (R_k - R_k^*)[\gamma_k(\tilde{N}_k - \tilde{N}_k^*) - \gamma_k(S_k - S_k^*) \\ &\quad -(\gamma_k \tilde{c}_k + \mu_{k3} + \delta_k)(R_k - R_k^*)] \\ &= -\gamma_k R_k^* S_k^*(z_k - 1)(x_k - 1) + \gamma_k R_k^* \tilde{N}_k^*(z_k - 1)(\tilde{n}_k - 1) \\ &\quad -(\gamma_k \tilde{c}_k + \mu_{k3} + \delta_k)(R_k^*)^2(z_k - 1)^2, \end{aligned} \tag{28}$$

Since $\tilde{N}_k = N_k + (\tilde{c}_k - 1)R_k$, directly from model (4) we have

$$\begin{aligned}
\frac{d\tilde{N}_k}{dt} &= b_k - \mu_{k1}S_k - \mu_{k2}I_k - \mu_{k3}R_k + (\tilde{c}_k - 1)[\gamma_k I_k - (\mu_{k3} + \delta_k)R_k] \\
&= -\mu_{k1}(S_k - S_k^*) - (\mu_{k2} + \gamma_k - \tilde{c}_k\gamma_k)(I_k - I_k^*) \\
&\quad -[\tilde{c}_k(\mu_{k3} + \delta_k) - \delta_k](R_k - R_k^*) \\
&= -\mu_{k1}[(\tilde{N}_k - \tilde{N}_k^*) - (I_k - I_k^*) - \tilde{c}_k(R_k - R_k^*)] \\
&\quad -(\mu_{k2} + \gamma_k - \tilde{c}_k\gamma_k)(I_k - I_k^*) \\
&\quad -[\tilde{c}_k(\mu_{k3} + \delta_k) - \delta_k](R_k - R_k^*) \\
&= -\mu_{k1}(\tilde{N}_k - \tilde{N}_k^*) - [(\mu_{k2} + \gamma_k - \mu_{k1}) - \tilde{c}_k\gamma_k](I_k - I_k^*) \\
&\quad -[\tilde{c}_k(\mu_{k3} + \delta_k - \mu_{k1}) - \delta_k](R_k - R_k^*) \\
&= -\mu_{k1}(\tilde{N}_k - \tilde{N}_k^*) - \tilde{\varepsilon}_k(R_k - R_k^*).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{(\tilde{N}_k - \tilde{N}_k^*)^2}{2}\right) &= (\tilde{N}_k - \tilde{N}_k^*)\frac{d\tilde{N}_k}{dt} \\
&= (\tilde{N}_k - \tilde{N}_k^*)[-\mu_{k1}(\tilde{N}_k - \tilde{N}_k^*) - \tilde{\varepsilon}_k(R_k - R_k^*)] \\
&= -\mu_{k1}(\tilde{N}_k^*)^2(\tilde{n}_k - 1)^2 - \tilde{\varepsilon}_k\tilde{N}_k^*R_k^*(\tilde{n}_k - 1)(z_k - 1).
\end{aligned} \tag{29}$$

(1) Assume $\mu_{k1} = \mu_{k2} = \mu_{k3} = \mu_k$, then $\tilde{c}_k = 1$ and $\tilde{\varepsilon}_k = 0$, for the W_k defined by (24), from (28) and (29), we have

$$\begin{aligned}
\frac{dW_k}{dt} &= \frac{d}{dt}\Big(\frac{(R_k - R_k^*)^2}{2}\Big) + \frac{\gamma_k^2}{4\mu_k(\mu_k + \gamma_k + \delta_k)}\frac{d}{dt}\Big(\frac{(N_k - N_k^*)^2}{2}\Big) \\
&= [-\gamma_k R_k^*S_k^*(z_k - 1)(x_k - 1) - (\gamma_k + \mu_k + \delta_k)(R_k^*)^2(z_k - 1)^2 \\
&\quad +\gamma_k R_k^*N_k^*(z_k - 1)(n_k - 1)] + \frac{\gamma_k^2}{4\mu_k(\mu_k + \gamma_k + \delta_k)}[-\mu_k(N_k^*)^2(n_k - 1)^2] \\
&= -\gamma_k R_k^*S_k^*(z_k - 1)(x_k - 1) - (\gamma_k + \mu_k + \delta_k)\Big[(R_k^*)^2(z_k - 1)^2 \\
&\quad -\frac{\gamma_k R_k^*N_k^*}{\gamma_k + \mu_k + \delta_k}(z_k - 1)(n_k - 1) + \frac{\gamma_k^2}{4(\mu_k + \gamma_k + \delta_k)^2}(N_k^*)^2(n_k - 1)^2\Big] \\
&= -(\mu_k + \gamma_k + \delta_k)\Big(R_k^*(z_k - 1) - \frac{\gamma_k N_k^*}{2(\mu_k + \gamma_k + \delta_k)}(n_k - 1)\Big)^2 \\
&\quad -\gamma_k R_k^*S_k^*(z_k - 1)(x_k - 1).
\end{aligned}$$

(2) Otherwise, for the W_k defined by (26), from (28) and (29) we have

$$
\begin{aligned}
\frac{dW_k}{dt} &= \frac{d}{dt}\Big(\frac{(R_k-R_k^*)^2}{2}\Big)+\frac{\gamma_k}{\tilde{\varepsilon}_k}\frac{d}{dt}\Big(\frac{(\tilde{N}_k-\tilde{N}_k^*)^2}{2}\Big)\\
&= [-\gamma_k R_k^* S_k^*(z_k-1)(x_k-1)-(\gamma_k\tilde{c}_k+\mu_{k3}+\delta_k)(R_k^*)^2(z_k-1)^2\\
&\quad +\gamma_k R_k^*\tilde{N}_k^*(z_k-1)(\tilde{n}_k-1)]\\
&\quad +\frac{\gamma_k}{\tilde{\varepsilon}_k}[-\mu_{k1}(\tilde{N}_k^*)^2(\tilde{n}_k-1)^2-\tilde{\varepsilon}_k\tilde{N}_k^* R_k^*(\tilde{n}_k-1)(z_k-1)]\\
&= -(\gamma_k\tilde{c}_k+\mu_{k3}+\delta_k)(R_k^*)^2(z_k-1)^2-\frac{\gamma_k\mu_{k1}}{\tilde{\varepsilon}_k}(\tilde{N}_k^*)^2(\tilde{n}_k-1)^2\\
&\quad -\gamma_k R_k^* S_k^*(z_k-1)(x_k-1).
\end{aligned}
$$

Therefore, we finally obtain (25) and (27). This completes the proof. ∎

Now, we consider the following function

$$U = U_1 + U_2,$$

where

$$U_2=\sum_{k=1}^{n} v_k\frac{\omega_k\delta_k}{\gamma_k}W_k,$$

where $\omega_k\,(k=1,2,\ldots,n)$ are positive constants which will be determined in the following assumption $(\mathbf{H}^*)$, and $v_k\,(k=1,2,\ldots,n)$ have been determined in above Lemma 4. We obtain the following lemma.

Lemma 6. *Assume that condition (7) holds. If $\tilde{R}_0>1$, then for $\mathbf{x}=(v_1,v_2,\ldots,v_n)^T$ given in Lemma 4, we have*

$$
\begin{aligned}
\frac{dU(t)}{dt} &= \sum_{k=1}^{n} v_k\Big[1-\frac{1}{g_k(y_k)}\Big]\Big[g_k(y_k)-y_k\Big]\Big\{\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\Big\}\\
&\quad -\sum_{k=1}^{n} v_k\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\Big[h\big(\frac{1}{f_k(x_k)}\big)+h\big(\frac{f_k(x_k)g_j(y_j)}{g_k(y_k)}\big)\Big]\\
&\quad +\sum_{k=1}^{n} v_k\Big(1-\frac{1}{f_k(x_k)}\Big)(S_k-S_k^*)\Big\{-\mu_{k1}+\delta_k(R_k-R_k^*)\\
&\quad \times\Big(\frac{1}{S_k-S_k^*}-\frac{\omega_k f_k(S_k)}{f_k(S_k)-f_k(S_k^*)}\Big)\Big\}-\sum_{k=1}^{n} v_k\frac{\omega_k\delta_k}{\gamma_k}W_{k0},
\end{aligned}
\tag{30}
$$

where for each $k=1,2,\ldots,n$. When $\mu_{k1}=\mu_{k2}=\mu_{k3}=\mu_k$, then

$$W_{k0}=(\mu_k+\gamma_k+\delta_k)\Big[R_k^*(z_k-1)-\frac{\gamma_k N_k^*}{2(\mu_k+\gamma_k+\delta_k)}(n_k-1)\Big]^2,$$

and otherwise, then

$$W_{k0} = (\gamma_k \tilde{c}_k + \mu_{k3} + \delta_k)(R_k^*)^2(z_k - 1)^2 + \frac{\gamma_k \mu_{k1}}{\tilde{\varepsilon}_k}(\tilde{N}_k^*)^2(\tilde{n}_k - 1)^2,$$

Proof. For $k = 1, 2, \ldots, n$, we consider the following two cases.
(1) When $\mu_{k1} = \mu_{k2} = \mu_{k3} = \mu_k$, then from (21), (22) and (25) we have

$$\begin{aligned}
\frac{dU(t)}{dt} =& \sum_{k=1}^{n} v_k \Big\{ -\mu_{k1} S_k^* \Big(1 - \frac{1}{f_k(x_k)}\Big)(x_k - 1) + \delta_k R_k^* \Big(1 - \frac{1}{f_k(x_k)}\Big)(z_k - 1) \Big\} \\
& + \sum_{k=1}^{n} v_k \Big[1 - \frac{1}{g_k(y_k)}\Big]\Big[g_k(y_k) - y_k\Big]\Big\{\sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*)\Big\} \\
& - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) \left[h\left(\frac{1}{f_k(x_k)}\right) + h\left(\frac{f_k(x_k) g_j(y_j)}{g_k(y_k)}\right)\right] \\
& + \sum_{k=1}^{n} v_k \frac{\omega_k \delta_k}{\gamma_k}[-\gamma_k R_k^* S_k^* (z_k - 1)(x_k - 1) - W_{k0}] \\
=& \sum_{k=1}^{n} v_k \Big[1 - \frac{1}{g_k(y_k)}\Big]\Big[g_k(y_k) - y_k\Big]\Big\{\sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*)\Big\} \\
& - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) \left[h\left(\frac{1}{f_k(x_k)}\right) + h\left(\frac{f_k(x_k) g_j(y_j)}{g_k(y_k)}\right)\right] \\
& + \sum_{k=1}^{n} v_k \Big\{ -\mu_{k1} S_k^* \Big(1 - \frac{1}{f_k(x_k)}\Big)(x_k - 1) + \delta_k R_k^* \Big(1 - \frac{1}{f_k(x_k)}\Big)(z_k - 1) \\
& - \omega_k \delta_k R_k^* S_k^* (z_k - 1)(x_k - 1) \Big\} - \sum_{k=1}^{n} v_k \frac{\omega_k \delta_k}{\gamma_k} W_{k0}.
\end{aligned} \tag{31}$$

For the last equation of (31), we further have

$$\begin{aligned}
& -\mu_{k1} S_k^* \Big(1 - \frac{1}{f_k(x_k)}\Big)(x_k - 1) + \delta_k R_k^* \Big(1 - \frac{1}{f_k(x_k)}\Big)(z_k - 1) \\
& -\omega_k \delta_k R_k^* S_k^* (z_k - 1)(x_k - 1) \\
=& \Big(1 - \frac{1}{f_k(x_k)}\Big)(S_k - S_k^*)\Big\{ -\mu_{k1} + \delta_k \frac{R_k - R_k^*}{S_k - S_k^*} - \omega_k \delta_k \frac{(R_k - R_k^*) f_k(S_k)}{f_k(S_k) - f_k(S_k^*)} \Big\} \\
=& \Big(1 - \frac{1}{f_k(x_k)}\Big)(S_k - S_k^*)\Big\{ -\mu_{k1} + \delta_k R_k \Big(\frac{1}{S_k - S_k^*} - \frac{\omega_k f_k(S_k)}{f_k(S_k) - f_k(S_k^*)}\Big) \\
& -\delta_k R_k^* \Big(\frac{1}{S_k - S_k^*} - \frac{\omega_k f_k(S_k)}{f_k(S_k) - f_k(S_k^*)}\Big) \Big\}.
\end{aligned}$$

Hence, we finally obtain

$$\begin{aligned}\frac{dU(t)}{dt} &= \sum_{k=1}^{n} v_k\Big[1-\frac{1}{g_k(y_k)}\Big]\Big[g_k(y_k)-y_k\Big]\Big\{\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\Big\}\\ &\quad -\sum_{k=1}^{n} v_k\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\left[h\left(\frac{1}{f_k(x_k)}\right)+h\left(\frac{f_k(x_k)g_j(y_j)}{g_k(y_k)}\right)\right]\\ &\quad +\sum_{k=1}^{n} v_k\Big(1-\frac{1}{f_k(x_k)}\Big)(S_k-S_k^*)\Big\{-\mu_{k1}+\delta_k(R_k-R_k^*)\\ &\quad \times\Big(\frac{1}{S_k-S_k^*}-\frac{\omega_k f_k(S_k)}{f_k(S_k)-f_k(S_k^*)}\Big)\Big\}-\sum_{k=1}^{n} v_k\frac{\omega_k\delta_k}{\gamma_k}W_{k0}.\end{aligned}$$

This shows that (30) holds.

(2) Otherwise, then a similar argument as in above, we also can obtain (30). This completes the proof. ∎

In order to obtain that the derivative $\frac{dU(t)}{dt}$ given in (30) is non-positive in region $\mathbf{\Gamma}$, we need to introduce the following assumption.

$(\mathbf{H}^*)$ There exist constants $\omega_k > 0$ such that for any $S_k > 0$ and $S_k \neq S_k^*\,(k = 1, 2, \ldots, n)$

$$\frac{1}{S_k-S_k^*}-\frac{\omega_k f_k(S_k)}{f_k(S_k)-f_k(S_k^*)}\leq 0,$$

$$\mu_{k1}+\delta_k R_k^*\Big(\frac{1}{S_k-S_k^*}-\frac{\omega_k f_k(S_k)}{f_k(S_k)-f_k(S_k^*)}\Big)\geq 0.$$

Therefore, by Lemmas 3-6, we can obtain the following theorem on the global stability of endemic equilibrium $\mathbf{E}^*$.

Theorem 2. *Suppose that assumptions (H), $(\mathbf{H}^*)$ and conditions (6), (7) hold. If $\tilde{\mathcal{R}}_0 > 1$, then endemic equilibrium $\mathbf{E}^*$ of model (4) is globally asymptotically stable in $\mathbf{\Gamma}^0$.*

Proof. From assumptions $(\mathbf{H})$ and $(\mathbf{H}^*)$, and $h(x) = x - 1 - \ln x > 0$ when $x \neq 1$, we can easily obtain from (30) that $\frac{dU(t)}{dt} \leq 0$ in $\mathbf{\Gamma}$. When $\frac{dU(t)}{dt} = 0$, we have

$$\Big[1-\frac{1}{g_k(y_k)}\Big]\Big[g_k(y_k)-y_k\Big]=0,\quad k=1,2,\ldots,n$$

and

$$f_k(x_k)=1,\quad k=1,2,\ldots,n$$

which imply that $x_k = 1$ and $y_k = 1$ for $k = 1, 2, \ldots, n$. From $x_k = 1$ and $y_k = 1$ for $k = 1, 2, \ldots, n$ we can further obtain that $S_k = S_k^*$, $I_k = I_k^*$ and $R_k = R_k^*$

for $k = 1, 2, \ldots, n$. Therefore, by the global asymptotic stability theorem on the Lyapunov functions we finally obtain that $\mathbf{E}^*$ is globally asymptotically stable in $\mathbf{\Gamma}^0$. This completes the proof. ∎

In model (4), when $f_k(S_k) = \frac{S_k}{1+\lambda_k S_k}$ $(k = 1, 2, \ldots, n)$, where λ_k is nonnegative constant, then by choosing $\omega_k = \frac{1}{S_k^*(1+\lambda_k S_k^*)}$ $(k = 1, 2, \ldots, n)$ we easily obtain

$$\frac{1}{S_k - S_k^*} - \frac{\omega_k f_k(S_k)}{f_k(S_k) - f_k(S_k^*)} = -\frac{1}{S_k^*} < 0,$$

and

$$\mu_{k1} + \delta_k R_k^* \left(\frac{1}{S_k - S_k^*} - \frac{\omega_k f_k(S_k)}{f_k(S_k) - f_k(S_k^*)} \right) = \frac{1}{S_k^*}(\mu_{k1} S_k^* - \delta_k R_k^*).$$

Therefore, as the corollary of Theorem 2 we have the following results.

Corollary 2. *Assume that conditions (6) and (7) hold, and* $f_k(S_k) = \frac{S_k}{1+\lambda_k S_k}$ $(k = 1, 2, \ldots, n)$ *in model (4) with* $\lambda_k \geq 0$ *is constant. If* $\tilde{\mathcal{R}}_0 > 1$ *and* $\mu_{k1} S_k^* - \delta_k R_k^* \geq 0$ $(k = 1, 2, \ldots, n)$*, then endemic equilibrium* $\mathbf{E}^*$ *of model (4) is globally asymptotically stable in* $\mathbf{\Gamma}^0$.

Remark 2. From Corollary 2 we easily see that the results obtained in this paper improve and extend the corresponding results given by Muroya et al. in [2].

Remark 3. From Theorem 1 we see that only under assumption $(\mathbf{H})$ and conditions (6), (7) we easily obtained the global asymptotical stability of the disease-free equilibrium and the uniform persistence of model (4). However, in Theorem 2, in order to obtain the global asymptotical stability of the endemic equilibrium we must introduce assumption $(\mathbf{H}^*)$. Therefore, an interesting and important open problem can be proposed, that is, whether we also can obtain the global asymptotical stability of the endemic equilibrium only under assumption $(\mathbf{H})$.

5. Conclusion

In this paper, we study the global asymptotic stability of the equilibria for a multi-group SIRS epidemic model with nonlinear incidence rate. By the theory on matrix and by constructing new Lyapunov functions we established the new results on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for model (4). That is, under conditions (6) and (7), the disease-free equilibrium $\mathbf{E^0}$ is globally asymptotically stable if the threshold value $\tilde{\mathcal{R}}_0 \leq 1$, and the endemic equilibrium $\mathbf{E}^*$ is globally asymptotically stable if the threshold value $\tilde{\mathcal{R}}_0 > 1$ and assumption $(\mathbf{H}^*)$ holds. As we can see, the results obtained in this paper improve the corresponding results given in [2].

However, from Corollary 2 we see that assumption $(\mathbf{H}^*)$ holds for $f_k(S_k) = \frac{S_k}{1+\lambda_k S_k}$ $(k = 1, 2, \ldots, n)$, where λ_k is nonnegative constant. Therefore, an important open problem is whether the results obtained in this paper can be extended to model (4) without condition (6) and assumption $(\mathbf{H}^*)$. We hope to get these results in in the future.

In addition, the results obtained in this paper whether can be extended to the following multi-group SEIRS epidemic model with nonlinear incidence rate

$$\begin{cases} \dfrac{dS}{dt} = \lambda - \displaystyle\sum_{j=1}^{n} \beta_j f(S) g_j(I_j) - \mu S + \sum_{i=1}^{n} \delta_i R_i, \\ \dfrac{dE}{dt} = \displaystyle\sum_{j=1}^{n} \beta_j f(S) g_j(I_j) + \sum_{j=1}^{n} r_j I_j - \sigma E, \\ \dfrac{dI_i}{dt} = \gamma_i E - \delta_i I_i, \\ \dfrac{dR_i}{dt} = \omega_i I_i - \mu R_i - \delta_i R_i, \quad i = 1, 2, \ldots, n \end{cases}$$

still is an interesting open problem.

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