TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 2, pp. 615-627, April 2015
DOI: 10.11650/tjm.19.2015.4135
This paper is available online at http://journal.taiwanmathsoc.org.tw

ON THE GENERALIZED CLASS OF CLOSE-TO-CONVEX MAPPINGS

Qinghua Xu* and Taishun Liu

Abstract. In this paper, a class of holomorphic mappings is introduced in the unit ball of complex Banach space which is the extension of subclass of close-to-convex functions in $\mathbb{C}$, and then, we give the sharp distortion theorems for this class of mappings in the unit ball of a complex Hilbert space X or the unit polydisc in $\mathbb{C}^n$. As an application, a sharp growth theorem for this class is obtained. These results generalize many known results.

1. Introduction

In the case of one complex variable, the following distortion theorem is well-known [20].

Theorem A. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be a normalized univalent holomorphic function on the unit disc U in $\mathbb{C}$. Then

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad z \in U.$$

However, in the case of several complex variables, Cartan [2] pointed out that the above theorem does not hold.

Until now, we only establish some distortion results for the class of normalized biholomorphic convex mappings in higher dimensions. Some best-possible results concerning the distortion theorems in several variables were obtained in work of Barnard, FitzGerald, and Gong [1], Gong and Liu [6], Hamada and Kohr [9, 10], Zhu and Liu [25].

Recently, Chu, Hamada, Honda and Kohr [3], Hamada, Honda and Kohr [12, 13, 14] obtained respectively the distortion results for convex mappings and linearly invariant families of locally biholomorphic mappings in the unit ball of a JB*-triple X.

Received December 3, 2013, accepted August 8, 2014.
Communicated by Der-Chen Chang.
2010 *Mathematics Subject Classification*: 30H02, 32A30.
Key words and phrases: Distortion theorem, Growth theorem, Close-to-convex mapping.
*Corresponding author.

It is natural to ask whether there exist other subclasses of biholomorphic mappings which have distortion results in several complex variables. The object of this paper is to give an affirmative answer to this problem.

In this paper, inspired by the above-cited works (especially [3, 12, 13, 14]) and [23], we introduce a subclass of holomorphic mappings in higher dimensions which is the generalization of a subclass of close-to-convex functions in one variable, and we obtain the sharp distortion and growth theorems for this class.

Let B be the unit ball in a complex Banach space X, Y be a complex Banach space, $\mathbb{N}$ be the set of all positive integers. A holomorphic mapping $f : B \to Y$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. A holomorphic mapping $f : B \to Y$ is said to be biholomorphic if $f(B)$ is a domain in Y, f^{-1} exists and holomorphic on $f(B)$. If $f : B \to X$ is a holomorphic mapping, then f is said to be normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X. Let $S(B)$ be the set of all normalized biholomorphic mappings, $H(B)$ be the set of all holomorphic mappings from B into X, $(\partial U)^n(0, r)$ be the distinguished boundary of the polydisc of radius r with the center 0. Let $G \subset X$ be a domain, $Aut(G)$ be the automorphism group of biholomorphic mappings of G onto itself. A domain G is called homogeneous if for any $x, y \in G$, there exists an $f \in Aut(G)$ such that $f(x) = y$. A complex Banach space is a JB*-triple if, and only if, its open unit ball is homogeneous [5, 15].

Let X^* be the dual space of X. For each $x \in X \setminus \{0\}$, we define

$$T(x) = \{T_x \in X^* : \|T_x\| = 1,\ T_x(x) = \|x\|\}.$$

By the Hahn-Banach theorem, $T(x)$ is nonempty.

We recall that a JB*-triple is a complex Banach space X equipped with a triple product $\{\cdot,\cdot,\cdot\} : X^3 \to X$ which is conjugate linear in the middle variable, but linear and symmetric in the other variables, and satisfies

(i) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$;

(ii) the map $a\Box a : x \in X \mapsto \{a, a, x\} \in X$ is hermitian with nonnegative spectrum;

(iii) $\|\{a, a, a\}\| = \|a\|^3$

for $a, b, x, y, z \in X$.

Besides the box operator $a\Box a$, a fundamental operator on a JB*-triple X is the Bergman operator $B(a, b) : X \to X$ defined by

$$B(a, b)(x) = x - 2\{a, b, x\} + \{a, \{b, x, b\}, a\} \quad (x \in X).$$

On the open unit ball B of a JB*-triple, each point $a \in B$ induces the Möbius transformation $\varphi_a \in Aut(B)$ given by

$$\varphi_a(x) = a + B(a, a)^{1/2}(I + x\Box a)^{-1}x \quad (x \in B),$$

where $x\Box a$ is the box operator $(x\Box a)(y) = \{x, a, y\}$. We have $\varphi_a(0) = a,\ \ \varphi_a^{-1} = \varphi_{-a}$ and

$$D\varphi_a(0) = B(a,a)^{1/2}, \quad D\varphi_{-a}(a) = B(a,a)^{-1/2}.$$

From [15, Corollary 3.6](see also [4, 11]), we have

$$\|B(a,a)^{-1/2}\| = \frac{1}{1-\|a\|^2}, \quad \|[D\varphi(0)]^{-1}\| = \frac{1}{1-\|a\|^2} \tag{1}$$

whenever $\varphi \in Aut(B)$ satisfies $\varphi(0) = a$.

Let G be a domain in a complex Banach space X. For any $x \in G, \xi \in X$, the infinitesimal Carathéodory pseudometric $\gamma_G(x, \xi)$ on G is defined by

$$\gamma_G(x,\xi) = \sup\{|Dh(x)\xi| : h \in H(G,U),\ h(x) = 0\},$$

where $H(G, U)$ denotes the family of holomorphic mappings which map G into U.

Each $\varphi \in Aut(G)$ is an isometry in this pseudometric:

$$\gamma_G(x,\xi) = \gamma_G(\varphi(x), D\varphi(x)\xi)$$

and for the open unit ball B in a complex Banach space X, one has $\gamma_B(0, \xi) = \|\xi\|$.

Let $K(B)$ denote the class of normalized convex mappings on B.

We first recall the following definition due to Suffridge [22].

Definition 1. Let $f : B \to X$ be a holomorphic mapping. We say that f is close-to-convex if there exists a convex mapping $g \in H(B)$ such that

$$\Re e\{T_u[Df(x)(Dg(x))^{-1}u]\} > 0, \quad x \in B,\ u \in X \setminus \{0\},\ T_u \in T(u). \tag{2}$$

Recently, Liu and Liu [19] introduced a class $C(B)$ of holomorphic mappings.

Definition 2. Let $f : B \to X$ be a normalized holomorphic mapping. We say that $f \in C(B)$ if there exists a mapping $g \in K(B)$ such that

$$\Re e\{T_x[(Dg(x))^{-1}Df(x)x]\} \geq 0, \quad x \in B \setminus \{0\}, \quad T_x \in T(x). \tag{3}$$

In one variable, the relations (2) and (3) are equivalent to $\Re e\frac{f'(z)}{g'(z)} > 0, \quad z \in U$. Therefore, Definitions 1 and 2 are the usual definition of close-to-convex functions on U.

Remark 1. Suffridge [22] showed that a close-to-convex mapping of B is univalent. However, examples show that the close-to-convex mappings do not satisfy a growth theorem and do not form a normal family on the Euclidean unit ball in $\mathbb{C}^2$ or on the unit polydisc in $\mathbb{C}^2$. On the other hand, Liu et al. [19] showed that mappings belong to the class $C(B)$ are not univalent. However, Xu et al. [24] show that at least in the case of the polydisc, the class $C(B)$ do form a normal family.

Definition 3. Suppose that $\alpha \in (0,1]$ and $f : U \to \mathbb{C}$ is a normalized holomorphic function. If there exists a function $g \in K(U)$ such that

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leq \frac{\alpha\pi}{2}, \quad z \in U,$$

then f is called a strongly close-to-convex function of order α on U.

Definition 3 was originally introduced by Pommerenke [21].

Recently, Xu et al. [24] extended Definition 3 to higher dimensions as follows.

Definition 4. Suppose that $\alpha \in (0,1]$ and $f : B \to X$ is a normalized holomorphic mapping. If there exists a mapping $g \in K(B)$ such that

$$\left|\arg \frac{1}{\|x\|} T_x[(Dg(x))^{-1} Df(x)x]\right| \leq \frac{\pi}{2}\alpha, \quad x \in B \setminus \{0\}, \ T_x \in T(x), \tag{4}$$

then f is called a strongly close-to-convex mapping of order α on B.

When $X = \mathbb{C}, \ B = U$, the relation (4) is equivalent to $\left|\arg \frac{f'(z)}{g'(z)}\right| \leq \frac{\alpha\pi}{2}, \ z \in U$, namely this is the definition of strongly close-to-convex functions of order α on U.

Let $C_\alpha(B)$ denote the class of strongly close-to-convex mappings of order α on B. It is obvious that $C_\alpha(B) \subseteq C(B)$ and $C_1(B) = C(B)$.

Now we introduce the following subclass of the mapping class $C(B)$.

Definition 5. Let $p \in H(U)$ be a biholomorphic function such that $p(0) = 1, \ p(\bar{\xi}) = \overline{p(\xi)}$, for $\xi \in U$, $\Re e p(\xi) > 0$ on $\xi \in U$, and assume p satisfies the following conditions for $r \in (0,1)$:

$$\begin{cases} \min\limits_{|\xi|=r} |p(\xi)| = \min\limits_{|\xi|=r} \Re e p(\xi) = p(-r) \\ \max\limits_{|\xi|=r} |p(\xi)| = \max\limits_{|\xi|=r} \Re e p(\xi) = p(r). \end{cases} \tag{5}$$

A mapping f is said to be $f \in \mathcal{M}_p(B)$ if there exists a mapping $g \in K(B)$ such that

$$\frac{1}{\|x\|} T_x[(Dg(x))^{-1} Df(x)x] \in p(U), \quad x \in B \setminus \{0\}, \quad T_x \in T(x). \tag{6}$$

Remark 2. There are many choices of the function p on U which would provide interesting subclasses of holomorphic mappings in several complex variables. For example, if we let

$$p(\xi) = \left(\frac{1+\xi}{1-\xi}\right)^\alpha, \quad \xi \in U, \quad 0 < \alpha \leq 1,$$

then it is easily verified that p satisfies the hypotheses of Definition 5. If $f \in \mathcal{M}_p(B)$, then

$$\left|\arg \frac{1}{\|x\|} T_x[(Dg(x))^{-1} Df(x)x]\right| \le \frac{\pi}{2}\alpha, \quad x \in B \setminus \{0\},\ T_x \in T(x),\ g \in K(B),$$

that is, the mapping class $\mathcal{M}_p(B)$ becomes the aforementioned mapping class $C_\alpha(B)$.

Definition 6. (see [17].) Suppose that Ω is a domain (connected open set) in X which contains 0. It is said that $x = 0$ is a zero of order k of $f(x)$ if $f(0) = 0, \cdots, D^{k-1}f(0) = 0$, but $D^k f(0) \neq 0$, where $k \in \mathbb{N}$.

2. Some Lemmas

In order to obtain the desired theorems, we first give the following lemmas.

Lemma 1. (see [7].) *Suppose $f \in H(U)$, g is a biholomorphic function on U, $f(0) = g(0), f'(0) = \cdots = f^{(k-1)}(0) = 0$, and $f \prec g$. Then*

$$f(rU) \subseteq g(r^k U), \quad r \in (0,1),\ rU = \{\xi \in \mathbb{C} : |\xi| < r\}.$$

Lemma 2. *Let $p : U \to \mathbb{C}$ satisfy the conditions of Definition 1. If*

$$\frac{T_x(h(x))}{\|x\|} \in p(U),\ x \in B \setminus \{0\},\ T_x \in T(x), \tag{7}$$

and $x = 0$ is the zero of order $k+1$ of $h(x) - x$, then

$$\|x\| p(-\|x\|^k) \le \Re e T_x((h(x)) \le |T_x(h(x))| \le \|x\| p(\|x\|^k). \tag{8}$$

Proof. Fix $x \in B\setminus\{0\}$, and denote $x_0 = \frac{x}{\|x\|}$. Let $q : U \longrightarrow \mathbb{C}$ be given by

$$q(\xi) = \begin{cases} \frac{T_x(h(\xi x_0))}{\xi}, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Then $q \in H(U),\ q(0) = p(0) = 1$, and from (7), we deduce that

$$q(\xi) = \frac{T_x(h(\xi x_0))}{\xi} = \frac{T_{x_0}(h(\xi x_0))}{\xi} = \frac{T_{\xi x_0}(h(\xi x_0))}{\|\xi x_0\|} \in p(U), \quad \xi \in U.$$

Therefore, $q \prec p$.

According to hypothesis of Lemma 4, we deduce that

$$q(\xi) = 1 + \sum_{m=k+1}^{\infty} \frac{T_x(D^m h(0)(x_0^m))}{m!} \xi^{m-1}.$$

It is easy to see that the function $q(\xi)$ satisfies the conditions of Lemma 1, hence we obtain

$$q(rU) \subseteq p(r^k U), \quad r \in (0,1),\ rU = \{\xi \in \mathbb{C} : |\xi| < r\}.$$

On the other hand, combining the maximum and minimum principles for harmonic functions with (2), we deduce that

$$p(-|\xi|^k) \leq \Re e q(\xi) \leq |q(\xi)| \leq p(|\xi|^k), \quad \xi \in U.$$

Setting $\xi = \|x\|$ in the above relation, we obtain (8), as desired. This completes the proof. ■

From Lemma 2, we obtain easily that the following lemma holds true (we omit the details of the proof here).

Lemma 3. *Suppose $g \in K(B)$, and $x = 0$ is the zero of order $k+1 (k \in \mathbb{N})$ of $g(x) - x$, $f \in \mathcal{M}_p$ (with respect to g), and $x = 0$ is the zero of order $k+1 (k \in \mathbb{N})$ of $f(x) - x$, then*

$$\begin{aligned}\|x\|p(-\|x\|) \leq \Re e T_x[(Dg(x))^{-1}Df(x)x] &\leq |T_x[(Dg(x))^{-1}Df(x)x]| \\ &\leq \|x\|p(\|x\|), \quad x \in B \setminus \{0\}, \quad T_x \in T(x).\end{aligned}$$

Lemma 4. (see [18].) *If $g \in K(B)$, and $x = 0$ is the zero of order $k+1 (k \in \mathbb{N})$ of $g(x) - x$, then*

$$\left(1 - \frac{2^{\frac{1}{k}}\|x\|}{(1+\|x\|^k)^{\frac{1}{k}}}\right)\gamma_B(x,y) \leq \|Dg(x)y\| \leq \left(1 + \frac{2^{\frac{1}{k}}\|x\|}{(1-\|x\|^k)^{\frac{1}{k}}}\right)\gamma_B(x,y)$$

for each $x \in B$ and $y \in X$.

Lemma 5. (see [19].)

$$\gamma_B(x,\xi) \geq \frac{|T_x(\xi)|}{1-\|x\|^2}, \quad x \in B,\ \xi \in X, \quad T_x \in T(x).$$

Lemma 6. (see [24].) *Let B be the open unit ball of a JB*-triple X, $g \in K(B)$, and $x = 0$ is the zero of order $k+1 (k \in \mathbb{N})$ of $g(x) - x$. Then for $(x,y) \in B \times X$, we have*

$$\|Dg(x)y\| \leq \left(1 + \frac{2^{\frac{1}{k}}\|x\|}{(1-\|x\|^k)^{\frac{1}{k}}}\right)\frac{\|y\|}{1-\|x\|^2}.$$

3. Main Results and Their Proofs

Theorem 1. *Suppose $g \in K(B)$, and $x = 0$ is the zero of order $k+1 (k \in \mathbb{N})$ of $g(x) - x$, $f \in \mathcal{M}_p$ (with respect to g), and $x = 0$ is the zero of order $k+1 (k \in \mathbb{N})$ of $f(x) - x$, then*

$$\|Df(x)x\| \geq \left(1 - \frac{2^{\frac{1}{k}}\|x\|}{(1+\|x\|^k)^{\frac{1}{k}}}\right) \frac{\|x\|}{1-\|x\|^2} p(-\|x\|), \quad x \in B.$$

Proof. Since $f \in \mathcal{M}_p$ (with respect to g), we have

$$\frac{1}{\|x\|} T_x[(Dg(x))^{-1} Df(x)x] \in p(U), \quad x \in B \setminus \{0\},\ T_x \in T(x).$$

Let $h(x) = (Dg(x))^{-1} Df(x)x$. Then, we have $Df(x)x = Dg(x)h(x)$. By Lemma 3, we obtain

$$|T_x(h(x))| \geq \|x\| p(-\|x\|), \quad x \in B \setminus \{0\}.$$

From the above relation and Lemmas 4 and 5, we have

$$\begin{aligned}
\|Df(x)x\| =& \|Dg(x)h(x)\| \geq \left(1 - \frac{2^{\frac{1}{k}}\|x\|}{(1+\|x\|^k)^{\frac{1}{k}}}\right) \gamma_B(x, h(x)) \\
&\geq \left(1 - \frac{2^{\frac{1}{k}}\|x\|}{(1+\|x\|^k)^{\frac{1}{k}}}\right) \frac{|T_x(h(x))|}{1-\|x\|^2} \\
&\geq \left(1 - \frac{2^{\frac{1}{k}}\|x\|}{(1+\|x\|^k)^{\frac{1}{k}}}\right) \frac{\|x\|}{1-\|x\|^2} p(-\|x\|), \quad x \in B \setminus \{0\}. \qquad \blacksquare
\end{aligned}$$

Theorem 2. *Suppose $g \in K(U^n)$, and $z = 0$ is the zero of order $k+1 (k \in \mathbb{N})$ of $g(z) - z$, $f \in \mathcal{M}_p$ (with respect to g), and $z = 0$ is the zero of order $k+1 (k \in \mathbb{N})$ of $f(z) - z$, then*

$$\|Df(z)z\| \leq \left(1 + \frac{2^{\frac{1}{k}}\|z\|}{(1-\|z\|^k)^{\frac{1}{k}}}\right) \frac{\|z\|}{1-\|z\|^2} p(\|z\|), \quad z \in U^n.$$

Proof. Need only consider $z \in U^n$ and $z \neq 0$. For any $\xi \in (\partial U)^n(0, \|z\|)$, we have

$$|\xi_1| = |\xi_2| = \cdots = |\xi_n| = \|z\|.$$

Note that

$$T_\xi = (0, \cdots, 0, \frac{\|\xi\|}{\xi_i}, 0, \cdots, 0).$$

Set $h(z) = (Dg(z))^{-1}Df(z)z$. Then there exists an i such that

$$\begin{aligned}\|h(z)\| = |h_i(z)| &\le \max_{\xi\in(\partial U)^n(0,\|z\|)} |h_i(\xi)| \\ &= \max_{\xi\in(\partial U)^n(0,\|z\|)} \left|\frac{\|\xi\|}{\xi_i}h_i(\xi)\right| \\ &= \max_{\xi\in(\partial U)^n(0,\|z\|)} |T_\xi[h(\xi)]| \\ &= \max_{\xi\in(\partial U)^n(0,\|z\|)} |T_\xi[(Dg(\xi))^{-1}Df(\xi)\xi]|.\end{aligned}$$

According to Lemma 3, we have

$$|T_\xi(Dg(\xi))^{-1}Df(\xi)\xi| \le \|\xi\|p(\|\xi\|) = \|z\|p(\|z\|),$$

and thus

$$\|h(z)\| = \|(Dg(z))^{-1}Df(z)z\| \le \|z\|p(\|z\|), \quad z \in U^n.$$

From the above relation and Lemma 6, we have

$$\begin{aligned}\|Df(z)z\| = \|Dg(z)h(z)\| &\le \left(1 + \frac{2^{\frac{1}{k}}\|z\|}{(1-\|z\|^k)^{\frac{1}{k}}}\right)\frac{\|h(z)\|}{1-\|z\|^2} \\ &\le \left(1 + \frac{2^{\frac{1}{k}}\|z\|}{(1-\|z\|^k)^{\frac{1}{k}}}\right)\frac{\|z\|}{1-\|z\|^2}p(\|z\|), \quad z \in U^n. \quad \blacksquare\end{aligned}$$

From Theorems 1 and 2 (the case of $k = 1$), we easily obtain the following sharp distortion theorems.

Theorem 3. *If $f \in \mathcal{M}_p(U^n)$, then*

(9) $$\frac{\|z\|}{(1+\|z\|)^2}p(-\|z\|) \le \|Df(z)z\| \le \frac{\|z\|}{(1-\|z\|)^2}p(\|z\|), \quad z \in U^n.$$

These estimates are sharp.

The following Example 1 shows that the estimations of Theorem 3 are sharp.

Example 1. Let

(10) $$f(z) = \left(\int_0^{z_1}\frac{1}{(1-t)^2}p(t)dt, \cdots, \int_0^{z_n}\frac{1}{(1-t)^2}p(t)dt\right)', \quad z \in U^n$$

and

$$g(z) = \left(\frac{z_1}{1-z_1}, \cdots, \frac{z_n}{1-z_n}\right)', \quad z \in U^n.$$

It is not difficult to check that

$$g \in K(U^n) \text{ and } f \in \mathcal{M}_p(U^n) \text{ (with respect to } g).$$

By using (10), we have

$$Df(z)z = \left(\frac{z_1}{(1-z_1)^2}p(z_1), \cdots, \frac{z_n}{(1-z_n)^2}p(z_n)\right)', \quad z \in U^n. \tag{11}$$

Taking $z = (r, 0, \cdots, 0)$ or $z = (-r, 0, \cdots, 0)(0 \le r < 1)$ in (11), then the equalities of the estimations (9) hold. ■

Theorem 4. *If $f \in \mathcal{M}_p(B)$, then*

$$\|Df(x)x\| \ge \frac{\|x\|}{(1+\|x\|)^2}p(-\|x\|), \quad x \in B. \tag{12}$$

When B is the open unit ball of a complex Hilbert space X, the above estimate is sharp.

In order to see that the estimation of Theorem 4 is sharp, it suffices to consider the following example.

Example 2. Let B be the open unit ball of a complex Hilbert space X. Suppose $\alpha \in (0, 1]$, if

$$f(x) = x\int_0^{\langle x,e\rangle} \frac{1}{(1-t)^2}p(t)dt/\langle x, e\rangle, \quad x \in B, \quad \|e\| = 1, \tag{13}$$

then $f \in \mathcal{M}_p(B)$ and the equality of (13) holds.

Proof. Let

$$h(z) = \int_0^z \frac{1}{(1-t)^2}p(t)dt, \quad z \in U.$$

Then

$$f(x) = \frac{h(\langle x, e\rangle)}{\langle x, e\rangle}x.$$

A short computation yields

$$Df(x)x = h'(\langle x, e\rangle)x = \frac{1}{(1-\langle x, e\rangle)^2}p(\langle x, e\rangle)x. \tag{14}$$

On the other hand, according to [8], we have

$$g(x) = \frac{x}{1-\langle x, e\rangle} \in K(B).$$

Thus, we conclude that

$$(Dg(x))^{-1} = (1 - \langle x, e\rangle)(I - \langle \cdot, e\rangle x). \tag{15}$$

By using (14) and (15), we deduce that

$$\frac{1}{\|x\|} T_x[(Dg(x))^{-1} Df(x)x] = p(\langle x, e\rangle),$$

which means that

$$f \in \mathcal{M}_p(B) \text{ (with respect to } g).$$

In addition, From (14), we obtain

$$\|Df(x)x\| = \left\| \frac{1}{(1-\langle x, e\rangle)^2} p(\langle x, e\rangle)x \right\|. \tag{16}$$

Taking $x = -re$ $(0 \le r < 1)$ in (14), then the equality of the estimation (10) holds. ■

Now, from the above theorem, we obtain the following growth result.

Theorem 5. *If $f \in \mathcal{M}_p(U^n)$, then*

$$\|f(z)\| \le \int_0^1 \frac{\|z\|}{(1-t\|z\|)^2} p(t\|z\|)dt, \quad z \in U^n. \tag{17}$$

The above estimate is sharp.

Proof. Suppose $z \in U^n \setminus \{0\}$ be such that $f(z) \neq 0$ and $f \in \mathcal{M}_p(U^n)$. Let $G : [0, 1] \longrightarrow \mathbb{R}$ be given by

$$G(t) = \frac{\|f(z)\|}{f_j(z)} f_j(tz), \quad 0 \le t \le 1,$$

where $|f_j(z)| = \|f(z)\| = \max\limits_{1\le k\le n}\{|f_k(z)|\}$.

From Theorem 3, we have

$$\begin{aligned} \|f(z)\| =& G(1) = \int_0^1 G'(t)dt = \int_0^1 \frac{\|f(z)\|}{f_j(z)} Df_j(tz)zdt \\ &\le \int_0^1 \|Df(tz)z\|dt \\ &\le \int_0^1 \frac{\|z\|}{(1-t\|z\|)^2} p(t\|z\|)dt. \end{aligned}$$

Taking $z = (r, 0, \cdots, 0)(0 \le r < 1)$ in (10), then the equality of the estimation (17) holds. ■

In view of Remark 2, if we set

$$p(\xi) = \left(\frac{1+\xi}{1-\xi}\right)^{\alpha}, \quad \xi \in U, \quad 0 < \alpha \leq 1$$

in Theorems 1, 2, 3, 4 and 5, respectively, we can readily deduce the following corollary, which we merely state here without proofs.

Corollary 1.

(i) *Suppose $g \in K(B)$, and $x = 0$ is the zero of order $k+1(k \in \mathbb{N})$ of $g(x) - x$, f is a strongly close-to-convex mapping of order α with respect to g, and $x = 0$ is the zero of order $k+1(k \in \mathbb{N})$ of $f(x) - x$, then*

$$\|Df(x)x\| \geq \left(1 - \frac{2^{\frac{1}{k}}\|x\|}{(1+\|x\|^k)^{\frac{1}{k}}}\right) \frac{\|x\|}{1-\|x\|^2} \left(\frac{1-\|x\|^k}{1+\|x\|^k}\right)^{\alpha}, \quad x \in B.$$

(ii) *Suppose $g \in K(U^n)$, and $z = 0$ is the zero of order $k+1(k \in \mathbb{N})$ of $g(z) - z$, f is a strongly close-to-convex mapping of order α with respect to g, and $z = 0$ is the zero of order $k+1(k \in \mathbb{N})$ of $f(z) - z$, then*

$$\|Df(z)z\| \leq \left(1 + \frac{2^{\frac{1}{k}}\|z\|}{(1-\|z\|^k)^{\frac{1}{k}}}\right) \frac{\|z\|}{1-\|z\|^2} \left(\frac{1+\|z\|^k}{1-\|z\|^k}\right)^{\alpha}, \quad z \in U^n.$$

(iii) *If $f \in C_\alpha(U^n)$, then*

$$\frac{\|z\|}{(1+\|z\|)^2} \left(\frac{1-\|z\|}{1+\|z\|}\right)^{\alpha} \leq \|Df(z)z\| \leq \frac{\|z\|}{(1-\|z\|)^2} \left(\frac{1+\|z\|}{1-\|z\|}\right)^{\alpha}, \quad z \in U^n.$$

These estimates are sharp.

(iv) *If $f \in C_\alpha(B)$, then*

$$\|Df(x)x\| \geq \frac{\|x\|}{(1+\|x\|)^2} \left(\frac{1-\|x\|}{1+\|x\|}\right)^{\alpha}, \quad x \in B.$$

When B is the open unit ball of a complex Hilbert space X, the above estimate is sharp.

(v) *If $f \in C_\alpha(U^n)$, then*

$$\|f(z)\| \leq \frac{1}{2(1+\alpha)} \left[\left(\frac{1+\|z\|}{1-\|z\|}\right)^{1+\alpha} - 1\right], \quad z \in U^n.$$

The above estimate is sharp.

Remark 3. These results in (i), (ii), (iii), (iv) and (v) of Corollary 1 were proven earlier by Xu et al. [24].

Acknowledgments

This work was supported by NNSF of China (Grant Nos. 11261022, 11471111), the Jiangxi Provincial Natural Science Foundation of China (Grant No. 20132BAB201004), Natural Science Foundation of Department of Education of Jiangxi Province, China (Grant No. GJJ12177).

References

1. R. W. Barnard, C. H. Fitzgerld and S. Gong, A distortion theorem of biholomorphic convex mappings in $\mathbb{C}^2$, *Trans. Amer. Math. Soc.*, **344** (1994), 907-924.
2. H. Cartan, *Sur la possibilité d'éntendre aux fonctions de plusieurs variables complexes la théorie des fonctions univalents*, Lecons sur les Fonctions Univalents on Mutivalents, by P. Montel, Gauthier-Villar, 1933, pp. 129-155.
3. C.-H. Chu, H. Hamada, T. Honda and G. Kohr, Distortion theorems for convex mappings on homogeneous balls, *J. Math. Anal. Appl.*, **369** (2010), 437-442.
4. C.-H. Chu, Jordan triples and Riemannian symmetric spaces, *Adv. Math.*, **219** (2008), 2029-2057.
5. C.-H. Chu and P. Mellon, Jordan structures in Banach spaces and symmetric manifolds, *Expo. Math.*, **16** (1998), 157-180.
6. S. Gong and T. S. Liu, Distortion theorems for biholomorphic convex mappings on bounded convex circular domains, *Chinese Ann. Math. Ser. B*, **20** (1999), 297-304.
7. I. Graham and D. Varolin, Bloch constants in one and several variables, *Pacific J. Math.*, **174** (1996), 347-357.
8. I. Graham and G. Kohr, *Geometric Function Theory in One and Higher Dimensions*, New York, Marcel Dekker, 2003.
9. H. Hamada and G. Kohr, Growth and distortion results for convex mappings in infinite dimensional spaces, *Complex Var. Theory Appl.*, **47** (2002), 291-301.
10. H. Hamada and G. Kohr, Φ-Like and convex mappings in infinite dimensional space, *Rev. Roumaine Math. Pures Appl.*, **47** (2002), 315-328.
11. H. Hamada, T. Honda and G. Kohr, Bohr's theorem for holomorphic mappings with values in homogeneous balls, *Israel J. Math.*, **173** (2009), 177-187.
12. H. Hamada, T. Honda and G. Kohr, Linear invariance of locally biholomorphic mappings in the unit ball of a JB*-triple, *J. Math. Anal. Appl.*, **385** (2012), 326-339.
13. H. Hamada, T. Honda and G. Kohr, Trace-order and a distortion theorem for linearly invariant families on the unit ball of a finite dimensional JB*-triple, *J. Math. Anal. Appl.*, **396** (2012), 829-843.
14. H. Hamada, T. Honda and G. Kohr, Growth and distortion theorems for linearly invariant families on homogenous unit balls in $\mathbb{C}^n$, *J. Math. Anal. Appl.*, **407** (2013), 398-412.

15. W. Kaup, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.*, **183** (1983), 503-529.

16. W. Kaup, Hermitian Jordan triple systems and the automorphisms of bounded symmetric domain, in: *Non-Associative Algebra and Its Applications*, S. González, (ed.), Oviedo, 1993, Math. Appl., Vol. 303, Kluwer Acad. Publ., Dordrecht, 1994, pp. 204-214.

17. Y. Y. Lin and Y. Hong, Some properties of holomorphic maps in Banach spaces, *Acta Math. Sinica.*, **38** (1995), 234-241 (in Chinese).

18. X. S. Liu and T. S. Liu, On the precise growth, covering, and distortion theorems for normalized biholomorphic mappings, *J. Math. Anal. Appl.*, **295** (2004), 404-417.

19. X. S. Liu and T. S. Liu, The sharp distortion theorem for a subclass of close-to-convex mappings, *Chinese Ann. Math. Ser. A*, **33** (2012), 91-100.

20. C. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

21. C. Pommerenke, On close-to-convex functions, *Trans. Amer. Math. Soc.*, **114** (1965), 176-186.

22. T. J. Suffridge, Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions, in: *Lecture Notes in Math.*, 599. Berlin-Heidelberg-New York: Springer-Verlag, 1976, pp. 146-159.

23. Q. H. Xu, T. S. Liu and X. S. Liu, The sharp estimates of homogeneous expansions for the generalized class of close-to-quasi-convex mappings, *J. Math. Anal. Appl.*, **389** (2012), 781-791.

24. Q. H. Xu, T. S. Liu and X. S. Liu, Sharp distortion theorems for a subclass of close-to-convex mappings, *Front. Math. China.*, **8(6)** (2013), 1425-1436.

25. Y. Zhu and M. Liu, Distortion theorems for biholomorphic convex mappings in Banach spaces, *Complex Var. Theory Appl.*, **50** (2005), 57-68.

Qinghua Xu
College of Mathematics and Information Science
Jiangxi Normal University
Nanchang, Jiangxi 330027
P. R. China
E-mail: xuqh@mail.ustc.edu.cn

Taishun Liu
Department of Mathematics
Huzhou Teachers College
Huzhou, Zhejiang 313000
P. R. China
E-mail: lts@ustc.edu.cn