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EXISTENCE, UNIQUENESS AND STABILITY OF PERIODIC SOLUTIONS OF A DUFFING EQUATION UNDER PERIODIC AND ANTI-PERIODIC EIGENVALUES CONDITIONS

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Abstract. Using periodic and anti-periodic eigenvalues, we present new criteria for guaranteeing the existence, uniqueness and asymptotic stability (in the sense of Lyapunov) of periodic solutions of a Duffing equation under conditions which are weaker than those used in the literature. The proof is based on the application of the existence theorem of Leray-Schauder type, Floquet theory, Lyapunov stability theory and some analytic techniques.

1. Introduction

In this paper, we consider the existence, uniqueness and stability of periodic solutions for the Duffing-type equation

$$x'' + cx' + g(t,x) = h(t), \tag{1.1}$$

where $c > 0$ is fixed, h is a T-periodic function and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a T-periodic function in t. We assume that g satisfies the following semilinear condition: there exist T-periodic functions $\phi, \Phi \in L^1(0,T)$ such that

$$\phi(t) \le g_x(t,x) \le \Phi(t), \quad \text{uniformly in } t \in [0,T]. \tag{1.2}$$

During the past three decades, the existence and stability of periodic solutions of (1.1) or more general types of nonlinear second-order differential equations have

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been studied by many authors (see [1, 2, 3, 4, 5, 8, 9, 12, 17] and the references therein) since Ortega initiated the study of the stability of periodic solutions of (1.1) with $h(t) = 0$ using the relation between topological degree and stability [13, 14, 15]. Besides topological degree, the method of upper and lower solutions is also successfully used to investigate stability properties. For example, it was shown in [12] that for (1.1), the solution lying between the well ordering lower- and upper-solution is usually unstable, but the solution lying between the reversed ordering lower- and upper-solution is stable when the derivative of the restoring force g with respect to x is small or the fractional constant c is large.

Recently, Chen and Li established the following theorems in [3, 4] for problem (1.1). In particular, bounds for the derivative of the restoring force are given that ensure the existence and uniqueness of a periodic solution. Furthermore, the stability of the unique periodic solution was analyzed and the sharp rate of exponential decay was determined for a solution that is near to the unique periodic solution. Recall that we say that the periodic solution x_0 of (1.1) is *asymptotically stable* if there exist constants $C > 0$ and $\alpha > 0$ such that if x is another solution with

$$|x(0) - x_0(0)| + |x'(0) - x_0'(0)| = d$$

sufficiently small, then

$$|x(t) - x_0(t)| + |x'(t) - x_0'(t)| < Cde^{-\alpha t}.$$

The super exponent α is called the *rate of decay* of x_0.

Theorem 1.1. *Suppose that $g \in C^1(\mathbb{R} \times \mathbb{R})$ satisfies the following conditions*

$$\alpha(t) < g_x(t, x) < \frac{\pi^2}{T^2} + \frac{c^2}{4}, \tag{1.3}$$

with $\overline{\alpha} > \frac{c^2}{4}$, here $\overline{\alpha}$ denotes the average of $\alpha(t)$ over a period. Then (1.1) has a unique T-periodic solution which is asymptotically stable with the rate of decay of $\frac{c}{2}$ for $c > 0$.

Theorem 1.2. *Suppose that $g \in C^1(\mathbb{R} \times \mathbb{R})$ satisfies the following condition*

$$\frac{n^2\pi^2}{T^2} + \frac{c^2}{4} \le g_x(t, x) \le \frac{(n+1)^2\pi^2}{T^2} + \frac{c^2}{4}, \tag{1.4}$$

for some $n \ge 1$. Then (1.1) has a unique T-periodic solution which is asymptotically stable with the rate of decay of $\frac{c}{2}$ for $c > 0$.

We call (1.3), (1.4) *the first stable condition*, *nth stable condition* because the corresponding linear equation lies in the first (nth) stable intervals when (1.3), (1.4) holds. See [18]. As remarked in [3, 4] that conditions (1.3), (1.4) in Theorems 1.1 and 1.2 are optimal in the following sense: for any $\varepsilon > 0$, there are unstable differential equations (1.1) in which

$$\alpha(t) < g_x(t,x) < \frac{\pi^2}{T^2} + \frac{c^2}{4} + \varepsilon, \quad \overline{\alpha} > \frac{c^2}{4}$$

or

$$\frac{n^2\pi^2}{T^2} + \frac{c^2}{4} - \varepsilon \le g_x(t,x) \le \frac{(n+1)^2\pi^2}{T^2} + \frac{c^2}{4} + \varepsilon.$$

However, we observe that the above stable conditions have some disadvantages: (1) they have no persistence, which means that when $g_x(t,x)$ has small perturbations, (1.3) and (1.4) may be no longer satisfied, because $g_x(t,x) - \varepsilon$ and $g_x(t,x) + \varepsilon$ may not satisfy conditions (1.3) and (1.4) for $\varepsilon > 0$; (2) conditions (1.3) and (1.4) naturally imply that $g_x(t,x) \in L^\infty(0,T)$, which can not deal with L^1-Carath$\acute{e}$odory functions. For example, Theorem 1.2 cannot be applicable to the following simple equation

$$x'' + cx' + \mu(1+\cos t)x + f(t,x) = h(t), \ t \in [0, 2\pi], \tag{1.5}$$

where f satisfies $f_x(t,x) = 0$.

The purpose of this paper is to present a generalization of Theorems 1.1 and 1.2, and our new results can deal with examples such as (1.5). The tools include Leray-Schauder type existence theorem, Floquet theory, Lyapunov stability theory and some analytic techniques. In particular, periodic and antiperiodic eigenvalues of equation $x'' + (\lambda + q(t))x = 0$ play the important role in Theorems 2.3 and 2.6.

2. Preliminaries

Let X, Z be real Banach spaces, $L : \operatorname{dom} L \subset X \to Z$ be a Fredholm map of index zero, there exist continuous projectors $P : X \to X, Q : Z \to Z$ such that Im P = Ker L, Ker $Q =$ Im L, and $X = \operatorname{Ker} L \oplus \operatorname{Ker} P,\ Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible, and we denote its inverse by K_P. If Ω is an open bounded subset of X such that dom $L \cap \Omega \neq \emptyset$, then the continuous map $N : X \to Z$ will be called L-compact on $\overline{\Omega}$ when $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

In order to prove the main results of this paper, we need the following existence result of Leray-Schauder type.

Lemma 2.1. [11, Theorem IV.5] *Let $A : X \to Z$ be L-compact and such that*
(1) Ker$(L + A) = \{0\}$,
(2) *for every* $(x, \lambda) \in (\operatorname{dom} L \cap \partial\Omega) \times (0, 1)$,

$$Lx + (1-\lambda)Ax + \lambda N x \neq 0,$$

and assume that $0 \in \Omega$. *Then* $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

Now we recall some facts on eigenvalues of Hill equations. Let $q(t)$ be a T-periodic potential such that $q \in L^1(0,T)$. Consider the eigenvalue problems

$$x'' + (\lambda + q(t))x = 0 \tag{2.1}$$

subject to the periodic boundary condition

$$x(0) - x(T) = x'(0) - x'(T) = 0, \tag{2.2}$$

or to the anti-periodic boundary condition

$$x(0) + x(T) = x'(0) + x'(T) = 0. \tag{2.3}$$

We use

$$\lambda_1^D(q) < \lambda_2^D(q) < \cdots < \lambda_n^D(q) < \cdots$$

to denote all eigenvalues of (2.1) with the Dirichlet boundary condition $x(0) = x(T) = 0$.

Theorem 2.2. [10] *There exist two sequences* $\{\underline{\lambda}_n(q) : n \in \mathbb{N}\}$ *and* $\{\overline{\lambda}_n(q) : n \in \mathbb{Z}^+\}$ *of the reals such that*

(P_1) *they have the following order:*

$$-\infty < \overline{\lambda}_0(q) < \underline{\lambda}_1(q) \le \overline{\lambda}_1(q) < \cdots < \underline{\lambda}_n(q) \le \overline{\lambda}_n(q) < \cdots$$

and $\underline{\lambda}_n(q) \to +\infty,\ \overline{\lambda}_n(q) \to +\infty$ *as* $n \to \infty$.

(P_2) λ *is an eigenvalue of (2.1)-(2.2) if and only if* $\lambda = \underline{\lambda}_n(q)$ *or* $\overline{\lambda}_n(q)$ *for some even integer* n*;* λ *is an eigenvalue of (2.1)-(2.3) if and only if* $\lambda = \underline{\lambda}_n(q)$ *or* $\overline{\lambda}_n(q)$ *for some odd integer* n.

(P_3) $\lambda_n^D(q), \underline{\lambda}_n(q)$, *and* $\overline{\lambda}_n(q)$ *are continuous functions of* q *with respect to the* L^1-*metric on* q*'s:* $d(q_1,q_2) = \int_0^T |q_1(t) - q_2(t)|dt$.

(P_4) *the eigenvalues* $\underline{\lambda}_n(q)$ *and* $\overline{\lambda}_n(q)$ *can be recovered from the Dirichlet eigenvalues in the following way: for any* $n \in \mathbb{N}$,

$$\underline{\lambda}_n(q) = \min\{\lambda_n^D(q_{t_0}) : t_0 \in \mathbb{R}\},\ \overline{\lambda}_n(q) = \max\{\lambda_n^D(q_{t_0}) : t_0 \in \mathbb{R}\},$$

here $q_{t_0}(t) \equiv q(t+t_0)$ *denotes the translation of* $q(t)$.

(P_5) *the comparison results hold for all of these eigenvalues. If* $q_1 \ge q_2$ *then*

$$\underline{\lambda}_n(q_1) \le \underline{\lambda}_n(q_2),\ \overline{\lambda}_n(q_1) \le \overline{\lambda}_n(q_1),\ \lambda_n^D(q_1) \le \lambda_n^D(q_2), \tag{2.4}$$

for any $n \in \mathbb{N}$. *If* $q_1(t) \ge q_2(t)$ *for all* t, *and* $q_1(t) > q_2(t)$ *for* t *in a subset of positive measure, then all of the inequalities in (2.4) are strict.*

Our first main result reads as follows.

Theorem 2.3. *Suppose that* $g \in C^1(\mathbb{R} \times \mathbb{R})$ *satisfies the semilinearity condition (1.2). Assume further that*

$$\overline{\phi} > \frac{c^2}{4}, \qquad \underline{\lambda}_1(\Phi) + \frac{c^2}{4} > 0. \tag{2.5}$$

Then (1.1) has a unique T*-periodic solution which is asymptotically stable with the rate of decay of* $\frac{c}{2}$.

Let $\Phi^+ = \max\{\Phi, 0\}$ denote the positive part of a function Φ. It was proved in [18] that

$$\underline{\lambda}_1(\Phi) \geq \Big(\frac{\pi}{T}\Big)^2 \Big(1 - \frac{\|\Phi^+\|_p}{K(2p^*)}\Big),$$

if there exists $p \in [1, +\infty]$ such that

$$\|\Phi^+\|_p \leq K(2p^*),\ \ p^* = p/(p-1),$$

here $K(q)$ is the best Sobolev constant in the following inequality:

$$C\|x\|_q^2 \leq \|x'\|_2^2, \qquad x \in H_0^1(0,T).$$

Explicitly,

$$K(q) = \begin{cases} \dfrac{2\pi}{qT^{1+2/q}} \Big(\dfrac{2}{2+q}\Big)^{1-2/q} \Big(\dfrac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2} + \frac{1}{q})}\Big)^2, & \text{if } 1 \leq q < \infty, \\ \dfrac{4}{T}, & \text{if } q = \infty. \end{cases}$$

See Talenti [16]. Now the following result is a direct consequence of Theorem 2.3 and the above fact.

Corollary 2.4. *Suppose that* $g \in C^1(\mathbb{R} \times \mathbb{R})$ *satisfies the semilinearity condition* (1.2). *Assume further that* $\overline{\phi} > \frac{c^2}{4}$ *and there exists* $p \in [1, +\infty]$ *such that*

$$\|\Phi^+\|_p < \Big(1 + \Big(\frac{cT}{2\pi}\Big)^2\Big) K(2p^*).$$

Then (1.1) has a unique T*-periodic solution which is asymptotically stable with the rate of decay of* $\frac{c}{2}$. *In particular, when* $p = +\infty$, *we arrive at the usual criterion*

$$\|\Phi^+\|_\infty < \Big(1 + \Big(\frac{cT}{2\pi}\Big)^2\Big) K(2) = \frac{\pi^2}{T^2} + \frac{c^2}{4}, \tag{2.6}$$

which was used in $[3,\ 4]$.

Example 2.5. Consider the equation

$$x'' + cx' + a(t)\left(x - \frac{1}{2}\arctan x\right) = h(t), \tag{2.7}$$

where a, h are 2π-periodic functions. It is easy to verify that (1.2) holds with

$$\phi(t) = \frac{a(t)}{2}, \quad \Phi(t) = a(t).$$

Theorem 1.1 shows that equation (2.7) has a unique asmptotically stable 2π-periodic solution when

$$\frac{c^2}{2} < a(t) < \frac{1}{4} + \frac{c^2}{4}, \tag{2.8}$$

which can also be improved as

$$\frac{c^2}{2} < a(t) < \frac{2}{\pi} + \frac{2c^2}{\pi} \approx 0.6367 + 0.6367c^2,$$

if we apply Theorem 2.3.

The following result is our second main result, which deal with the case when n stable condition holds.

Theorem 2.6. *Suppose that $g(t, x)$ satisfies the semilinearity condition (1.2). Assume further that there exists $n \in \mathbb{N}$ such that*

$$\overline{\lambda}_n(\phi) + \frac{c^2}{4} < 0 \quad \textit{and} \quad \underline{\lambda}_{n+1}(\Phi) + \frac{c^2}{4} > 0. \tag{2.9}$$

Then (1.1) has a unique T-periodic solution which is asymptotically stable with the rate of decay of $\frac{c}{2}$ for $c > 0$.

Example 2.7. Consider the equation (1.5). Obviously, (1.5) satisfies (1.2) with $\phi(t) = \Phi(t) = \varpi(t)$, where $\varpi(t) = \mu(1 + \cos t)$. As explained in Section 1, Theorem 1.2 is not applicable to this example since $\varpi(t)$ has zeros in t. However, by Theorem 2.6, we can obtain higher asymptotically stable results if $0 \in (\overline{\lambda}_n(\varpi) + \frac{c^2}{4}, \underline{\lambda}_{n+1}(\varpi) + \frac{c^2}{4})$ for some $n \in \mathbb{N}$.

3. Proof of Main Results

In this section we present the proof of Theorem 2.3 and omit the proof of Theorem 2.6 since it can be proved by the same method.

To prove the existence of solutions of (1.1), we will apply Lemma 2.1. To do this, we consider the following equation

$$x'' + cx' + \Phi(t)x = 0. \tag{3.1}$$

Lemma 3.1. *Assume that*

$$\overline{\Phi} > \frac{c^2}{4} \quad and \quad \underline{\lambda}_1(\Phi) + \frac{c^2}{4} > 0. \tag{3.2}$$

Then problem (3.1)-(2.2) has only the trivial T-periodic solution.

Proof. On the contrary, suppose that (3.1)-(2.2) has a nontrivial T-periodic solution x. Let

$$x(t) = e^{-\frac{1}{2}ct}u(t).$$

Then u satisfies the equation

$$u''(t) + \left[\Phi(t) - \frac{c^2}{4}\right]u(t) = 0. \tag{3.3}$$

First we claim that there exists $t_0 \in [0, T]$ such that

$$u'(t_0) = 0. \tag{3.4}$$

On the contrary, suppose that (3.4) does not hold. Then x does not change sign in $\mathbb{R}$, neither does u. Dividing (3.3) by $u(t)$ and integrating it from 0 to T, and noting that

$$\frac{u'(T)}{u(T)} = \frac{u'(0)}{u(0)},$$

we have that

$$\int_0^T \frac{u'(t)^2}{u(t)^2}dt + \int_0^T \left[\Phi(t) - \frac{c^2}{4}\right]dt = 0,$$

which is impossible by the hypothesis of lemma and therefore (3.4) holds. Then u is a nontrivial solution of the following Dirichlet boundary value problem

$$\begin{cases} u'' + [\Phi(t) - \frac{c^2}{4}]u = 0, \\ u(t_0) = u(t_0 + T) = 0, \end{cases} \tag{3.5}$$

which implies that there exists an n such that $\lambda_n^D(\Phi - \frac{c^2}{4}) = 0$. By (3.2) and [$(P_5)$, Theorem 2.2], we obtain that

$$\lambda_n^D(\Phi - \frac{c^2}{4}) = \lambda_n^D(\Phi) + \frac{c^2}{4} \geq \underline{\lambda}_n(\Phi) + \frac{c^2}{4} \geq \underline{\lambda}_1(\Phi) + \frac{c^2}{4} > 0,$$

which is a contradiction. ∎

Next we recall a principle of linearized stability for periodic systems [6]. Consider the periodic boundary value problem

$$\begin{cases} x' = F(t, x), \\ x(0) = x(T), \end{cases} \tag{3.6}$$

where $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function that is T-periodic in t, and has continuous first-order partial derivative with x. Let x_0 be a T-periodic solution of (3.6), consider the linearized equation associated to x_0

$$y' = F_x(t, x_0)y. \tag{3.7}$$

Lemma 3.2. [6] *If the characteristic exponents associated with (3.7) all have negative real parts, then the T-periodic solution x_0 of (3.6) is asymptotically stable.*

Let $x_0(t)$ be the unique T-periodic solution of (1.1), it is easy to see that the linearized equation of (1.1) is

$$x''(t) + cx'(t) + g_x(t, x_0(t))x(t) = 0. \tag{3.8}$$

Lemma 3.3. *Under the conditions of Theorem 2.3, equation (3.8) does not admit real Floquet multipliers.*

Proof. On the contrary, suppose that there exist a real Floquet multiplier ρ and a non-trivial solution x such that $x(t + T) = \rho x(t)$. Combining [(P_5), Theorem 2.2] with conditions (1.2) and (2.5), we have that

$$\overline{g_x(t, x_0(t))} \geq \overline{\phi} > \frac{c^2}{4}, \tag{3.9}$$

$$\underline{\lambda}_1(g_x(t, x_0(t))) + \frac{c^2}{4} \geq \underline{\lambda}_1(\Phi) + \frac{c^2}{4} > 0. \tag{3.10}$$

In the same way as in the proof of Lemma 3.1, we can show that (3.9) and (3.10) imply that problem (3.8)-(2.2) has only the trivial T-periodic solution, which is a contraction. Thus the result is proved. ∎

Proof of Theorem 2.3. The proof will be divided into three steps.

Step 1. *Existence.* Let $X = Z = C[0, T]$ with the supremum norm $\|x\| = \max_{t \in [0,T]} |x(t)|$. We shall use the following notations:

$$\operatorname{dom} L = \{x : [0, T] \to \mathbb{R} \text{ is } C^1 \text{ on } [0, T] \text{ and satisfies } (2.2)\},$$

$$L : \operatorname{dom} L \subset X \to Z,\ x \to x'' + cx',$$

$$A : X \to Z,\ x \to \Phi(t)x,$$

$$N : X \to Z,\ x \to g(t, x) - h(t).$$

One may check that A and N are well defined and L-compact on bounded subsets of X, and that L is a linear Fredholm map of index zero. By the linearity of the operator

$L+A$ and Lemma 3.1, we have $\mathrm{Ker}(L+A)=\{0\}$. Thus condition (1) of Lemma 2.1 is satisfied.

Without loss of generality, we may assume that $g(t,0)=0$, otherwise we can reduce both sides of (1.1) by $g(t,0)$. This leads to the following homotopy H : $\operatorname{dom} L\times[0,1]\to Z$ defined by

$$H(x,\lambda)=x''(t)+cx'(t)+g_\lambda(t,x)-\lambda h(t), \tag{3.11}$$

where $g_\lambda(t,x)=\lambda g(t,x)+(1-\lambda)\Phi(t)x$.

In order to apply Lemma 2.1 with $\Omega=\{x\in X:\|x\|<R\}$, we have only to show that there exists $R>0$ for which $H(x,\lambda)\neq 0$ when $\lambda\in[0,1]$ and $x\in$ dom L with $\|x\|\geq R$. On the contrary, suppose that there exist sequences $x_n\in X$ and $\lambda_n\in[0,1]$ such that

$$x_n''(t)+cx_n'(t)+g_{\lambda_n}(t,x_n)=\lambda_n h(t) \tag{3.12}$$

and

$$\max_t x_n(t)\to\infty \quad \text{as} \quad n\to\infty.$$

Let $z_n(t)=\frac{x_n(t)}{\|x_n\|}, n=1,2,\ldots$. Dividing (3.12) by $\|x_n\|$, then multiplying by $\varphi(t)\in C_T^2$ and integrating by parts, we get

$$\int_0^T\Big(z_n\varphi''-cz_n\varphi'+\frac{g_{\lambda_n}(t,x_n)}{\|x_n\|}\cdot z_n\varphi\Big)dt=\lambda_n\int_0^T\frac{\varphi h(t)}{\|x_n\|}dt. \tag{3.13}$$

Observe that the right-hand side of (3.13) converges to zero. Let us study the left-hand side. The condition of Theorem 2.3 implies that

$$\omega_n(t)=\frac{g_{\lambda_n}(t,x_n)}{\|x_n\|}$$

is bounded. It is pre-compact in the weak* topology in $L^1(0,T)$. Thus $\{\frac{g(t,x_n)}{x_n}\}$ contains a subsequence which converges weakly to $\alpha(t)$ and $\lambda_n\to\lambda$. Taking the limit in (3.13), one obtains that

$$\int_0^T(z\varphi''-cz\varphi'+z\omega(t)\varphi)dt=0,$$

where $\omega(t)=\lambda\alpha(t)+(1-\lambda)\Phi(t)$ satisfying the conditions of Lemma 3.1. It follows from Lemma 3.1 that $z(t)\equiv 0$, which contradicts $\|z(t)\|=1$, which is a contradiction. Now we have proved the existence.

Step 2. *Uniqueness.* Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1). Then

$$[x_1(t)-x_2(t)]''+c[x_1(t)-x_2(t)]'+[g(t,x_1(t))-g(t,x_2(t))]=0. \tag{3.14}$$

Setting $\widetilde{x}(t) = x_1(t) - x_2(t)$, we obtain, from (3.14), that

$$\widetilde{x}''(t) + c\widetilde{x}'(t) + \beta(t)\widetilde{x}(t) = 0, \tag{3.15}$$

where $\beta(t) = \frac{g(t,x_1)-g(t,x_2)}{x_1-x_2}$. It follows from Lemma 3.1 that $\widetilde{x} \equiv 0$, which implies that $x_1(t) \equiv x_2(t)$ for all $t \in \mathbb{R}$.

Step 3. *Asymptotic stability.* Consider the planar system associated with equation (1.1)

$$\begin{cases} x' = y - cx, \\ y' = h(t) - g(t,x). \end{cases} \tag{3.16}$$

Let $X_0(t) = (x_0(t), y_0(t))^T$ be the unique T-periodic solution determined by the initial condition $X_0(0) = (x_0, y_0)^T$. Then X_0 corresponds to the unique fixed point of the Poincaré mapping $PX = U(T,X)$, here $U(T,X)$ is the initial value solution of (3.16) with $U(0,X) = X$.

Next, we show that the characteristic exponents associated with (3.8) all have negative real parts. To this end we consider a system equivalent to (3.8)

$$X'(t) = A(t)X(t), \tag{3.17}$$

where the column vector function $X(t) = (x(t), x'(t))^T$ and $A(t)$ is the matrix function

$$A(t) = \begin{bmatrix} 0 & 1 \\ -g_x(t, x_0(t)) & -c \end{bmatrix}$$

Let $\rho_1 = e^{T\mu_1}$ and $\rho_2 = e^{T\mu_2}$ be the Floquet multipliers of (3.17) and μ_1 and μ_2 be the characteristic exponents associate with ρ_1 and ρ_2. Then it follows from the above claim that ρ_1 and ρ_2 are a pair of complex conjugates. Thus the eigenvectors that are associate with different eigenvalues are linearly independent. Therefore $x_i = p_i(t)e^{\mu_i t}$ (for $i = 1, 2$) form the fundamental solutions of equation (3.17). On the other hand, by applying the Jacobi-Liouville formula, we have

$$|\rho_1|^2 = \rho_1\rho_2 = e^{\int_0^T \text{ trace } A(t)dt} = e^{-cT} \tag{3.18}$$

and

$$\mathrm{Re}(\mu_1) = \mathrm{Re}(\mu_2) = \frac{1}{2}\mathrm{Re}(\mu_1 + \mu_2) = \frac{1}{2T}\ln(\rho_1\rho_2) = -\frac{c}{2} < 0.$$

Applying Lemma 3.2, we obtain that $x_0(t)$ is asymptotically stable. Since every solution is linear combination of $x_1(t)$ and $x_2(t)$, $p_i(t)$ is T-periodic, hence it is bounded. Therefore every nonzero solution of the equation (3.17) decays at the same exponential rate of $\frac{c}{2}$. Let $M(t)$ be the fundamental matrix solution of (3.17). By the differentiability of $X(t)$ with respect to the initial value, the Poincaré mapping can be expressed in terms of the initial value X by the following formula:

$$PX - X_0 = M(T)(X - X_0) + o(X - X_0).$$

Using (3.18), we have that $M(T)$ has a pair of conjugate eigenvalues ρ_1, ρ_2 with $|\rho_1| = |\rho_2| = e^{-cT/2}$. Thus PX is a contraction mapping. According to the Hartman-Grobman theorem [7], there is a C^1 diffeomorphism φ is near enough to the identity that $PX - X_0$ is conjugate equivalent to $M(T)$. There is an invertible constant matrix C such that

$$C^{-1}M(T)C = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} = D(\rho),$$

and we may suppose that

$$\frac{1}{2}|X - X_0| < |\varphi(X) - \varphi(X_0)| < 2|X - X_0|,$$

for $X - X_0$ small, since φ is near the identity. Therefore, the Lyapunov exponent is given by

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{nT} \ln |P^n X - X_0| \\ = & \lim_{n\to\infty} \frac{1}{nT} \ln |\varphi \circ M^n(T) \circ \varphi^{-1}(X) - \varphi \circ M^n(T) \circ \varphi^{-1}(X_0)| \\ = & \lim_{n\to\infty} \frac{1}{nT} \ln |CD(\rho)^n C^{-1}[\varphi^{-1}(X) - \varphi^{-1}(X_0)]| = -\frac{c}{2}. \end{aligned}$$

Hence the rate of decay of the solution to the unique T-periodic solution x_0 is $\frac{c}{2}$, independently of the initial value X. This completes the proof of Theorem 2.3.

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