INFINITELY MANY SOLUTIONS FOR A CLASS OF SUBLINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this paper, we deal with the existence of infinitely many solutions for a class of sublinear Schrödinger equation

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Under the assumptions that $\inf_{\mathbb{R}^N} V(x) > 0$ and $f(x,t)$ is indefinite sign and sublinear as $|t| \to +\infty$, we establish some existence criteria to guarantee that the above problem has at least one or infinitely many nontrival solutions by using the genus properties in critical point theory.

1. INTRODUCTION

Consider the following semilinear Schrödinger equation

$$(1.1) \qquad \begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$.

In the past several decades, the existence and multiplicity of nontrivial solutions for problem (1.1) have been extensively investigated in the literature with the aid of critical point theory and variational methods. Many papers deal with the autonomous case where the potential V and the nonlinearity f are independent of x, or with the radially symmetric case where V and f depend on $|x|$, see for instance [1, 2, 5, 12, 18, 19] and the references therein. If the radial symmetry is lost, the problem becomes very different because of the lack of compactness. Ever since the work of Ding and Ni [7],

Li [11] and Rabinowitz [16], this situation has been treated in a great number of papers under various growth conditions on V and f. When the nonlinearity f is superlinear as $|t| \to \infty$, the so-called Ambrosetti-Rabinowitz superquadratic condition is usually assumed, see [1, 2, 5, 7, 11, 12, 16, 18, 19]. Weaker superlinear conditions on f were obtained in [3, 21, 22, 23, 24, 27] and the references therein.

Compared to the superlinear case, as far as the authors are aware, there are few papers [6, 8-10, 14, 26] concerning the case where f is sublinear as $|t| \to \infty$.

In 1992, Brezis and Kamin [6] gave a sufficient and necessary condition for the existence of bounded positive solutions of problem (1.1) with $V(x) = 0$. With a strong coercive condition on the potential $V(x)$, Ding and Li [8] (1994) and Ding [9] (1997) proved the existence and multiplicity of nontrivial solutions for a class of sublinear elliptic systems corresponding to (1.1). In 2007, Kristály [10] considered (1.1) with parameter λ in $f(x, u)$ and proved the finite multiplicity of the solutions for some uncertain values of λ. This result was improved in the recent paper [14].

When $f(x, t) = \mu a(x)|t|^{\mu-2}t$, where $\mu \in (1, 2)$ is a constant and $a : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function such that $a \in L^{2/(2-\mu)}(\mathbb{R}^N, [0, +\infty))$, by using variant fountain theorem [27], Zhang and Wang [26] established the following theorem on the existence of infinitely many nontrivial solutions of problem (1.1) under the assumptions that V satisfies some weaker conditions than those in [3], which have been given in [4].

Theorem 1.1. ([26]). *Assume that V and f satisfy the following conditions:*

(S1) $V \in C(\mathbb{R}^N, \mathbb{R})$ *and* $\inf_{\mathbb{R}^N} V(x) = \beta > 0$;

(S2) *There exists a constant $l_0 > 0$ such that*

$$\lim_{|y| \to +\infty} \text{meas}\left\{ x \in \mathbb{R}^N : |x - y| \leq l_0, \ V(x) \leq M \right\} = 0, \quad \forall M > 0,$$

where $\text{meas}(\cdot)$ *denotes the Lebesgue measure in $\mathbb{R}^N$;*

(S3) $f(x, t) = \mu a(x)|t|^{\mu-2}t$, *where* $\mu \in (1, 2)$ *is a constant and* $a : \mathbb{R}^N \to \mathbb{R}$ *is a positive continuous function such that* $a \in L^{2/(2-\mu)}(\mathbb{R}^N, [0, +\infty))$.

Then problem (1.1) *possesses infinitely many nontrivial solutions.*

In the above theorem, assumption (S2) is a coercive condition on L, and (S3) is a strict restriction on f. There are much sublinear functions in mathematical physics in problem like (1.1) except for $f(x, t) = \mu a(x)|t|^{\mu-2}t$ in (S3). In the present paper, motivated by paper [20, 26], we will use the genus properties in critical point theory to generalize Theorem 1.1 by removing assumption (S2) and relaxing assumption (S3).

Theorem 1.2. *Assume that V and f satisfy* (S1) *and the following conditions:*

(F1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ *and there exist two constants* $1 < \gamma_1 < \gamma_2 < 2$ *and two functions* $a_1 \in L^{2/(2-\gamma_1)}(\mathbb{R}^N, [0, +\infty))$ *and* $a_2 \in L^{2/(2-\gamma_2)}(\mathbb{R}^N, [0, +\infty))$ *such that*

$$|f(x,t)| \leq \gamma_1 a_1(x)|t|^{\gamma_1-1} + \gamma_2 a_2(x)|t|^{\gamma_2-1}, \quad \forall \ (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(F2) *There exist an open set* $J \subset \mathbb{R}^N$ *and three constants* $\delta, \eta > 0$ *and* $\gamma_3 \in (1,2)$ *such that*

$$F(x,t) \geq \eta|t|^{\gamma_3}, \quad \forall \ (x,t) \in J \times [-\delta, \delta],$$

where $F(x,t) := \int_0^t f(x,s)\mathrm{d}s, \ x \in \mathbb{R}^N, \ t \in \mathbb{R}.$

Then problem (1.1) *possesses at least one nontrivial solution.*

Theorem 1.3. *Assume that* V *and* f *satisfy* (S1), (F1), (F2) *and the following condition:*

(F3) $f(x,-t) = -f(x,t), \quad \forall \ (x,t) \in \mathbb{R}^N \times \mathbb{R}.$

Then problem (1.1) *possesses infinitely many nontrivial solutions.*

It is easy to see that assumption (F2) is satisfied if the following condition holds:

(F2′) *There exist an open set* $J \subset \mathbb{R}^N$ *and three constants* $\delta, \eta > 0$ *and* $\gamma_3 \in (1,2)$ *such that*

$$tf(x,t) \geq \gamma_3 \eta|t|^{\gamma_3}, \quad \forall \ (x,t) \in J \times [-\delta, \delta].$$

Hence, by Theorems 1.1 and 1.2, we have the following corollary.

Corollary 1.4. *In Theorems* 1.2 *and* 1.3*, if assumption* (F2) *is replaced by* (F2′), *then the conclusions still hold.*

Remark 1.5. If $f(x,t) = \mu a(x)|t|^{\mu-2}t$, then $F(x,t) = a(x)|t|^{\mu}$. Hence, assumption (S3) implies that (F1), (F2) and (F3) with $\gamma_1 = \gamma_3 = \mu < \gamma_2 < 2$, $a_1(x) = a(x)$ and $a_2(x) = 0$.

Remark 1.6. Our results can be applied to the following indefinite sign sublinear functions:

$$(1.2) \qquad f(x,t) = \frac{1 + \sin^2 x_1}{1 + |x|^{N/2}} \left(|t|^{-3/4}t - 3|t|^{-1/2}t\right)$$

and

$$(1.3) \qquad f(x,t) = \frac{4\cos x_1}{3\left(1 + |x|^{N/2}\right)}|t|^{-2/3}t + \frac{3\sin x_2}{2\left(1 + |x|^{N/3}\right)}|t|^{-1/2}t,$$

where $x = (x_1, x_2, \cdots, x_N)$. See Examples 4.1 and 4.2 in Section 4.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. The proofs of our main results are given in Section 3. Some examples to illustrate our results are given in Section 4.

2. PRELIMINARIES

Let

$$E = \left\{ u \in H^1(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} \left[|\nabla u|^2 + V(x)u^2 \right] \mathrm{d}x < +\infty \right\}$$

equipped with the norm

$$\|u\| = \left\{ \int_{\mathbb{R}^N} \left[|\nabla u|^2 + V(x)u^2 \right] \mathrm{d}x \right\}^{1/2}, \quad u \in E,$$

and the inner product

$$(u, v) = \int_{\mathbb{R}^N} \left[(\nabla u, \nabla v) + V(x)uv \right] \mathrm{d}x, \quad u, v \in E.$$

Then E is a Hilbert space with this inner product. As usual, for $1 \leq p < +\infty$, we let

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p \mathrm{d}x \right)^{1/p}, \quad u \in L^p(\mathbb{R}^N),$$

and

$$\|u\|_\infty = \mathrm{ess} \sup_{x \in \mathbb{R}^N} |u|, \quad u \in L^\infty(\mathbb{R}^N).$$

Lemma 2.1. *Assume that* (S1) *and* (F1) *hold. Then the functional* $\varphi : E \to \mathbb{R}$ *defined by*

$$(2.1) \qquad \varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u)\mathrm{d}x$$

is well defined and of class $C^1(E, \mathbb{R})$ *and*

$$(2.2) \qquad \langle \varphi'(u), v \rangle = (u, v) - \int_{\mathbb{R}^N} f(x, u) \, v(x)\mathrm{d}x.$$

Furthermore, the critical points of φ *in* E *are solutions of problem* (1.1).

Proof. By virtue of (F1), one has

$$(2.3) \qquad |F(x, t)| \leq a_1(x)|t|^{\gamma_1} + a_2(x)|t|^{\gamma_2}, \quad \forall \ (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

For any $u \in E$, it follows from (S1), (2.3) and the Hölder inequality that

$$\int_{\mathbb{R}^N} |F(x, u)| \mathrm{d}x$$

$$\leq \int_{\mathbb{R}^N} [a_1(x)|u|^{\gamma_1} + a_2(x)|u|^{\gamma_2}] \, \mathrm{d}x$$

$$\leq \sum_{i=1}^{2} \beta^{-\gamma_i/2} \left(\int_{\mathbb{R}^N} |a_i(x)|^{2/(2-\gamma_i)} \mathrm{d}x \right)^{(2-\gamma_i)/2} \left(\int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x \right)^{\gamma_i/2}$$

$$\leq \sum_{i=1}^{2} \beta^{-\gamma_i/2} \|a_i\|_{2/(2-\gamma_i)} \|u\|^{\gamma_i},$$

and so φ defined by (2.1) is well defined on E.

Next, we prove that (2.2) holds. For any function $\theta : \mathbb{R}^N \to (0, 1)$, by (F1) and the Hölder inequality, we have

$$\int_{\mathbb{R}^N} \max_{h \in [0,1]} |f(x, u + \theta(x)hv)v| \mathrm{d}x$$

$$\leq \int_{\mathbb{R}^N} \max_{h \in [0,1]} |f(x, u + \theta(x)hv)||v| \mathrm{d}x$$

$$\leq \sum_{i=1}^{2} \gamma_i \int_{\mathbb{R}^N} a_i(x)(|u| + |v|)^{\gamma_i - 1} |v| \mathrm{d}x$$

$$\leq \sum_{i=1}^{2} \gamma_i \int_{\mathbb{R}^N} a_i(x) \left(|u|^{\gamma_i - 1} + |v|^{\gamma_i - 1} \right) |v| \mathrm{d}x$$

(2.4)

$$\leq \sum_{i=1}^{2} \gamma_i \beta^{-\gamma_i/2} \left(\int_{\mathbb{R}^N} |a_i(x)|^{2/(2-\gamma_i)} \mathrm{d}x \right)^{\frac{2-\gamma_i}{2}}$$

$$\left(\int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x \right)^{\frac{\gamma_i - 1}{2}} \left(\int_{\mathbb{R}^N} V(x)v^2 \mathrm{d}x \right)^{\frac{1}{2}}$$

$$+ \sum_{i=1}^{2} \gamma_i \beta^{-\gamma_i/2} \left(\int_{\mathbb{R}^N} |a_i(x)|^{2/(2-\gamma_i)} \mathrm{d}x \right)^{\frac{2-\gamma_i}{2}} \left(\int_{\mathbb{R}^N} V(x)v^2 \mathrm{d}x \right)^{\frac{\gamma_i}{2}}$$

$$\leq \sum_{i=1}^{2} \gamma_i \beta^{-\gamma_i/2} \|a_i\|_{2/(2-\gamma_i)} \left(\|u\|^{\gamma_i - 1} + \|v\|^{\gamma_i - 1} \right) \|v\| < +\infty.$$

Then by (2.1), (2.4) and Lebesgue's Dominated Convergence Theorem, we have

$$
\begin{aligned}
\langle \varphi'(u), v \rangle &= \lim_{h \to 0^+} \frac{\varphi(u + hv) - \varphi(u)}{h} \\
&= \lim_{h \to 0^+} \frac{1}{h} \left\{ \frac{\|u + hv\|^2 - \|u\|^2}{2} - \int_{\mathbb{R}^N} [F(x, u + hv) - F(x, u)] \mathrm{d}x \right\} \\
&= \lim_{h \to 0^+} \left[(u, v) + \frac{h\|v\|^2}{2} - \int_{\mathbb{R}^N} f(x, u + \theta(x)hv)v \mathrm{d}x \right] \\
&= (u, v) - \int_{\mathbb{R}^N} f(x, u)v \mathrm{d}x.
\end{aligned}
$$

This shows that (2.2) holds. Furthermore, by a standard argument, it is easy to show that the critical points of φ in E are solutions of problem (1.1).

Let's prove now that φ' is continuous. Let $u_k \to u$ in E. Then $u_k \to u$ in $L^2(\mathbb{R}^N)$, and so

$$
(2.5) \qquad \lim_{k \to \infty} u_k(x) = u(x), \quad a.e.\ x \in \mathbb{R}^N.
$$

We claim that

$$
(2.6) \qquad \lim_{k \to \infty} \int_{\mathbb{R}^N} |f(x, u_k) - f(x, u)|^2 \mathrm{d}x = 0.
$$

If (2.6) is not true, then there exist a constant $\varepsilon_0 > 0$ and a subsequence $\{u_{k_i}\}$ such that

$$
(2.7) \qquad \int_{\mathbb{R}^N} |f(x, u_{k_i}) - f(x, u)|^2 \mathrm{d}x \geq \varepsilon_0, \quad \forall\, i \in \mathbb{N}.
$$

Since $u_k \to u$ in $L^2(\mathbb{R}^N)$, passing to a subsequence if necessary, it can be assumed that $\sum_{i=1}^{\infty} \|u_{k_i} - u\|_2^2 < +\infty$. Set $w(x) = \left[\sum_{i=1}^{\infty} |u_{k_i}(x) - u(x)|^2 \right]^{1/2}$, $x \in \mathbb{R}^N$. Then $w \in L^2(\mathbb{R}^N)$. Note that

$$
\begin{aligned}
&|f(x, u_{k_i}) - f(x, u)|^2 \\
&\leq 2|f(x, u_{k_i})|^2 + 2|f(x, u)|^2 \\
&\leq 4\gamma_1^2 |a_1(x)|^2 \left[|u_{k_i}|^{2(\gamma_1 - 1)} + |u|^{2(\gamma_1 - 1)} \right] \\
&\quad + 4\gamma_2^2 |a_2(x)|^2 \left[|u_{k_i}|^{2(\gamma_2 - 1)} + |u|^{2(\gamma_2 - 1)} \right] \\
&\leq \sum_{j=1}^{2} \left(4^{\gamma_j} + 4 \right) \gamma_j^2 |a_j(x)|^2 \left[|u_{k_i} - u|^{2(\gamma_j - 1)} + |u|^{2(\gamma_j - 1)} \right] \\
&\leq \sum_{j=1}^{2} \left(4^{\gamma_j} + 4 \right) \gamma_j^2 |a_j(x)|^2 \left[|w|^{2(\gamma_j - 1)} + |u|^{2(\gamma_j - 1)} \right] \\
&:= g(x), \quad \forall\, i \in \mathbb{N}, \ x \in \mathbb{R}^N
\end{aligned}
$$

(2.8)

and

$$\int_{\mathbb{R}^N} g(x)\mathrm{d}x$$

$$(2.9) \qquad = \sum_{j=1}^{2} (4^{\gamma_j} + 4)\,\gamma_j^2 \int_{\mathbb{R}^N} |a_j(x)|^2 \left[|w|^{2(\gamma_j-1)} + |u|^{2(\gamma_j-1)}\right]\mathrm{d}x$$

$$\leq \sum_{j=1}^{2} (4^{\gamma_j} + 4)\,\gamma_j^2\|a_j\|_{2/(2-\gamma_j)}^2 \left(\|w\|_2^{2(\gamma_j-1)} + \|u\|_2^{2(\gamma_j-1)}\right) < +\infty.$$

Then by (2.5), (2.8), (2.9) and Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{i\to\infty} \int_{\mathbb{R}^N} |f(x, u_{k_i}) - f(x, u)|^2 \mathrm{d}x = 0,$$

which contradicts (2.7). Hence (2.6) holds. From (2.2), (2.6) and the Hölder inequality, we have

$$|\langle \varphi'(u_k) - \varphi'(u), v\rangle|$$

$$= \left|(u_k - u, v) - \int_{\mathbb{R}^N} [f(x, u_k) - f(x, u)]v\mathrm{d}x\right|$$

$$\leq \|u_k - u\|\,\|v\| + \int_{\mathbb{R}^N} |f(x, u_k) - f(x, u)|\,|v|\mathrm{d}x$$

$$\leq \|u_k - u\|\,\|v\| + \beta^{-1/2} \left(\int_{\mathbb{R}^N} |f(x, u_k) - f(x, u)|^2\mathrm{d}x\right)^{1/2}\|v\|$$

$$\leq o(1), \quad k \to +\infty,$$

which implies the continuity of φ'. The proof is complete. ∎

Lemma 2.2. ([13]). *Let X be a real Banach space and $\psi \in C^1(X, \mathbb{R})$ satisfy the (PS)-condition. If ψ is bounded from below, then $c = \inf_X \psi$ is a critical value of ψ.*

To order to find nontrivial critical points of ψ, we will use the "genus" properties, so we recall the following definitions and results (see [15] and [17]).

Let X be a Banach space, $\psi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We set

$$\Sigma = \{A \subset X - \{0\} : A \text{ is closed in } X \text{ and symmertric with respect to } 0\},$$

$$K_c = \{u \in X : \psi(u) = c,\ \psi'(u) = 0\}, \quad \psi^c = \{u \in X : \psi(u) \leq c\}.$$

Definition 2.3. ([15]). For $A \in \Sigma$, we say genus of A is n (denoted by $\gamma(A) = n$) if there is an odd map $\phi \in C(A, \mathbb{R}^n \setminus \{0\})$ and n is the smallest integer with this property.

Lemma 2.4. ([15]). *Let ψ be an even C^1 functional on X and satisfy the (PS)-condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma : \gamma(A) \geq n\}, \quad c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \psi(u).$$

(i) *If $\Sigma_n \neq \emptyset$ and $c_n \in \mathbb{R}$, then c_n is a critical value of ψ;*

(ii) *If there exists $r \in \mathbb{N}$ such that*

$$c_n = c_{n+1} = \cdots = c_{n+r} = c \in \mathbb{R},$$

and $c \neq \psi(0)$, then $\gamma(K_c) \geq r + 1$.

3. Proofs of Theorems

Proof of Theorem 1.2. In view of Lemma 2.1, $\varphi \in C^1(E, \mathbb{R})$. In what follows, we first show that φ is bounded from below. By (2.1), (2.3) and the Hölder inequality, we have

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u)\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} a_1(x)|u|^{\gamma_1}\mathrm{d}x - \int_{\mathbb{R}^N} a_2(x)|u|^{\gamma_2}\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|^2 - \sum_{i=1}^{2} \beta^{-\gamma_i/2} \left(\int_{\mathbb{R}^N} |a_i(x)|^{2/(2-\gamma_i)}\mathrm{d}x \right)^{(2-\gamma_i)/2} \\
&\quad \left(\int_{\mathbb{R}^N} V(x)u^2\mathrm{d}x \right)^{\gamma_i/2} \\
&\geq \frac{1}{2}\|u\|^2 - \sum_{i=1}^{2} \beta^{-\gamma_i/2} \|a_i\|_{2/(2-\gamma_i)} \|u\|^{\gamma_i}.
\end{aligned}
$$

(3.1)

Since $1 < \gamma_1 < \gamma_2 < 2$, (3.1) implies that $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$. Consequently, φ is bounded from below.

Next, we prove that φ satisfies the (PS)-condition. Assume that $\{u_k\}_{k \in \mathbb{N}} \subset E$ is a sequence such that $\{\varphi(u_k)\}_{k \in \mathbb{N}}$ is bounded and $\varphi'(u_k) \to 0$ as $k \to +\infty$. Then by (3.1), there exists a constant $A > 0$ such that

$$(3.2) \qquad \|u_k\|_2 \leq \beta^{-1/2}\|u_k\| \leq A, \quad k \in \mathbb{N}.$$

So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in E. For any given number $\varepsilon > 0$, by (F1), we can choose $R_\varepsilon > 0$ such that

$$(3.3) \qquad \left(\int_{|x|>R_\varepsilon} |a_i(x)|^{2/(2-\gamma_i)}\mathrm{d}x \right)^{(2-\gamma_i)/2} < \varepsilon, \quad i = 1, 2.$$

We claim that

$$(3.4) \qquad \lim_{k \to \infty} \int_{|x| \le R_\varepsilon} |u_k - u_0|^2 \mathrm{d}x = 0.$$

In fact, if (3.4) is not true, then there exist a constant $\varepsilon_0 > 0$ and a subsequence $\{u_{k_i}\}$ such that

$$(3.5) \qquad \int_{|x| \le R_\varepsilon} |u_{k_i} - u_0|^2 \mathrm{d}x \ge \varepsilon_0, \quad \forall\, i \in \mathbb{N}.$$

$\{u_{k_i}\}$ has a convergent subsequence in $L^2(\bar{B}_{R_\varepsilon})$, which is still denoted by $\{u_{k_i}\}$ for the sake of simplicity. Let $\{u_{k_i}\}$ converge to u_* in $L^2(\bar{B}_{R_\varepsilon})$, i.e.

$$(3.6) \qquad \lim_{i \to \infty} \int_{|x| \le R_\varepsilon} |u_{k_i} - u_*|^2 \mathrm{d}x = 0.$$

Since $u_{k_i} \rightharpoonup u_0$ in $E \subset L^2(\bar{B}_{R_\varepsilon})$, it follows that $u_{k_i} \rightharpoonup u_0$ in $L^2(\bar{B}_{R_\varepsilon})$, which, together with (3.6), implies that $u_*(x) = u_0(x)$, a.e. $x \in \bar{B}_{R_\varepsilon}$. Hence,

$$\lim_{i \to \infty} \int_{|x| \le R_\varepsilon} |u_{k_i} - u_0|^2 \mathrm{d}x = 0,$$

which contradicts (3.5), and so (3.4) holds. By (3.4), there exists $k_0 \in \mathbb{N}$ such that

$$(3.7) \qquad \int_{|x| \le R_\varepsilon} |u_k - u_0|^2 \mathrm{d}x < \varepsilon^2 \quad \text{for } k \ge k_0.$$

Hence, by (F1), (3.2), (3.7) and the Hölder inequality, we have

$$
\begin{aligned}
&\int_{|x| \le R_\varepsilon} |f(x, u_k) - f(x, u_0)||u_k - u_0|\mathrm{d}x \\
&\le \left(\int_{|x| \le R_\varepsilon} |f(x, u_k) - f(x, u_0)|^2 \mathrm{d}x \right)^{1/2} \left(\int_{|x| \le R_\varepsilon} |u_k - u_0|^2 \mathrm{d}x \right)^{1/2} \\
&\le \left[\int_{|x| \le R_\varepsilon} 2 \left(|f(x, u_k)|^2 + |f(x, u_0)|^2 \right) \mathrm{d}x \right]^{1/2} \varepsilon \\
(3.8) \quad &\le 2 \left[\sum_{i=1}^{2} \gamma_i^2 \int_{|x| \le R_\varepsilon} |a_i|^2 \left(|u_k|^{2(\gamma_i - 1)} + |u_0|^{2(\gamma_i - 1)} \right) \mathrm{d}x \right]^{1/2} \varepsilon \\
&\le 2 \left[\sum_{i=1}^{2} \gamma_i^2 \|a_i\|_{2/(2-\gamma_i)}^2 \left(\|u_k\|_2^{2(\gamma_i - 1)} + \|u_0\|_2^{2(\gamma_i - 1)} \right) \right]^{1/2} \varepsilon \\
&\le 2 \left[\sum_{i=1}^{2} \gamma_i^2 \|a_i\|_{2/(2-\gamma_i)}^2 \left(A^{2(\gamma_i - 1)} + \|u_0\|_2^{2(\gamma_i - 1)} \right) \right]^{1/2} \varepsilon, \quad k \ge k_0.
\end{aligned}
$$

On the other hand, it follows from (F1), (3.2), (3.3) and the Hölder inequality that

$$\int_{|x|>R_\varepsilon} |f(x,u_k) - f(x,u_0)||u_k - u_0|\mathrm{d}x$$

$$\leq \sum_{i=1}^{2} \gamma_i \int_{|x|>R_\varepsilon} |a_i(x)|(|u_k|^{\gamma_i-1} + |u_0|^{\gamma_i-1})(|u_k| + |u_0|)\mathrm{d}x$$

$$\leq 2\sum_{i=1}^{2} \gamma_i \int_{|x|>R_\varepsilon} |a_i(x)|(|u_k|^{\gamma_i} + |u_0|^{\gamma_i})\mathrm{d}x$$

(3.9)

$$\leq 2\sum_{i=1}^{2} \gamma_i \left(\int_{|x|>R_\varepsilon} |a_i(x)|^{2/(2-\gamma_i)}\mathrm{d}x\right)^{(2-\gamma_i)/2} (\|u_k\|_2^{\gamma_i} + \|u_0\|_2^{\gamma_i})$$

$$\leq 2\sum_{i=1}^{2} \gamma_i \left(\int_{|x|>R_\varepsilon} |a_i(x)|^{2/(2-\gamma_i)}\mathrm{d}x\right)^{(2-\gamma_i)/2} (A^{\gamma_i} + \|u_0\|_2^{\gamma_i})$$

$$\leq 2\sum_{i=1}^{2} \gamma_i \left(A^{\gamma_i} + \|u_0\|_2^{\gamma_i}\right)\varepsilon, \quad k \in \mathbb{N}.$$

Since ε is arbitrary, combining (3.8) with (3.9), one gets

$$(3.10) \qquad \int_{\mathbb{R}^N} [f(x,u_k) - f(x,u_0)][u_k - u_0]\mathrm{d}x \to 0 \quad \text{as} \ \ k \to \infty.$$

It follows from (2.2) that

$$(3.11) \ \ \langle \varphi'(u_k) - \varphi'(u_0), u_k - u_0 \rangle = \|u_k - u_0\|^2 - \int_{\mathbb{R}^N} [f(x,u_k) - f(x,u_0)][u_k - u_0]\mathrm{d}x.$$

Since $\langle \varphi'(u_k) - \varphi'(u_0), u_k - u_0 \rangle \to 0$, it follows from (3.10) and (3.11) that $u_k \to u_0$ in E. Hence, φ satisfies the (PS)-condition.

By Lemma 2.2, $c = \inf_E \varphi(u)$ is a critical value of φ, that is there exists a critical point $u^* \in E$ such that $\varphi(u^*) = c$.

Finally, we show that $u^* \neq 0$. Let $u_0 \in \left(W_0^{1,2}(J) \bigcap E\right) \setminus \{0\}$ and $\|u_0\|_\infty \leq 1$, then by (2.1) and (F2), we have

$$\varphi(su_0) = \frac{s^2}{2}\|u_0\|^2 - \int_{\mathbb{R}^N} F(x,su_0)\mathrm{d}x$$

(3.12)

$$= \frac{s^2}{2}\|u_0\|^2 - \int_J F(x,su_0)\mathrm{d}x$$

$$\leq \frac{s^2}{2}\|u_0\|^2 - \eta s^{\gamma_3} \int_J |u_0|^{\gamma_3}\mathrm{d}x, \quad 0 < s < \delta.$$

Since $1 < \gamma_3 < 2$, it follows from (3.12) that $\varphi(su_0) < 0$ for $s > 0$ small enough. Hence $\varphi(u^*) = c < 0$, therefore u^* is a nontrivial critical point of φ, and so u^* is a nontrivial solution of problem (1.1). The proof is complete. ∎

Proof of Theorem 1.3. In view of Lemma 2.1 and the proof of Theorem 1.2, $\varphi \in C^1(E, \mathbb{R})$ is bounded from below and satisfies the (PS)-condition. By (F3), it is obvious that φ is even and $\varphi(0) = 0$. In order to apply Lemma 2.4, we prove now that

(3.13) $\qquad$ for any $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\gamma(\varphi^{-\varepsilon}) \geq n$.

For any $n \in \mathbb{N}$, we take n disjoint open sets J_i such that

$$\bigcup_{i=1}^{n} J_i \subset J.$$

For $i = 1, 2, \ldots, n$, let $u_i \in \left(W_0^{1,2}(J_i) \bigcap E\right) \setminus \{0\}$, $\|u_i\|_{\infty} < +\infty$ and $\|u_i\| = 1$, and

$$E_n = \operatorname{span}\{u_1, u_2, \cdots, u_n\}, \qquad S_n = \{u \in E_n : \|u\| = 1\}.$$

For any $u \in E_n$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ such that

(3.14) $$u = \sum_{i=1}^{n} \lambda_i u_i \quad \text{for} \ x \in \mathbb{R}^N.$$

Then

(3.15) $$\|u\|_{\gamma_3} = \left(\int_{\mathbb{R}^N} |u|^{\gamma_3} \mathrm{d}x\right)^{1/\gamma_3} = \left(\sum_{i=1}^{n} |\lambda_i|^{\gamma_3} \int_{J_i} |u_i|^{\gamma_3} \mathrm{d}x\right)^{1/\gamma_3},$$

and

(3.16)
$$\|u\|^2 = \int_{\mathbb{R}^N} \left[|\nabla u|^2 + V(x)u^2\right] \mathrm{d}x = \sum_{i=1}^{n} \lambda_i^2 \int_{J_i} \left[|\nabla u_i|^2 + V(x)u_i^2\right] \mathrm{d}x$$
$$= \sum_{i=1}^{n} \lambda_i^2 \int_{\mathbb{R}^N} \left[|\nabla u_i|^2 + V(x)u_i^2\right] \mathrm{d}x = \sum_{i=1}^{n} \lambda_i^2 \|u_i\|^2 = \sum_{i=1}^{n} \lambda_i^2.$$

Since all norms of a finite dimensional normed space are equivalent, so there is a constant $c' > 0$ such that

(3.17) $$c'\|u\| \leq \|u\|_{\gamma_3} \quad \text{for} \ u \in E_n.$$

By (F3), (2.1), (3.14), (3.15), (3.16) and (3.17), we have

$$
\begin{aligned}
\varphi(su) &= \frac{s^2}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, su)\mathrm{d}x = \frac{s^2}{2}\|u\|^2 - \sum_{i=1}^{n} \int_{J_i} F(x, s\lambda_i u_i)\mathrm{d}x \\
&\leq \frac{s^2}{2}\|u\|^2 - \eta s^{\gamma_3} \sum_{i=1}^{n} |\lambda_i|^{\gamma_3} \int_{J_i} |u_i|^{\gamma_3}\mathrm{d}x = \frac{s^2}{2}\|u\|^2 - \eta s^{\gamma_3}\|u\|_{\gamma_3}^{\gamma_3} \\
&\leq \frac{s^2}{2}\|u\|^2 - \eta (c's)^{\gamma_3}\|u\|^{\gamma_3} \\
&= \frac{s^2}{2} - \eta(c's)^{\gamma_3}, \quad \forall\, u \in S_n, \ 0 < s \leq \delta \left(\max_{1 \leq i \leq n} \|u_i\|_\infty \right)^{-1}.
\end{aligned}
$$

(3.18)

(3.18) implies that there exist $\varepsilon > 0$ and $\sigma > 0$ such that

$$
\varphi(\sigma u) < -\varepsilon \quad \text{for}\ \ u \in S_n. \tag{3.19}
$$

Let

$$
S_n^\sigma = \{\sigma u : u \in S_n\}, \quad \Omega = \left\{ (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i^2 < \sigma^2 \right\}.
$$

Then it follows from (3.19) that

$$
\varphi(u) < -\varepsilon \quad \text{for}\ \ u \in S_n^\sigma,
$$

which, together with the fact that $\varphi \in C^1(E, \mathbb{R})$ and is even, implies that

$$
S_n^\sigma \subset \varphi^{-\varepsilon} \in \Sigma. \tag{3.20}
$$

On the other hand, it follows from (3.14) and (3.16) that there exists an odd homeomorphism mapping $\phi \in C(S_n^\sigma, \partial\Omega)$. By some properties of the genus (see 3° of Proposition 7.5 and Proposition 7.7 in [15]), we have

$$
\gamma(\varphi^{-\varepsilon}) \geq \gamma(S_n^\sigma) = n, \tag{3.21}
$$

so the proof of (3.13) follows. Set

$$
c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \varphi(u).
$$

It follows from (3.21) and the fact that φ is bounded from below on E that $-\infty < c_n \leq -\varepsilon < 0$, that is for any $n \in \mathbb{N}$, c_n is a real negative number. By Lemma 2.4, φ has infinitely many nontrivial critical points, and so problem (1.1) possesses infinitely many nontrivial solutions. The proof is complete. ∎

4. EXAMPLES

Example 4.1. In problem (1.1), let $V(x) = 1 + \sin^2 x_1$, and let $f(x, t)$ be as in (1.2). Then

$$|f(x,t)| \leq \frac{2}{1+|x|^{N/2}} \left(|t|^{1/4} + 3|t|^{1/2} \right), \quad \forall \ (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

$$F(x,t) = \frac{1+\sin^2 x_1}{1+|x|^{N/2}} \left(\frac{4}{5}|t|^{5/4} - 2|t|^{3/2} \right)$$

$$\geq \frac{3^{(N+2)/2}}{10\left(3^{N/2} + \pi^{N/2}\right)} |t|^{5/4}, \quad \forall \ (x,t) \in B_{\pi/3} \times \left[-4^{-4}, 4^{-4}\right].$$

Thus all conditions of Theorem 1.3 are satisfied with

$$\frac{5}{4} = \gamma_1 = \gamma_3 < \gamma_2 = \frac{3}{2}, \quad a_1(x) = \frac{2}{1+|x|^{N/2}}, \quad a_2(x) = \frac{6}{1+|x|^{N/2}},$$

$$\delta = \frac{1}{4^4}, \quad \eta = \frac{3^{(N+2)/2}}{10\left(3^{N/2} + \pi^{N/2}\right)}, \quad J = B_{\pi/3}.$$

By Theorem 1.3, problem (1.1) has infinitely many nontrivial solutions.

Example 4.2. In problem (1.1), let $V(x) = 1 + \cos^2 x_1$, and let $f(x, t)$ be as in (1.3). Then

$$|f(x,t)| \leq \frac{4}{3\left(1+|x|^{N/2}\right)}|t|^{1/3} + \frac{3}{2\left(1+|x|^{N/3}\right)}|t|^{1/2}, \quad \forall \ (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

$$F(x,t) = \frac{\cos x_1}{1+|x|^{N/2}}|t|^{4/3} + \frac{\sin x_2}{1+|x|^{N/3}}|t|^{3/2}$$

$$\geq \frac{1}{2\left(1+N^{N/4}\right)}|t|^{4/3}, \quad \forall \ (x,t) \in (-1,1)^N \times [-1,1].$$

Thus all conditions of Theorem 1.3 are satisfied with

$$\frac{4}{3} = \gamma_1 = \gamma_3 < \gamma_2 = \frac{3}{2}, \quad a_1(x) = \frac{4}{3\left(1+|x|^{N/2}\right)}, \quad a_2(x) = \frac{3}{2\left(1+|x|^{N/3}\right)},$$

$$\delta = 1, \quad \eta = \frac{1}{2\left(1+N^{N/4}\right)}, \quad J = (-1,1)^N.$$

By Theorem 1.3, problem (1.1) has infinitely many nontrivial solutions.

Example 4.3. In problem (1.1), let $V(x) = \ln(3 + |x_1|)$, and let

$$f(x,t) = \frac{3 \sin x_2}{2 \left(1 + |x|^{N/3}\right) \left(1 + |t|^{3/2}\right)} |t|^{-1/2} z.$$

Then

$$|f(x,t)| \leq \frac{3}{2 \left(1 + |x|^{N/3}\right)} |t|^{1/2}, \quad \forall \ (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

$$F(x,t) = \frac{\sin x_2}{1 + |x|^{N/3}} \ln \left(1 + |t|^{3/2}\right)$$

$$\geq \frac{1}{4 \left[1 + (4N)^{N/6}\right]} |t|^{3/2}, \quad \forall \ (x,t) \in (1,2)^N \times [-1,1].$$

These show that all conditions of Theorem 1.2 are satisfied, where

$$\gamma_1 = \gamma_3 = \frac{3}{2} < \gamma_2 < 2, \quad a_1(x) = \frac{3}{2 \left(1 + |x|^{N/3}\right)}, \quad a_2(x) = 0,$$

$$\delta = 1, \quad \eta = \frac{1}{4 \left[1 + (4N)^{N/6}\right]}, \quad J = (1,2)^N.$$

By Theorem 1.3, problem (1.1) has infinitely many nontrivial solutions.

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