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ON EXISTENCE OF THREE SOLUTIONS FOR $p(x)$-KIRCHHOFF TYPE DIFFERENTIAL INCLUSION PROBLEM VIA NONSMOOTH CRITICAL POINT THEORY

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Abstract. In this paper, we study a class of differential inclusion problems driven by the $p(x)$-Kirchhoff with non-standard growth depending on a real parameter. Working within the framework of variable exponent spaces, a new existence result of at least three solutions for the considered problem is established by using the nonsmooth version three critical points theorem.

1. INTRODUCTION

In recent years, various Kirchhoff type problems have been extensively investigated by many authors due to their theoretical and practical importance, such problems are often referred to as being nonlocal because of the presence of the integral over the entire domain Ω. It is well known that this problem is analogous to the stationary problem of a model introduced by Kirchhoff [1]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho u_{tt} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L u_x^2 dx\right)u_{xx} = 0, \tag{1.1}$$

where ρ, ρ_0, h, E, L are all positive constants. Equation (1.1) is an extension of the classical D'Alembert wave equation by considering the changes in the length of the string during the vibrations. For a bounded domain Ω, the problem

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\triangle u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

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is related to the stationary analogue of (1.1). Nonlocal elliptic problems like (1.2) have received a lot of attention due to the fact that they can model several physical and biological systems and some interesting results have been established in, for example, [2,3] and the references therein. Moreover, the study of the Kirchhoff type equation has already been extended to the case involving the p-Laplacian operator or $p(x)$-Laplacian operator. For instance, G. Dai and R. Hao in [4] were concerned with the existence and multiplicity of solutions to the following Kirchhoff type equation involving the p-Laplacian operator

$$\begin{cases} -K\left(\dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p dx\right)\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

they established conditions ensuring the existence and multiplicity of solutions for the addressed problem by means of a direct variational approach and the theory of the variable exponent Sobolev spaces. In [5], by using a direct variational approach, G. Dai and R. Hao established conditions ensuring the existence and multiplicity of solutions for the following problem

$$\begin{cases} -K\left(\dfrac{1}{p(x)}\displaystyle\int_\Omega |\nabla u|^{p(x)} dx\right)\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

Very recently, G. Dai and R. Ma in [6] studied the existence and multiplicity of solutions to the following $p(x)$-Kirchhoff type problem

$$\begin{cases} -K\left(\displaystyle\int_\Omega \dfrac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\right)\big(\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) & \\ \quad -|u|^{p(x)-2}u\Big) = f(x,u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

we refer the reader to, e.g., [7,8] for other interesting results and further research on this subject.

However, the nonlinearity in the above mentioned Kirchhoff type problems is continuous or differentiable. As pointed out by K.C. Chang in [9-11], many free boundary problems and obstacle problems arising in mathematical physics may be reduced to partial differential equations with discontinuous nonlinearities, among these problems, we have the obstacle problem [9], the seepage surface problem [10], and the Elenbaas equation [11], and so forth. Associated with this development, the theory of nonsmooth variational has been given extensive attention, for a comprehensive treatment, we refer to the monographs of [12,13], as well as for updated list of references [14-21] and the references therein. Therefore, it is natural from both a physical and biological standpoint as well as a theoretical view to give considerable attention to a synthesis involving $p(x)$-Kirchhoff type differential inclusion problem.

Motivated by the above discussions, the main purpose of this paper is to establish a new result for the existence of at least three solutions to the following differential inclusion problem

$$(P_\lambda) \qquad \begin{cases} -K(t)\big(\Delta_{p(x)}u - |u|^{p(x)-2}u\big) \in \partial F_1(x,u) + \lambda \partial F_2(x,u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where $\Delta_{p(x)}u = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be $p(x)$-Laplacian operator, $K(t)$ is a continuous function with $t := \displaystyle\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx$, $\lambda > 0$ is a parameter, $\Omega \subseteq \mathbb{R}^N (N > 2)$ is a nonempty bounded domain with a boundary $\partial\Omega$ of class C^1, $p(x) > 0, p(x) \in C(\overline{\Omega})$ with $1 < p^- = \inf\limits_{x\in\Omega} p(x), p^+ = \sup\limits_{x\in\Omega} p(x)$, $F_1(x,t)$ and $F_2(x,t)$ are locally Lipschitz functions in the t-variable integrand, $\partial F_1(x,t)$ and $\partial F_2(x,t)$ are the subdifferential with respect to the t-variable in the sense of Clarke [22], and n is the outward unit normal on $\partial\Omega$.

To study problem (P_λ), we should overcome the difficulties as follows: Firstly, the $p(x)$-Laplacian possesses more complicated nonlinearities than p-Laplacian, in general, the property of the first eigenvalue of $p(x)$-Laplacian is not the same as the p-Laplacian, namely the first eigenvalue is not isolated (see [23]), therefore, the first difficulty is that we cannot use the eigenvalue property of $p(x)$-Laplacian. Secondly, the lack of differentiability of nonlinearity causes several technical obstructions, that is to say, we can not use variational methods for C^1 functional, because in our case, the energy functional is only locally Lipschitz continuous, and so, our approach, which is variational, is based on the nonsmooth critical point theory as was developed by K. C. Chang [9] and S.A. Marano et al. [15]. On the other hand, to the best of our knowledge, there are no papers concerning the problem (P_λ) by the nonsmooth three critical points theorem. So even in the case of a constant exponent, our results are also new.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Lebesgue and Sobolev spaces and generalized gradient of locally Lipschitz function. In Section 3, we give the main result and its proof of this paper.

2. Preliminaries

2.1. Variable exponent spaces and $p(x)$-Kirchhoff-Laplacian operator

In this subsection, we recall some preliminary results about Lebesgue and Sobolev variable exponent spaces and the properties of $p(x)$-Kirchhoff-Laplace operator, which are useful for discussing problem (P_λ). Set

$$C_+(\overline{\Omega}) = \big\{h | h \in C(\overline{\Omega}) : h(x) > 1 \quad \text{for all} \quad x \in \overline{\Omega}\big\},$$

for any $h \in C_+(\overline{\Omega})$, we will denote

$$h^- = \min_{x\in\overline{\Omega}} h(x), \quad h^+ = \max_{x\in\overline{\Omega}} h(x).$$

Let $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u(x)|^{p(x)} dx < +\infty\right\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega |\frac{u}{\lambda}|^{p(x)} dx \le 1\right\},$$

and the variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Proposition 2.1. (See [24]). *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2. (See [24]). (1) If $q(x) \in C_+(\overline{\Omega})$ and $q(x) \le p^*(x),$ for all $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is continuous and if $q(x) < p^*(x),$ for all $x \in \overline{\Omega}$, the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact, where

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & p(x) < N, \\ +\infty, & p(x) \ge N. \end{cases}$$

(2) If $p_1(x), p_2(x) \in C_+(\overline{\Omega})$, and $p_1(x) \le p_2(x),$ $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, and the embedding is continuous.

In the following, we will discuss the $p(x)$-Kirchhoff-Laplace operator

$$J_p(u) = -K\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\right)\big(\text{div}|\nabla u|^{p(x)-2}\nabla u - |u|^{p(x)-2}u\big).$$

Denote

$$J(u) := \widehat{K}\bigg(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\bigg)$$

where $\widehat{K}(t) = \displaystyle\int_0^t K(\tau)d\tau$, $K(t)$ satisfies the following condition:

(K_0) $K(t) : [0, +\infty) \to (k_0, k_1)$ is a continuous and increasing function with $k_1 > k_0 > 0$.

For simplicity, we write $X = W^{1,p(x)}(\Omega)$, denote by $\|\cdot\|_X$ the norm of X, $u_n \rightharpoonup u$ and $u_n \to u$ the weak convergence and strong convergence of sequence $\{u_n\}$ in X, respectively. It is obvious that the functional J is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $u \in X$ is the functional $J'(u) \in X^*$, given by

$$\langle J'(u), v\rangle = K\bigg(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})dx\bigg)\int_\Omega \big(|\nabla u|^{p(x)-2}\nabla u\nabla v+|u|^{p(x)-2}uv\big)dx,$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X. Therefore, the $p(x)$-Kirchhoff-Laplace operator is up to the minus sign the derivative operator of J in the weak sense. We have the following properties about the derivative operator of J.

Lemma 2.1. (See [6]). *If* (K_0) *holds, then*

(i) $J' : X \to X^*$ *is a continuous, bounded and strictly monotone operator;*

(ii) J' *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* X *and* $\limsup\limits_{n\to+\infty}\langle J'(u_n) - J'(u), u_n - u\rangle \leq 0$, *then* $u_n \to u$ *in* X*;*

(iii) $J' : X \to X^*$ *is a homeomorphism;*

(iv) J *is weakly lower semicontinuous.*

2.2. Generalized gradient

Let $(X, \|\cdot\|)$ be a real Banach space and X^* be its topological dual. A function $\varphi : X \to \mathbb{R}$ is called locally Lipschitz if each point $u \in X$ possesses a neighborhood U and a constant $L_u > 0$ such that

$$|\varphi(u_1) - \varphi(u_2)| \leq L_u\|u_1 - u_2\|, \quad \text{for all } u_1, u_2 \in U.$$

We know from convex analysis that a proper, convex and lower semicontinuous function $g : X \to \overline{R} = R\bigcup\{+\infty\}$ is locally Lipschitz in the interior of its effective domain dom $g = \{u \in X : g(u) < +\infty\}$. We define the generalized directional derivative of a locally Lipschitz function ϕ at $x \in X$ in the direction $h \in X$ by

$$\phi^\circ(x; h) = \limsup_{\substack{x'\to x\\ \lambda\to 0^+}} \frac{\phi(x' + \lambda h) - \phi(x')}{\lambda}.$$

It is easy to see that the function $h \mapsto \phi^\circ(x;h)$ is sublinear and continuous on X, so by Hahn-Banach Theorem, it is the support function of a nonempty, convex, w^*-compact set

$$\partial\phi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \phi^\circ(x;h) \text{ for all } \quad h \in X\}.$$

The multifunction $x \to \partial\phi(x)$ is known as the generalized(or Clarke) subdifferential of ϕ. Now, we list some fundamental properties of the generalized grandient and directional derivative which will be used throughout this paper.

Proposition 2.3. (See [22]). (i) $(-\phi)^\circ(u;z) = \phi^\circ(u;-z)$ *for all* $u, z \in X$*;*

(ii) $\phi^\circ(u;z) = \max\{\langle x^*, z\rangle : x^* \in \partial\phi(u)\}$ *for all* $u, z \in X$*;*

(iii) *Let* κ *be a locally Lipschitz function,* $(\phi+\kappa)^\circ(u;z) \leq \phi^\circ(u;z) + \kappa^\circ(u;z)$, *for all* $u, z \in X$*;*

(iv) *Let* $j : X \to \mathbb{R}$ *be a continuously differentiable function. Then* $\partial j(u) = \{j'(u)\}, j^\circ(u;z) = \langle j'(u), z\rangle$, *and* $(\phi + j)^\circ(u;z) = \phi^\circ(u;z) + \langle j'(u), z\rangle$ *for all* $u, z \in X$*;*

(v) *(Lebourg's mean value theorem) Let* u *and* v *be two points in* X. *Then there exists a point* w *in the open segment between* u *and* v, *and* $x_w^* \in \partial\phi(w)$ *such that*

$$\phi(u) - \phi(v) = \langle x_w^*, u - v\rangle;$$

(vi) *The function* $(u,h) \to \phi^\circ(u;h)$ *is upper semicontinuous.*

Let I be a function on X satisfying the following structure hypothesis:
(H) $I = \Phi + \Psi$, where $\Phi : X \to \mathbb{R}$ is locally Lipschitz while $\Psi : X \to \mathbb{R}\bigcup\{+\infty\}$ is convex, proper, and lower semicontinuous.

We say that $u \in X$ is a critical point of I if it satisfies the following inequality

$$\Phi^\circ(u; v-u) + \Psi(v) - \Psi(u) \geq 0, \quad \text{for all} \quad v \in X.$$

Set

$$K := \{u \in X | u \text{ is a critical point of } I\}$$

and

$$K_c = K \bigcap I^{-1}(c).$$

A number $c \in \mathbb{R}$ such that $K_c \neq \emptyset$ is called a critical value of I.

Definition 2.1. (See [15]). $I = \Phi + \Psi$ is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ ($(PS)_c$ for short) if every sequence $\{u_n\}$ in X satisfying $I(u_n) \to c$ and

$$\Phi^\circ(u_n; v - u_n) + \Psi(v) - \Psi(u_n) \geq -\epsilon_n\|v - u_n\|, \quad \text{for all } v \in X,$$

for a sequence $\{\epsilon_n\}$ in $[0,\infty)$ with $\epsilon_n \to 0^+$, contains a convergent subsequence. If $(PS)_c$ is verified for all $c \in \mathbb{R}$, I is said to satisfy the Palais-Smale condition (shortly, (PS)).

Finally, we recall a non-smooth version three critical points theorem due to S. A. Marano and D. Motreanu, which represents the main tool to investigate problem (P_λ).

Theorem 2.1. (See [15]). *Let X be a separable and reflexive Banach space, let $I_1 := \Phi_1 + \Psi_1$ and $I_2 := \Phi_2$ be like in (H), let Λ be a real interval. Suppose that*

- (b_1) *Φ_1 is weakly sequentially lower semicontinuous while Φ_2 is weakly sequentially continuous.*
- (b_2) *For every $\lambda \in \Lambda$ the function $I_1 + \lambda I_2$ fulfills $(PS)_c, c \in \mathbb{R}$, together with $\lim\limits_{\|u\|\to+\infty} \big(I_1(u) + \lambda I_2(u)\big) = +\infty$.*
- (b_3) *There exists a continuous concave function $h : \Lambda \to \mathbb{R}$ satisfying*

$$\sup_{\lambda\in\Lambda} \inf_{u\in X} \big(I_1(u) + \lambda I_2(u) + h(\lambda)\big) < \inf_{u\in X} \sup_{\lambda\in\Lambda} \big(I_1(u) + \lambda I_2(u) + h(\lambda)\big).$$

Then there is an open interval $\Lambda_0 \subseteq \Lambda$ such that for each $\lambda \in \Lambda_0$ the function $I_1 + \lambda I_2$ has at least three critical points in X. Moreover, if $\Psi_1 \equiv 0$, then there exist an open interval $\Lambda_1 \subseteq \Lambda$ and a number $\sigma > 0$ such that for each $\lambda \in \Lambda_0$ the function $I_1 + \lambda I_2$ has at least three critical points in X having norms less than $\sigma > 0$.

Theorem 2.2. (See [25]). *Let X be a non-empty set and Φ, Ψ two real functionals on X. Assume that there are $r > 0, u_0, u_1 \in X$, such that*

$$\Phi(u_0) = \Psi(u_0) = 0, \qquad \Phi(u_1) > r,$$

and

$$\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}.$$

Then, for each ρ satisfying

$$\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) < \rho < r\frac{\Psi(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\geq 0} \inf_{u\in X} \big(\Phi(u) + \lambda(\rho - \Psi(u))\big) < \inf_{u\in X} \sup_{\lambda\geq 0} \big(\Phi(u) + \lambda(\rho - \Psi(u))\big).$$

3. Main Result

In this section, we shall prove the existence of at least three solutions to problem (P_λ). Let us first introduce some notations. It is clear that (P_λ) is the Euler-Lagrange equation of the functional $\varphi : X \to \mathbb{R}$ defined by

$$\begin{aligned}\varphi(u) = \widehat{K}\bigg(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\bigg)\\ - \int_\Omega F_1(x,u)dx - \lambda\int_\Omega F_2(x,u)dx,\end{aligned} \tag{3.1}$$

where $\widehat{K}(t) = \displaystyle\int_0^t K(\tau)d\tau$. Set

$$J(u) = \widehat{K}\bigg(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\bigg), \quad \mathscr{F}_1(u) = -\int_\Omega F_1(x,u)dx,$$

$$\mathscr{F}_2(u) = -\int_\Omega F_2(x,u)dx, \tag{3.2}$$

and

$$h_1 = J + \mathscr{F}_1, \qquad h_2 = \mathscr{F}_2, \tag{3.3}$$

then, under these notations, $\varphi = h_1 + \lambda h_2$. Let $V_{p(x)} = \{u \in W^{1,p(x)}(\Omega) : \displaystyle\int_\Omega u dx = 0\}$, then $V_{p(x)}$ is a closed linear subspace of $W^{1,p(x)}(\Omega)$ with codimension 1, and we have $W^{1,p(x)}(\Omega) = \mathbb{R} \oplus V_{p(x)}$ (see [23]). If we define the norm by

$$\|u\|' = \inf\bigg\{\lambda > 0 : \int_\Omega \bigg(\bigg|\frac{\nabla u}{\lambda}\bigg|^{p(x)} + \bigg|\frac{u}{\lambda}\bigg|^{p(x)}\bigg)dx \le 1\bigg\},$$

by Proposition 2.6 in [23], it is easy to see that $\|u\|'$ is an equivalent norm in $V_{p(x)}$. Thereafter, we will also use $\|\cdot\|$ to denote the equivalent norm $\|u\|'$ in $V_{p(x)}$ and one can easily see that when $u \in V_{p(x)}$, we have

(i) if $\|u\| < 1$, then $\dfrac{k_0}{p^+}\|u\|^{p^+} \le J(u) \le \dfrac{k_1}{p^-}\|u\|^{p^+}$;

(ii) if $\|u\| > 1$, then $\dfrac{k_0}{p^+}\|u\|^{p^-} \le J(u) \le \dfrac{k_1}{p^-}\|u\|^{p^+}$.

Moreover, let $q^+ < p(x)$ and denote

$$\lambda_1 := \inf_{u \in V_{p(x)}\setminus\{0\}, \|u\|>1} \frac{\displaystyle\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx}{\displaystyle\int_\Omega |u|^{q(x)}dx},$$

by continuous embedding of $W^{1,p(x)}(\Omega)$ in $L^{p^-}(\Omega)$ and, by Proposition 2.2(2), the above inequalities of (i), (ii), one has $\lambda_1 > 0$.

Now we are in a position to present our main result. In order to reduce our statements, we need the following assumptions:
(A_1): $F_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $F_1(x,t)$ satisfies $F_1(x,0) = 0$ and also
$(A_1)_1$ for all $t \in \mathbb{R}, \Omega \ni x \to F_1(x,t) \in \mathbb{R}$ is measurable;
$(A_1)_2$ for almost all (shortly, a.a.) $x \in \Omega, \mathbb{R} \ni t \to F_1(x,t) \in \mathbb{R}$ is locally Lipschitz;
$(A_1)_3$ for a.a. $x \in \Omega$, all $t \in \mathbb{R}$ and all $w \in \partial F_1(x,t)$, there holds

$$|w| \leq a(x)|t|^{p^- - 1}, \quad \text{with} \quad a(x) \in L^\infty(\Omega)_+;$$

$(A_1)_4$ there exist $q(x), s(x) \in C_+(\overline{\Omega})$ satisfying $q^+ < p(x) < s^-$, for all $x \in \Omega$, such that

$$\limsup_{|t| \to 0} \frac{F_1(x,t)}{|t|^{q(x)}} < -2\lambda_1,$$

and

$$\limsup_{|t| \to +\infty} \frac{F_1(x,t) - \hat{a}(x)|t|^{p^-}}{|t|^{s^-}} < 0, \quad \text{with} \quad \hat{a}(x) \in L^\infty(\Omega)_+,$$

uniformly for a.a. $x \in \Omega$.
(A_2): $F_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $F_2(x,t)$ satisfies $F_2(x,0) = 0$ and also
$(A_2)_1$ for all $t \in \mathbb{R}, \Omega \ni x \to F_2(x,t) \in \mathbb{R}$ is measurable;
$(A_2)_2$ for a.a. $x \in \Omega, \mathbb{R} \ni t \to F_2(x,t) \in \mathbb{R}$ is locally Lipschitz;
$(A_2)_3$ for a.a. $x \in \Omega$, all $t \in \mathbb{R}$ and all $v \in \partial F_2(x,t)$, there holds

$$|v| \leq b(x)|t|^{p(x)-1}, \quad \text{with} \quad b(x) \in L^\infty(\Omega)_+;$$

$(A_2)_4$ there exists $\mathcal{R} > 0$, for all $0 < |t| < \mathcal{R}$, such that $p(x)F_2(x,t) > 0$, and

$$p(x)F_2(x,t) = o(|t|^{p^+}) \quad \text{as} \quad |t| \to 0,$$

and

$$\limsup_{|t| \to +\infty} \frac{F_2(x,t)}{|t|^{p^+}} < 0,$$

uniformly for a.a. $x \in \Omega$.

Remark 3.1. A simple example of nonsmooth locally Lipschitz function satisfying hypothesis (A_1) and (A_2) is (for simplicity we drop the x-dependence):

$$F_1(t) = \begin{cases} (-2\lambda_1 - 1)|t|^{q(x)}, & \text{if} \quad |t| \leq 1, \\ \hat{a}(x)|t|^{p^-} - \ln|t|^{s^-}, & \text{if} \quad |t| > 1, \end{cases}$$

and

$$F_2(t) = \begin{cases} 2|t|^{p(x)}, & \text{if} \quad |t| \leq 1, \\ -\ln |t|^{p^+}, & \text{if} \quad |t| > 1. \end{cases}$$

One can easily check that all conditions in (A_1) and (A_2) are satisfied.

The main result can be formulated as follows:

Theorem 3.1. *Suppose that the assumptions* $(K_0), (A_1)$ *and* (A_2) *hold, then there exists an open interval* $\Lambda_1 \subseteq [0, +\infty)$*, such that for each* $\lambda \in \Lambda_1$*, problem* (P_λ) *has at least three nontrivial solutions in* $V_{p(x)}$.

Before proving Theorem 3.1, we give some preliminary lemmas which are useful to prove the main result.

Lemma 3.1. *Since* F_i *are locally Lipschitz functions which satisfy* $(A_i)_3$*, then* $\mathscr{F}_i$ *in (3.2) are well defined and they are locally Lipschitz. Moreover, let* E *be a closed subspace of* X *and* $\mathscr{F}_i|_E$ *the restriction of* $\mathscr{F}_i$ *to* E*, where* $i = 1$ *or* 2*. Then*

$$(\mathscr{F}_i|_E)^\circ(u; v) \leq \int_\Omega (-F_i)^\circ(x, u(x); v(x))dx, \quad \text{for all } u, v \in E.$$

The proof of the above lemma is similar to that of [26, Lemma 4.2], we omit the details here.

Lemma 3.2. *For any* $\epsilon > 0$*, and* $q(x)$ *as mentioned in* $(A_1)_4$*, there exists a* $u_\epsilon \in V_{p(x)}$ *with* $\|u_\epsilon\| > 1$*, such that*

$$\frac{k_0}{p^+}\|u_\epsilon\|^{p^-} + \lambda_1 \int_\Omega |u_\epsilon|^{q(x)}dx \geq \frac{(1+k_0)\lambda_1 + k_0\epsilon}{p^+(\lambda_1+\epsilon)}.$$

Proof. Since $\lambda_1 := \inf\limits_{u \in V_{p(x)} \setminus \{0\}, \|u\|>1} \dfrac{\displaystyle\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx}{\displaystyle\int_\Omega |u|^{q(x)}dx}$, then by the definition of infimum, for any $\epsilon > 0$, there exists a $u_\epsilon \in V_{p(x)}$ with $\|u_\epsilon\| > 1$, such that

$$\frac{\displaystyle\int_\Omega \frac{1}{p(x)}(|\nabla u_\epsilon|^{p(x)} + |u_\epsilon|^{p(x)})dx}{\displaystyle\int_\Omega |u_\epsilon|^{q(x)}dx} < \lambda_1 + \epsilon,$$

which produces

$$\frac{\lambda_1}{\lambda_1+\epsilon} < \frac{\lambda_1 \displaystyle\int_\Omega |u_\epsilon|^{q(x)}dx}{\displaystyle\int_\Omega \frac{1}{p(x)}(|\nabla u_\epsilon|^{p(x)} + |u_\epsilon|^{p(x)})dx} \leq \frac{\lambda_1 \displaystyle\int_\Omega |u_\epsilon|^{q(x)}dx}{\frac{1}{p^+}\|u_\epsilon\|^{p^-}}.$$

Consequently, we have

$$\frac{k_0}{p^+}\|u_\epsilon\|^{p^-} + \lambda_1 \int_\Omega |u_\epsilon|^{q(x)}dx \geq \frac{1}{p^+}\left(k_0 + \frac{\lambda_1}{\lambda_1+\epsilon}\right)\|u_\epsilon\|^{p^-}.$$

Lemma 3.3. *There exists a $\bar{u} \in V_{p(x)}$ with $\bar{u} \neq 0$ and $\bar{r} > 0$ such that $h_1(\bar{u}) > \bar{r}$, where $h_1 = J + \mathscr{F}_1$ is as mentioned in (3.3).*

Proof. In view of $(A_1)_4$, there exists a $\delta_0 > 0$ small enough (without loss of generality we assume $\delta_0 < 1$), such that for a.a. $x \in \Omega$, we have

$$F_1(x,t) \leq -2\lambda_1|t|^{q(x)}, \quad \text{for all} \quad |t| \leq \delta_0, \tag{3.4}$$

we also know from $(A_1)_4$ that there exists $\tau > 0$ such that

$$\limsup_{|t|\to+\infty} \frac{F_1(x,t) - \hat{a}(x)|t|^{p^-}}{|t|^{s^-}} < -2\tau,$$

uniformly for a.a. $x \in \Omega$. Then we can find a $M > 1$ large enough such that for a.a. $x \in \Omega$, all $|t| \geq M$ such that

$$\limsup_{|t|\to+\infty} \frac{F_1(x,t) - \hat{a}(x)|t|^{p^-}}{|t|^{s^-}} < -\tau,$$

which leads to

$$F_1(x,t) \leq \hat{a}(x)|t|^{p^-} - \tau|t|^{s^-}, \quad \text{for all} \quad |t| > M. \tag{3.5}$$

On the other hand, since $F_1(x,0) = 0$ and $(A_1)_3$ holds, from Lebourg's mean value theorem, for a.a. $x \in \Omega$, it follows that

$$F_1(x,t) \leq a(x)|t|^{p^-}, \quad \text{for all} \quad \delta_0 < |t| \leq M. \tag{3.6}$$

Note that $a(x) \in L^\infty(\Omega)_+$, $p^- > q^+$ and, $\delta_0 < 1$ imply $-\frac{2\lambda_1}{\delta_0^{p^-}} < -2\lambda_1$, for a.a. $x \in \Omega$, we obtain from (3.4) and (3.6) that

$$F_1(x,t) \leq -2\lambda_1|t|^{q(x)} + \left(a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|t|^{p^-}, \quad \text{for all} \quad |t| \leq M. \tag{3.7}$$

When $|t| > M > \delta_0$, since $\delta_0 < 1$ and $p^- > q^+$, then $(\frac{|t|}{\delta_0})^{p^-} > |t|^{q(x)}$, which shows that

$$-2\lambda_1|t|^{q(x)} + \left(a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|t|^{p^-} > 0. \tag{3.8}$$

Thus, through (3.5), for a.a. $x \in \Omega$, and (3.8), we have

$$F_1(x,t) \le \left(a(x)+\hat{a}(x)+\frac{2\lambda_1}{\delta_0^{p^-}}\right)|t|^{p^-} - \tau |t|^{s^-} - 2\lambda_1 |t|^{q(x)}, \text{ for all } |t| > M. \tag{3.9}$$

Due to $q^+ < p^- < s^-$, it follows from (3.7) that

$$\begin{aligned} F_1(x,t) &\le -2\lambda_1 |t|^{q(x)} + \left(a(x)+\frac{2\lambda_1}{\delta_0^{p^-}}\right)|t|^{p^-} - \frac{a(x)}{M^{p^-}}|t|^{p^-} \\ &\quad - \frac{\tau}{M^{s^-}}|t|^{s^-} + a(x)\Big|\frac{t}{M}\Big|^{q(x)} + \tau\Big|\frac{t}{M}\Big|^{q(x)} \\ &\le \left(-2\lambda_1 + \frac{a(x)+\tau}{M^{q(x)}}\right)|t|^{q(x)} - \frac{\tau}{M^{s^-}}|t|^{s^-} \\ &\quad + \left(a(x) - \frac{a(x)}{M^{p^-}} + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|t|^{p^-}, \quad \text{for all} \quad |t| \le M. \end{aligned} \tag{3.10}$$

Therefore, note that $M > 1$, from (3.9) and (3.10), for a.a. $x \in \Omega$ and all $t \in R$, one can derive

$$\begin{aligned} F_1(x,t) \le \left(-2\lambda_1 + \frac{a(x)+\tau}{M^{q(x)}}\right)|t|^{q(x)} \\ + \left(a(x)+\hat{a}(x)+\frac{2\lambda_1}{\delta_0^{p^-}}\right)|t|^{p^-} - \frac{\tau}{M^{s^-}}|t|^{s^-}, \end{aligned} \tag{3.11}$$

we take $M_0 > M$ sufficiently large such that $\dfrac{a(x)+\tau}{M_0^{q(x)}} < \lambda_1$, without loss of generality we suppose that $a(x) > \hat{a}(x) > 0$, for all $x \in \Omega$, it follows from (3.11) that

$$F_1(x,t) \le -\lambda_1 |t|^{q(x)} + \left(2a(x)+\frac{2\lambda_1}{\delta_0^{p^-}}\right)|t|^{p^-} - \frac{\tau}{M_0^{s^-}}|t|^{s^-}. \tag{3.12}$$

We now distinguish two cases to complete the proof.

Case 1. We claim that there exists a $u_1 \in V_{p(x)}$ with $\|u_1\| > 1$ such that $h_1(u_1) > r_1$. In fact, for any $u \in V_{p(x)}$ with $\|u\| > 1$, we obtain from (3.12) that

$$\begin{aligned} h_1(u) &= J(u) + \mathscr{F}_1(u) \\ &= \widehat{K}\left(\int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx\right) - \int_\Omega F_1(x,u)dx \\ &\ge \frac{k_0}{p^+}\|u\|^{p^-} + \lambda_1 \int_\Omega |u|^{q(x)}dx + \int_\Omega \frac{\tau}{M_0^{s^-}}|u|^{s^-}dx \\ &\quad - \int_\Omega \left(2a(x)+\frac{2\lambda_1}{\delta_0^{p^-}}\right)|u|^{p^-}dx. \end{aligned} \tag{3.13}$$

Since the imbedding $X \hookrightarrow L^{p^-}$ is compact, then there exists a constant $C_1 > 0$, such that $|u|_{p^-} \leq C_1\|u\|$, for all $u \in X$. By virtue of Lemma 3.2, taking a ϵ_0 small enough, we can reach

$$\frac{\lambda_1}{p^+(\lambda_1+\epsilon_0)} > \left(2\|a\|_\infty + \frac{2\lambda_1}{\delta_0^{p^-}}\right)C_1. \tag{3.14}$$

Recall that $a(x) \in L^\infty(\Omega)_+$, where $\|\cdot\|_\infty$ denotes the norm of $L^\infty(\Omega)$, there exists a $u_{\epsilon_0} \in V_{p(x)}$ with $\|u_{\epsilon_0}\| > 1$, combining (3.13) with (3.14) yields

$$\begin{aligned}
h_1(u_{\epsilon_0}) &= J(u_{\epsilon_0}) + \mathscr{T}_1(u_{\epsilon_0}) \\
&\geq k_0\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u_{\epsilon_0}|^{p(x)} + |u_{\epsilon_0}|^{p(x)}\right)dx\right) + \lambda_1\int_\Omega |u_{\epsilon_0}|^{q(x)}dx \\
&\quad + \int_\Omega \frac{\tau}{M_0^{s^-}}|u_{\epsilon_0}|^{s^-}dx - \int_\Omega\left(2a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|u_{\epsilon_0}|^{p^-}dx \\
&\geq \frac{1}{p^+}\left(k_0 + \frac{\lambda_1}{\lambda_1+\epsilon_0}\right)\|u_{\epsilon_0}\|^{p^-} + \int_\Omega \frac{\tau}{M_0^{s^-}}|u_{\epsilon_0}|^{s^-}dx \\
&\quad - \left(2\|a\|_\infty + \frac{2\lambda_1}{\delta_0^{p^-}}\right)C_1\|u_{\epsilon_0}\|^{p^-} \\
&\geq \frac{k_0}{p^+}\|u_{\epsilon_0}\|^{p^-}.
\end{aligned}$$

Taking $u_1 = u_{\epsilon_0}$ and a constant $0 < r_1 < \frac{k_0}{p^+}$, we have $h_1(u_1) > r_1$.

Case 2. We also claim that there exists a $u_2 \in V_{p(x)}$ with $\|u_2\| \leq 1$ such that $h_1(u_2) > r_2$. In fact, for any $u \in V_{p(x)}$ with $\|u\| \leq 1$, we know from (3.13) that

$$\begin{aligned}
h_1(u) \geq{}& \frac{k_0}{p^+}\|u\|^{p^+} + \int_\Omega \frac{\tau}{M_0^{s^-}}|u|^{s^-}dx + \lambda_1\int_\Omega |u|^{q(x)}dx \\
&- \int_\Omega\left(2\|a\|_\infty + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|u|^{p^-}dx.
\end{aligned} \tag{3.15}$$

Set

$$K = \left\{t \in \left| |t| < \min\left(1, \left(\frac{\lambda_1\delta_0^{p^-}}{2\|a\|_\infty\delta_0^{p^-} + 2\lambda_1}\right)^{\frac{1}{p^- - q^-}}\right)\right.\right\},$$

then we can choose a $u_2 \in K \cap V_{p(x)}$ with $\|u_2\| < 1$, satisfying

$$\lambda_1|u_2|^{q(x)} - \left(2\|a\|_\infty + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|u_2|^{p^-} > 0,$$

which, together with (3.15), leads to

$$h_1(u_2) \geq \frac{k_0}{p^+}\|u\|^{p^+} + \int_\Omega \frac{\tau}{M_0^{s^-}}|u|^{s^-}dx \geq \frac{k_0}{p^+}\|u\|^{p^+} > 0. \tag{3.16}$$

We can easily see from (3.16) that there exists a $r_2 > 0$ satisfying $h_1(u_2) > r_2$. Combining the above two cases, there exist a $\bar{u} \in V_{p(x)}$ with $\bar{u} \neq 0$ and a $\bar{r} > 0$ such that $h_1(\bar{u}) > \bar{r}$. The proof of Lemma 3.3 is completed.

Remark 3.2. From the proof of Lemma 3.3, it is easy to see that h_1 is coercive, i.e., $h_1(u) \to +\infty$ as $\|u\| \to +\infty$. In fact, since $p^- < s^-$, by Young inequality, we have

$$\begin{aligned}\left(2a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|u|^{p^-} &\leq \frac{p^-\epsilon}{s^-}\left||u|^{p^-}\right|^{\frac{s^-}{p^-}} + \frac{s^- - p^-}{s^-}\epsilon^{-\frac{p^-}{s^- - p^-}}c_1^{\frac{s^-}{s^- - p^-}} \\ &\leq \frac{p^-\epsilon}{s^-}|u|^{s^-} + c_2,\end{aligned} \tag{3.17}$$

where $c_1 = 2\|a\|_\infty + \frac{2\lambda_1}{\delta_0^{p^-}} > 0, c_2 = c_2(c_1, \epsilon) > 0$. Therefore, we know from (3.13) that

$$h_1(u) \geq \frac{k_0}{p^+}\|u\|^{p^-} + \lambda_1\int_\Omega |u|^{q(x)}dx + \int_\Omega \frac{\tau}{M_0^{s^-}}|u|^{s^-}dx - \int_\Omega \frac{p^-\epsilon}{s^-}|u|^{s^-}dx - c_2|\Omega|.$$

Let $\epsilon < \frac{s^-\tau}{p^-M_0^{s^-}}$, one can easily get the coercivity of h_1.

Lemma 3.4. *There exists a $r > 0$ with $r < \bar{r}$ such that*

$$\sup_{u \in h_1^{-1}((-\infty, r]) \bigcap V_{p(x)}} \big(-h_2(u)\big) < r\frac{-h_2(\bar{u})}{h_1(\bar{u})},$$

where $\bar{u}, \bar{r}$ and h_2 are as mentioned in Lemma 3.3 and (3.3), respectively.

Proof. Firstly, from the assumptions of $(A_2)_4$, for any $\epsilon > 0$, there exists a $\delta' > 0$, for any $0 < |t| \leq \delta_1 = \min\{\delta', 1\}$, for a.a. $x \in \Omega$, we have

$$F_2(t, x) \leq \frac{\epsilon}{p(x)}|t|^{p^+}. \tag{3.18}$$

By considering again $(A_2)_4$, for the above $\epsilon > 0$, there exists a $N > 1$ large enough, for a.a. $x \in \Omega$, such that

$$F_2(t, x) \leq \epsilon|t|^{p^+}, \quad \text{for all} \quad |t| > N. \tag{3.19}$$

Secondly, since $F_2(x,0)=0$ and taking into account $(A_2)_3$, for a.a. $x\in\Omega$, by the Lebourg mean value theorem, it follows that

$$F_2(x,t)\le c(x)|t|^{\alpha(x)},\quad \text{for all}\quad \delta_1<|t|\le N, \tag{3.20}$$

where $c(x)\in L^\infty(\Omega)_+$ and $p^+<\alpha(x)<p^*(x)$, for all $x\in\Omega$. Combining (3.19) and (3.20), for a.a. $x\in\Omega$, we have

$$F_2(x,t)\le \epsilon|t|^{p^+}+c(x)|t|^{\alpha(x)},\quad \text{for all}\quad |t|>\delta_1, \tag{3.21}$$

we get from (3.18) and (3.21) that

$$F_2(x,t)\le \frac{\epsilon}{p(x)}|t|^{p^+}+\epsilon|t|^{p^+}+c(x)|t|^{\alpha(x)},\quad \text{for all}\quad |t|\le\delta_1, \tag{3.22}$$

and

$$F_2(x,t)\le \frac{\epsilon}{p(x)}|t|^{p^+}+\epsilon|t|^{p^+}+c(x)|t|^{\alpha(x)},\quad \text{for all}\quad |t|>\delta_1, \tag{3.23}$$

and hence, for a.a. $x\in\Omega$ and all $t\in\mathbb{R}$, from (3.22) and (3.23), we have

$$F_2(x,t)\le \left(1+\frac{1}{p(x)}\right)\epsilon|t|^{p^+}+c(x)|t|^{\alpha(x)}. \tag{3.24}$$

Define the function $g:[0,+\infty)\to\mathbb{R}$ by

$$g(t)=\sup\Big\{-h_2(u): u\in V_{p(x)} \text{ with } \|u\|^{p^+}\le \eta t\Big\},$$

where η is an arbitrary constant satisfying $\eta>1$. Note that $\alpha(x)<p^*(x)$, for all $x\in\Omega$, due to the embedding $X\hookrightarrow L^{\alpha(x)}$ is compact and the embedding $X\hookrightarrow L^{p^+}$ is continuous, then there exist constants $C_2>0$ and $C_3>0$ such that

$$|u|_{\alpha(x)}\le C_2\|u\|,\quad |u|_{p^+}\le C_3\|u\|,\quad \text{for all}\quad u\in X. \tag{3.25}$$

Since $h_2(u)=\mathscr{F}_2(u)$, then from (3.24) and (3.25), we have

$$\begin{aligned} g(t)&\le \left(1+\frac{1}{p^-}\right)\epsilon|u|_{p^+}^{p^+}+c_3\max\left\{|u|_{\alpha(x)}^{\alpha^+},|u|_{\alpha(x)}^{\alpha^-}\right\}\\ &\le \left(1+\frac{1}{p^-}\right)\epsilon C_3^{p^+}\eta t+c_3\max\left\{C_2^{\alpha^+}\eta^{\frac{\alpha^+}{p^+}}t^{\frac{\alpha^+}{p^+}},C_2^{\alpha^-}\eta^{\frac{\alpha^-}{p^+}}t^{\frac{\alpha^-}{p^+}}\right\}\end{aligned} \tag{3.26}$$

where $c_3=\|c\|_\infty$. On the other hand, by virtue of $(A_2)_4$, $g(t)>0$ for $t>0$. Furthermore, due to $\alpha^->p^+$ and the arbitrariness of $\epsilon>0$, we deduce

$$\lim_{t\to 0^+}\frac{g(t)}{t}=0. \tag{3.27}$$

By Lemma 3.3, we know $h_1(\bar{u}) > 0$, it is obvious that $\bar{u} \neq 0$. Thus, by $(A_2)_4$, it shows that $-h_2(\bar{u}) > 0$. In view of (3.27), for $\frac{-h_2(\bar{u})}{h_1(\bar{u})} > 0$, there exists a $t_0 > 0$ such that

$$\frac{g(t)}{t} < \frac{-h_2(\bar{u})}{h_1(\bar{u})}, \quad \text{for all} \quad t < t_0,$$

that is,

$$\sup_{u \in \left\{ \|u\|^{p^+} \leq \eta t \right\} \bigcap V_{p(x)}} -h_2(u) < t \frac{-h_2(\bar{u})}{h_1(\bar{u})}. \tag{3.28}$$

Now, we choose a constant r with $0 < r < \min\{t_0, \bar{r}\}$(where $\bar{r}$ is the one in Lemma 3.3). If $h_1(u) \leq r$, then by the coercivity of h_1, there exists a constant $c_4 > 1$ such that $\|u\|^{p^+} \leq c_4 r$, therefore, we have

$$h_1^{-1}\big((-\infty, r]\big) \bigcap V_{p(x)} \subseteq \Big\{ u \in V_{p(x)}, \|u\|^{p^+} \leq c_4 r \Big\},$$

from which it follows that

$$\sup_{u \in h_1^{-1}\left((-\infty, r]\right) \bigcap V_{p(x)}} -h_2(u) \leq \sup_{u \in \{\|u\|^{p^+} \leq c_4 r\} \bigcap V_{p(x)}} -h_2(u). \tag{3.29}$$

In view of the fact that $r < t_0$ and $c_4 > 1$, one can deduce from (3.28) and the arbitrariness of $\eta > 1$ that

$$\sup_{u \in \{\|u\|^{p^+} \leq c_4 r\} \bigcap V_{p(x)}} -h_2(u) < r \frac{-h_2(\bar{u})}{h_1(\bar{u})}. \tag{3.30}$$

Therefore, by (3.29) and (3.30), the desired inequality of Lemma 3.4 is obtained. This completes the proof of Lemma 3.4.

By a standard argument, one can easily show that h_1 and h_2 are locally Lipschitz under the assumptions of (A_1) and (A_2). Here, we consider the indicator function of closed subspace $V_{p(x)}$, i.e., $\psi_1 : X \to (-\infty, +\infty]$,

$$\psi_1(u) = \begin{cases} 0, & \text{if} \quad u \in V_{p(x)}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Obviously, ψ_1 is convex, proper, and lower semicontinuous. Denote

$$I_1(u) := h_1(u) + \psi_1(u), \quad \text{and} \quad I_2(u) := h_2(u), \quad u \in X.$$

It is clear that I_1 and I_2 satisfy (H) in Section 2. Therefore, for every $\lambda > 0$ the function $I_1 + \lambda I_2$ complies with (H) as well.

Lemma 3.5. *If the assumptions of (A_1) and (A_2) hold, then for every $\lambda > 0$ the function $I_1 + \lambda I_2$ satisfies $(PS)_c$ in $V_p(x), c \in \mathbb{R}$, and*

$$\lim_{\|u\|\to+\infty} \big(I_1 + \lambda I_2\big)|_{V_{p(x)}}(u) = +\infty, \quad \text{for all} \quad u \in V_{p(x)}.$$

Proof. Firstly, we shall prove that $I_1 + \lambda I_2$ is coercive in $V_p(x)$, for every $\lambda > 0$. Indeed, for every $u \in V_{p(x)}$ and without loss of generality we assume $\|u\| > 1$, because of the definition of ψ_1 and (3.13), we obtain

$$\begin{aligned}(I_1 + \lambda I_2)|_{V_p(x)}(u) &= \widehat{K}\bigg(\int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx\bigg)\\ &\quad - \int_\Omega F_1(x,u)dx - \lambda \int_\Omega F_2(x,u)dx\\ &\geq \frac{k_0}{p^+}\|u\|^{p^-} + \lambda_1 \int_\Omega |u|^{q(x)}dx + \int_\Omega \frac{\tau}{M_0^{s^-}}|u|^{s^-}dx\\ &\quad - \int_\Omega \bigg(2a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\bigg)|u|^{p^-}dx - \lambda \int_\Omega F_2(x,u)dx.\end{aligned} \tag{3.31}$$

On the one hand, by assumption of $(A_2)_4$, there exists a $M_1 > 0$, for a.a. $x \in \Omega$ and all $|t| > M_1$, such that

$$F_2(x,t) < 0.$$

On the other hand, from the Lebourg mean value theorem, for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$, one has

$$\big|F_2(x,t) - F_2(x,0)\big| \leq |v_1||t|, \tag{3.32}$$

for some $v_1 \in \partial F_2(x, \theta t), 0 < \theta < 1$. On account of $(A_2)_3$ and $F_2(x,0) = 0$, then for a.a. $x \in \Omega$ and all t such that $|t| \leq M_1$, it follows from (3.32) that

$$\big|F_2(x,t)\big| \leq c_5,$$

where $c_5 = c_5(M_1, \|b\|_\infty) > 0$. Then it follows that

$$\int_\Omega F_2(x,u)dx = \int_{\{|u|>M_1\}} F_2(x,u)dx + \int_{\{|u|\leq M_1\}} F_2(x,u)dx \leq c_5|\Omega|. \tag{3.33}$$

By applying (3.17) and (3.33) to (3.31), we have

$$\begin{aligned}(I_1 + \lambda I_2)|_{V_p(x)}(u) &\geq \frac{k_0}{p^+}\|u\|^{p^-} + \lambda_1 \int_\Omega |u|^{q(x)}dx\\ &\quad + \int_\Omega \frac{\tau}{M_0^{s^-}}|u|^{s^-}dx - \frac{p^-\epsilon}{s^-}\int_\Omega |u|^{s^-}dx - (c_2 + \lambda c_5)|\Omega|.\end{aligned}$$

Let $\epsilon < \dfrac{s^-\tau}{p^- M_0^{s^-}}$, then it is easy from (3.33) to see that $I_1 + \lambda I_2$ is coercive in $V_{p(x)}$, for every $\lambda > 0$.

Next, we prove that $(I_1 + \lambda I_2)|_{V_{p(x)}}(u)$ satisfies $(PS)_c, c \in \mathbb{R}$. Let $\{u_n\} \subset V_{p(x)}$ be a sequence such that

$$I_1(u_n) + \lambda I_2(u_n) \to c \tag{3.34}$$

and for every $v \in V_{p(x)}$, we have

$$(h_1 + \lambda h_2)^\circ(u_n; v - u_n) + \psi_1(v) - \psi_1(u_n) \geq -\epsilon_n \|v - u_n\|, \tag{3.35}$$

for a sequence $\{\epsilon_n\}$ in $[0, +\infty)$ with $\epsilon_n \to 0^+$. By the coerciveness of the function $I_1 + \lambda I_2$ in $V_p(x)$, (3.34) implies that the sequence $\{u_n\}$ is bounded in $V_{p(x)}$. Therefore, there exists an element $u \in V_{p(x)}$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad V_{p(x)}.$$

Since J is a continuous differentiable function in (3.2), then by Proposition 2.3(iii) and (iv) we have

$$\begin{aligned} &(h_1 + \lambda h_2)^\circ(u_n; v - u_n) \leq h_1^\circ(u_n; v - u_n) + \lambda h_2^\circ(u_n; v - u_n) \\ = &\langle J'(u_n), v - u_n \rangle_X + \mathscr{F}_1^\circ(u_n; v - u_n) + \lambda \mathscr{F}_2^\circ(u_n; v - u_n). \end{aligned} \tag{3.36}$$

Choose in particular $v = u$ in (3.35) and the definition of ψ_1, (3.36) becomes

$$\langle J'(u_n), u_n - u \rangle_X \leq \epsilon_n \|u - u_n\| + \mathscr{F}_1^\circ(u_n; u - u_n) + \lambda \mathscr{F}_2^\circ(u_n; u - u_n).$$

Since the embedding $X \hookrightarrow L^{p^-}(\Omega)$ and $X \hookrightarrow L^{p(x)}(\Omega)$ are compact, passing to a subsequence if necessary, we may assume that $\{u_n\}$ converges strongly to u in L^{p^-} and $L^{p(x)}$. By Lemma 3.1, Proposition 2.3, $(A_1)_3$ and Hölder inequality, one can deduce that

$$\begin{aligned} \mathscr{F}_1^\circ(u_n; u - u_n) \leq &\int_\Omega (-F_1)^\circ(x, u_n; u - u_n)dx \\ = &\int_\Omega (F_1)^\circ(x, u_n; u_n - u)dx \\ = &\int_\Omega \max\big\{\langle \zeta_n(x), u_n - u \rangle_X : \zeta_n(x) \in \partial F_1(x, u_n(x))\big\}dx \\ \leq &\int_\Omega a(x)|u_n|^{p^- - 1}|u_n - u|dx \\ \leq &\|a\|_\infty |u_n|_{p^-}^{p^- - 1}|u_n - u|_{p^-}. \end{aligned}$$

Because $\{u_n\}$ is bounded in X, it follows that $\{u_n\}$ is bounded in $L^{p^-}(\Omega)$, and recall that $\{u_n\}$ converges strongly to u in L^{p^-}. Thus,

$$|u_n - u|_{p^-} \to 0, \quad \text{as} \quad n \to +\infty$$

and so

$$\mathscr{F}_1^\circ(u_n; u - u_n) \to 0, \quad \text{as} \quad n \to +\infty. \tag{3.37}$$

By a similar argument as above, it follows from Lemma 3.1, Proposition 2.3, $(A_2)_3$ and Hölder inequality that

$$\begin{aligned}
\mathscr{F}_2^\circ(u_n; u - u_n) &\le \int_\Omega (-F_2)^\circ(x, u_n; u - u_n)dx \\
&= \int_\Omega (F_2)^\circ(x, u_n; u_n - u)dx \\
&= \int_\Omega \max\left\{\langle \xi_n(x), u_n - u\rangle_X : \xi_n(x) \in \partial F_2(x, u_n(x))\right\}dx \\
&\le \int_\Omega b(x)|u_n|^{p(x)-1}|u_n - u|dx \\
&\le \|b\|_\infty |u_n|_{p(x)}^{p(x)-1}|u_n - u|_{p(x)},
\end{aligned}$$

also by the compact embedding of $X \hookrightarrow L^{p(x)}$, we get

$$\mathscr{F}_2^\circ(u_n; u - u_n) \to 0, \quad \text{as} \quad n \to +\infty. \tag{3.38}$$

Because of the sequence $\epsilon_n \to 0$ as $n \to +\infty$, together with (3.37) and (3.38), we conclude that

$$\limsup_{n\to+\infty} \langle J'(u_n), u_n - u\rangle_X \le 0. \tag{3.39}$$

From $u_n \rightharpoonup u$, one can easily see that

$$\lim_{n\to+\infty} \langle J'(u), u_n - u\rangle_X = 0,$$

which, together with (3.39), implies that

$$\limsup_{n\to+\infty} \langle J'(u_n) - J'(u), u_n - u\rangle_X \le 0.$$

We know from Lemma 2.1 that $u_n \to u$ as $n \to +\infty$, which implies that $I_1 + \lambda I_2$ satisfies $(PS)_c$ in $V_{p(x)}, c \in \mathbb{R}$.

We are now in a position to prove the main result of this section.

Proof of Theorem 3.1. Firstly, from Lemma 2.1 (iv) and a standard argument, the function h_1 is locally Lipschitz and weakly sequentially lower semicontinuous. Since $(A_2)_3$ holds and X is compactly embedded in $L^{p(x)}(\Omega)$, the assertion remains true regarding h_2 as well, thus (b_1) of Theorem 2.1 is satisfied. Secondly, it is clear from Lemma 3.5 that (b_2) in Theorem 2.1 holds. Finally, by Lemmas 3.3 and 3.4, and the definition of ψ_1, we know that there exist a $r > 0$ and $\bar{u} \in V_{p(x)}$ such that $I_1(\bar{u}) > r$. Moreover, keep in mind that $F_1(x, 0) = F_2(x, 0) = 0$, then we have $I_1(0) = -I_2(0) = 0$. Choose ρ satisfying

$$\sup_{u \in I_1^{-1}((-\infty, r]) \bigcap V_{p(x)}} \big(- I_2(u) \big) < \rho < r \frac{-I_2(\bar{u})}{I_1(\bar{u})},$$

by Theorem 2.2, one has

$$\sup_{\lambda \geq 0} \inf_{u \in V_{p(x)}} \big(I_1(u) + \lambda(\rho + I_2(u)) \big) \leq \inf_{u \in V_{p(x)}} \sup_{\lambda \geq 0} \big(I_1(u) + \lambda(\rho + I_2(u)) \big), \tag{3.40}$$

it is easy to see from Lemma 3.4 that $\rho > 0$. If we define $h : [0, +\infty) \to \mathbb{R}$ by $h(\lambda) = \rho\lambda$ and $\Lambda = [0, +\infty)$, then h and (3.40) satisfy the condition (b_3) in Theorem 2.1 . Therefore, all conditions in Theorem 2.1 are satisfied, then there is an open interval $\Lambda_0 \subseteq \Lambda$ such that for each $\lambda \in \Lambda_0$ the function $I_1 + \lambda I_2$ has at least three critical points in $V_{p(x)}$, then the energy function $\varphi = h_1 + \lambda h_2$ corresponding to problem (P_λ) possesses at least three critical points in $V_{p(x)}$, which implies that problem (P_λ) has at least three nontrivial solutions in $V_{p(x)}$. The proof of Theorem 3.1 is completed.

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