TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 3, pp. 927-942, June 2015
DOI: 10.11650/tjm.19.2015.3944
This paper is available online at http://journal.taiwanmathsoc.org.tw

INVERSE PROBLEM FOR DIFFERENTIAL PENCILS WITH INCOMPLETELY SPECTRAL INFORMATION

Yongxia Guo and Guangsheng Wei*

Abstract. In this paper we are concerned with the inverse spectral problems for energy-dependent Sturm-Liouville problems (that is, differential pencils) defined on interval $[0,1]$ with two potentials known on a subinterval $[a_1, a_2] \subset [0,1]$. We prove that the potentials on the entire interval $[0,1]$ and the boundary condition at $x = 1$ are uniquely determined in terms of partial knowledge of the spectrum in the situation of $a_1 = 0$ and $a_2 \geq 1/2$. Furthermore, in the situation of $a_1 > 0$ and $1/2 \in [a_1, a_2]$ we need additional information on the eigenfunctions at some interior point to obtain the uniqueness of the potentials on $[0,1]$ and two boundary conditions at $x = 0, 1$.

1. INTRODUCTION

We are concerned with the inverse problems for energy-dependent Sturm-Liouville problems, denoted by $L := L(q_0, q_1, h, H)$, of the form

$$-y'' + q_0(x)y + 2\rho q_1(x)y = \rho^2 y \tag{1.1}$$

on $[0,1]$ subject to the Robin boundary conditions

$$y'(0) - hy(0) = 0, \tag{1.2}$$

$$y'(1) + Hy(1) = 0. \tag{1.3}$$

Here $\rho \in \mathbb{C}$ is a spectral parameter, two potentials $q_0 \in L_2[0,1]$ and $q_1 \in W_2^1[0,1]$ are complex-valued functions. It is well known [19, 27] that the problem L can be regarded as the spectral problem for a quadratic operator pencil and has a discrete spectrum consisting of simple eigenvalues with finitely many exceptions, denoted by $\sigma(L) = \{\rho_n\}_{n=-\infty}^{\infty}$.

Received October 23, 2013, accepted September 2, 2014.
Communicated by Jenn-Nan Wang.
2010 *Mathematics Subject Classification*: Primary 34L05, 34A55; Secondary 34B09.
Key words and phrases: Differential pencil, Inverse spectral problem, Interior spectral data.
*Corresponding author.

Sturm-Liouville spectral problems with potentials depending on the spectral parameter arise in various models of quantum and classical mechanics. For instance, the evolution equations that are used to model interactions between colliding relativistic spineless particles can be reduced to the form (1.1). Then ρ^2 is associated with the energy of the system (see [12, 19]). Another typical example is related to vibrations of mechanical systems in viscous media, see [24].

The main aim of this paper is to investigate in detail the uniqueness problem of the determination of two potentials q_0 and q_1 under the following two cases: Given the potentials q_0 and q_1 on a subinterval $[a_1, a_2] \subset [0, 1]$ with either $a_1 = 0$ or $a_1 > 0$, respectively. This problem is connected with the inverse spectral problem with incompletely spectral information (cf. [2, 6, 9]).

The above problem for classical Sturm-Liouville equations (i.e., $q_1 \equiv 0$ in (1.1)) was first introduced by Hochstadt and Lieberman [9] in 1978. They proved that the known potential q_0 on half of the interval $[0, 1]$ and the spectrum of the problem determine q_0 everywhere, which is now called the half-inverse spectral theorem. This result has been further generalized to different settings (see [2, 4, 6, 15, 22] and references therein). In particular, Gesztesy and Simon [6] in 2000 provided a remarkable generalization of the half-inverse spectral theorem, which shows that the known q_0 on part of $[0, 1]$ (i.e., $[0, a_2]$ with $a_2 \in [1/2, 1)$) and certain part of the spectrum $\sigma(L)$ completely determine q_0 on the entire interval $[0, 1]$ and the boundary condition (1.3). On the other hand, the inverse spectral problems for differential pencil (1.1)-(1.3) were studied in [3, 4, 5, 7, 8, 11, 14, 16, 21, 24, 26] and other papers. Let us mention here that in [2, 15, 22], the half-inverse problems for the pencil L were considered in different situations, which show that if q_0 and q_1 are known on half of the interval $[0, 1]$, then one spectrum suffices to determine them uniquely on the other half. Moreover, it was proved in [2] that an arbitrary finite number of eigenvalues can be missing provided that q_0 and q_1 are known on an interval $(0, a)$ for any $a > 1/2$.

In this paper, we shall prove that the result of Gesztesy and Simon [6] remains valid for differential pencils. More precisely, we show that knowing q_0 and q_1 on the subinterval $[0, a_2]$ of $[0, 1]$ with $1/2 \leq a_2 < 1$ and fractions of the spectrum $\sigma(L)$ one can uniquely determine q_0 and q_1 on the interval $[0, 1]$ and the boundary condition (1.3). Furthermore, we consider the same problem in the situation of the potentials q_0 and q_1 given on the interior subinterval $[a_1, a_2]$ of $[0, 1]$, and solve it by virtue of the known eigenvalues and some information on the eigenfunctions at some interior point $a \in [a_1, a_2]$. The later is called interior spectral data, which together with the associated eigenvalues has been used to recover the q_0 and/or q_1 uniquely for the Sturm-Liouville problems and differential pencils, etc. (see [20, 23]).

Note that the case of Dirichlet boundary conditions where formally $h = \infty$ and/or $H = \infty$ would demand a separate treatment. Nevertheless, one expects that the above similar results may be obtained also for these situations.

The paper is organized as follows. In the next section, we recall some classical results in order to prove our main results in this paper. We give the uniqueness theorems in cases of $a_1 = 0$ and $a_1 > 0$, respectively, in Sections 3 and 4.

2. Preliminaries

We begin by recalling some classical results, which will be needed later. Let the functions $\varphi(x,\rho)$ and $\psi(x,\rho)$ be solutions of Eq. (1.1) with the initial-valued conditions

$$\varphi(0,\rho) = 1,\ \varphi'(0,\rho) = h; \qquad \psi(1,\rho) = 1,\ \psi'(1,\rho) = -H. \tag{2.1}$$

As is known ([1, 12, 13, 25]), for each fixed $x \in [0,1]$ the functions $\varphi(x,\rho)$ and $\psi(x,\rho)$ together with their derivatives with respect to x are entire in ρ, and are of exponential type [25]. For $|\rho| \to \infty$ the following asymptotics hold uniformly with respect to $x \in [0,1]$

$$\begin{cases} \varphi(x,\rho) = \cos(\rho x - Q(x)) + O\left(\exp(|\mathrm{Im}\rho| x)/\rho\right), \\ \varphi'(x,\rho) = -\rho\sin(\rho x - Q(x)) + O(\exp(|\mathrm{Im}\rho| x)), \end{cases} \tag{2.2}$$

and

$$\begin{cases} \psi(x,\rho) = \cos(\rho(1-x) - \omega_0 + Q(x)) + O\left(\exp(|\mathrm{Im}\rho|(1-x))/\rho\right), \\ \psi'(x,\rho) = \rho\sin(\rho(1-x) - \omega_0 + Q(x)) + O(\exp(|\mathrm{Im}\rho|(1-x))), \end{cases}$$

where

$$Q(x) = \int_0^x q_1(t)dt, \qquad \omega_0 = \int_0^1 q_1(t)dt. \tag{2.3}$$

Since the spectral problem (1.1)-(1.3) can be regarded as the spectral problem for a quadratic operator pencil, it follows from [19] that its spectrum is a discrete subset of $\mathbb{C}$ and consists entirely of eigenvalues of finite algebraic multiplicity. In general the pencil L can possess non-real and/or non-simple eigenvalues. Moreover, the eigenvalues ρ_n with account of multiplicity m_n of L coincide with the zeros of its *characteristic function*

$$\Delta(\rho) := \langle \varphi(x,\rho), \psi(x,\rho) \rangle, \tag{2.4}$$

where $\langle y, z\rangle := yz' - y'z$. Clearly, $\Delta(\rho) = U(\psi) = -V(\varphi)$. It should be noted [18, 19] that, different from classical Sturm-Liouville problems, the characteristic function $\Delta(\rho)$ for differential pencil is of exponential type. Denote $G_\delta = \{\rho : |\rho - n\pi - \omega_0| \geq \delta, n \in \mathbb{Z}\}$ with fixed $\delta > 0$. Then for sufficiently large $|\rho|$

$$|\Delta(\rho)| \geq C_\delta |\rho| \exp(|\mathrm{Im}\rho|), \quad \rho \in G_\delta. \tag{2.5}$$

It is known [1, 2, 19] that if the real parts of all the eigenvalues of problem (1.1)-(1.3) are arranged as a nondecreasing sequence, that is,

$$\cdots \le \mathrm{Re}(\rho_{-n}) \le \cdots \le \mathrm{Re}(\rho_{-1}) \le \mathrm{Re}(\rho_{-0}) \le \mathrm{Re}(\rho_{+0}) \le \mathrm{Re}(\rho_1) \le \cdots$$

for $n \in \mathbb{N}_0$, then the sequence $\{\rho_n\}$ satisfies the classical asymptotic form [1]

$$\rho_n = n\pi + \omega_0 + \frac{\omega_1}{n\pi} + o\left(\frac{1}{n}\right) \tag{2.6}$$

as $|n| \to \infty$, where ω_0 is defined by (2.3) and

$$\omega_1 = h + H + \frac{1}{2}\int_0^1 (q_0 + q_1^2)(x)\mathrm{d}x.$$

This shows that there exist at most finite number of non-simple eigenvalues, that is, $m_n = 1$ for sufficiently large $|n|$, and all the eigenvalues locate in a parallel region of the real axis.

Lemma 2.1. *If* $0 \in \sigma(L)$, *then there exist a constant* $a \in \mathbb{C}$ *and differential pencil* $\hat{L}_a(\hat{q}_0, \hat{q}_1, h, H)$ *such that* $0 \notin \sigma(\hat{L}_a)$, *where*

$$\hat{q}_0(x) = q_0(x) + 2aq_1(x) - a^2, \qquad \hat{q}_1(x) = q_1(x) - a. \tag{2.7}$$

Proof. Making the shift $q_1(x) \longmapsto (q_1(x) - a) =: \hat{q}_1(x)$, it is easy to see that Eq. (1.1) equivalently reduces to the following equation

$$-y'' + (q_0(x) + 2aq_1(x) - a^2)y + 2\hat{\rho}(q_1(x) - a)y = \hat{\rho}^2 y, \tag{2.8}$$

where $\hat{\rho} = \rho - a$. Note that there exists $a \in \mathbb{C}$ such that a is not an eigenvalue of the differential pencil consisting of equation $-y'' + q_0(x)y + 2aq_1(x)y = a^2 y$ and the boundary conditions (1.2)-(1.3). This implies that $\hat{\rho} = 0 \notin \sigma(\hat{L}_a)$ for this constant a and the proof is complete. ∎

According to the above lemma, throughout of this paper we always assume that 0 is not an eigenvalue of differential pencil L defined by (1.1)-(1.3).

Next we analyze the growth properties of some infinite products in order to prove our main theorems. Consider a sequence $\Lambda := \{\lambda_n\}_{n\in\mathbb{Z}}$ of arbitrary values satisfying the following asymptotics

$$\lambda_n = n\pi + O(1) \tag{2.9}$$

as $n \to \pm\infty$. Let $\Lambda_0 := \{\lambda_{n_k}\}_{k\in\mathbb{Z}} \subset \Lambda$ be almost symmetric with respect to the origin, which means that if $n_k \in S$ then $-n_k \in S$ with finitely many exceptions, where

$S := \{n_k : \lambda_{n_k} \in \Lambda_0\}$. Denote

$$N_{\Lambda_0}(t) = \begin{cases} \displaystyle\sum_{0\le \mathrm{Re}(\lambda_{n_k})\le t} 1 & \text{if } t > 0, \\ \displaystyle -\sum_{t<\mathrm{Re}(\lambda_{n_k})<0} 1 & \text{if } t < 0. \end{cases} \tag{2.10}$$

It is known [10, 25] that if $\lambda_{n_k} \neq 0$ for all $n_k \in S$ then the product

$$W_{\Lambda_0}(z) = \text{p.v.} \prod_{\lambda_{n_k}\in\Lambda_0} \left(1 - \frac{z}{\lambda_{n_k}}\right) := \lim_{N\to\infty} \prod_{k=-N,\lambda_{n_k}\in\Lambda_0}^{N} \left(1 - \frac{z}{\lambda_{n_k}}\right) \tag{2.11}$$

converges locally uniformly and defines an entire function with zeros $\lambda_{n_k}, n_k \in S$. The next lemma can be found in [10, Lemmas 2.5-2.7].

Lemma 2.2. *Let $z = x + iy$ with $x, y \in \mathbb{R}$ and all $\lambda_{n_k} \neq 0$. Then*

$$\begin{aligned} \ln|W_{\Lambda_0}(z)| &= \text{p.v.} \int_{-\infty}^{\infty} \frac{N_{\Lambda_0}(t)}{t} \frac{y^2 - x(t-x)}{y^2 + (t-x)^2} \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \frac{N_{\Lambda_0}(t)}{t} \frac{y^2}{y^2+t^2} \mathrm{d}t + O(1) \end{aligned} \tag{2.12}$$

as $|y| \to \infty$, where O-term is locally uniform in x. Moveover, if $\lambda^*_{n_k} = \lambda_{n_k} + O(1/n_k)$ with $\lambda^*_{n_k} \neq 0$, then for every $\delta > 0$,

$$|W_{\Lambda_0}(z)| \asymp |W_{\Lambda_0^*}(z)|, \tag{2.13}$$

where z satisfies $|z - \lambda_{n_k}| > \delta$ and $|z - \lambda^*_{n_k}| > \delta$ with $\delta > 0$ being a constant. Here $W_{\Lambda_0^*}(z)$ is defined by (2.11) with replacing λ_{n_k} by $\lambda^*_{n_k}$ and the notation $\asymp$ means that both $|W_{\Lambda_0^*}(z)|/|W_{\Lambda_0}(z)|$ and $|W_{\Lambda_0}(z)|/|W_{\Lambda_0^*}(z)|$ are bounded.

Lemma 2.3. *Let $\Lambda_0 := \{\lambda_{n_k}\}_{k\in\mathbb{Z}} \subset \Lambda$ be almost symmetric with respect to the origin. If there is $t_0 > 0$ such that*

$$N_{\Lambda_0}(t) \begin{cases} \ge AN_{\Lambda}(t) + B_+ & \text{if } t > t_0, \\ \le AN_{\Lambda}(t) - B_- & \text{if } t < -t_0, \end{cases} \tag{2.14}$$

and $\ln|W_{\Lambda}(iy)| = |y| + \ln|y| + O(1)$ as y (real) $\to \pm\infty$, then we have

$$\ln|W_{\Lambda_0}(iy)| \ge A|y| + (B_+ + B_- + A)\ln|y| + O(1), \tag{2.15}$$

where $N_{\Lambda}(t)$ and $W_{\Lambda}(z)$ are defined by (2.10) and (2.11) with replacing λ_{n_k} by λ_n, respectively.

Proof. Without loss of generality, we assume that $N_\Lambda(t) = 0$ for $-1 \le t \le 1$, which implies that $N_{\Lambda_0}(t) = 0$ also holds in the same interval. It follows from (2.12) that

$$\begin{aligned} \ln|W_{\Lambda_0}(iy)| &= \int_{-\infty}^{\infty} \frac{N_{\Lambda_0}(t)}{t} \frac{y^2}{y^2+t^2} \mathrm{d}t \\ &= \int_{-\infty}^{-1} \frac{N_{\Lambda_0}(t)}{t} \frac{y^2}{y^2+t^2} \mathrm{d}t + \int_{1}^{\infty} \frac{N_{\Lambda_0}(t)}{t} \frac{y^2}{y^2+t^2} \mathrm{d}t. \end{aligned} \tag{2.16}$$

By (2.14), it is easy to infer that there exists a constant $C_0 \ge 0$ satisfying

$$N_{\Lambda_0}(t) \begin{cases} \ge A N_\Lambda(t) - C_0 & \text{if } 1 < t \le t_0, \\ \le A N_\Lambda(t) + C_0 & \text{if } -t_0 \le t < -1. \end{cases} \tag{2.17}$$

Substituting the above inequalities and (2.14) into (2.16), one yields that

$$\begin{aligned} &\ln|W_{\Lambda_0}(iy)| \\ &\ge \int_{-\infty}^{-t_0} \frac{(AN_\Lambda(t) - B_-)y^2}{ty^2+t^3} \mathrm{d}t + \int_{t_0}^{\infty} \frac{(AN_\Lambda(t) + B_+)y^2}{ty^2+t^3} \mathrm{d}t + O(1) \\ &= A\int_{-\infty}^{-1} \frac{N_\Lambda(t)}{t} \frac{y^2}{y^2+t^2} \mathrm{d}t + A\int_{1}^{\infty} \frac{N_\Lambda(t)}{t} \frac{y^2}{y^2+t^2} \mathrm{d}t \\ &\quad -B_- \int_{-\infty}^{-1} \frac{y^2}{t^3+ty^2} \mathrm{d}t + B_+ \int_{1}^{\infty} \frac{y^2}{t^3+y^2} \mathrm{d}t + O(1). \end{aligned} \tag{2.18}$$

Here we have used the following formula

$$\int_1^{t_0} \frac{y^2}{t^3+ty^2} \mathrm{d}t = -\frac{1}{2} \ln\left(1 + \frac{y^2}{t^2}\right)\Bigg|_1^{t_0} = O(1)$$

as y (real) $\to \pm\infty$. Since $\ln|W_\Lambda(iy)| = |y| + \ln|y| + O(1)$, it follows from the condition of the lemma and (2.12) that

$$\int_{-\infty}^{-1} \frac{N_\Lambda(t)}{t} \frac{y^2}{t^2+y^2} \mathrm{d}t + \int_{1}^{\infty} \frac{N_\Lambda(t)}{t} \frac{y^2}{t^2+y^2} \mathrm{d}t = |y| + \ln|y| + O(1). \tag{2.19}$$

Moreover, by direct calculation we have

$$\begin{aligned} \int_1^{\infty} \frac{y^2}{t^3+ty^2} \mathrm{d}t &= -\frac{1}{2} \ln\left(1 + \frac{y^2}{t^2}\right)\Bigg|_{t=1}^{\infty} \\ &= \ln|y| + O(1). \end{aligned} \tag{2.20}$$

The similar result also holds for the integral from $t = -\infty$ to -1. Thus, by virtue of (2.18)-(2.20), we arrive at (2.15) and the proof is complete. ■

We also need a Phragmén-Lindelöf-type result.

Lemma 2.4. (Levin [17]). *Let $F(z)$ be an entire function of zero exponential type, that is,*

$$\limsup_{r\to\infty}\frac{\ln M(r)}{r}\le 0,\qquad M(r)=\max_{\theta}|F(re^{i\theta})|. \tag{2.21}$$

If $F(z)$ is bounded along a line, then $F(z)$ is constant. In particular, if $F(z)\to 0$ when $|z|\to\infty$ along a line, then $F(z)\equiv 0$.

With the above preliminaries, we shall give our uniqueness results of this paper through the following two sections.

3. The Case Where $a_1 = 0$

In this section we mainly study the inverse eigenvalue problem for differential pencil (1.1)-(1.3) with the potentials q_0 and q_1 known on a left-hand subinterval $[0,a_2]\subset[0,1]$ with $a_2\ge 1/2$. Basing on this condition, we shall employ the partial knowledge on the spectrum of pencil L and the boundary condition parameter h in (1.2) to determine the whole potentials q_0 and q_1 and boundary condition parameter H in (1.3) uniquely.

Our main result of this section is as follows.

Theorem 3.1. *Let $a_2\in[1/2,1)$. Let $\sigma_0\subset\sigma:=\sigma(L)$ be almost symmetric with respect to the origin. Suppose that the limit*

$$\lim_{t\to\infty}\frac{N_{\sigma_0}(t)}{t}=\gamma \tag{3.1}$$

exists and there are constants $\varepsilon>0$, $t_0>0$ and $\mu\in\mathbb{R}$ such that

$$N_{\sigma_0}(t)\begin{cases}\ge 2(1-a_2)N_\sigma(t)-(1-a_2)+\mu+\varepsilon & \text{if } t\ge t_0,\\ \le 2(1-a_2)N_\sigma(t)+(1-a_2)+\mu & \text{if } t\le -t_0,\end{cases} \tag{3.2}$$

where $N_\sigma(t)$ and $N_{\sigma_0}(t)$ are defined by (2.10) with replacing Λ_0 by σ and σ_0 respectively.

Then q_j on $(0,a_2)$, h and σ_0 uniquely determine H and q_j on $[0,1]$ for $j=0,1$.

Remark 3.2. The obtained result here is a natural generalization of the result of Buterin and Shieh [2] there the case $a_2=1/2$ was treated. Moreover, Theorem 3.1 is optimal in the sense that one can dispense with a finite density of eigenvalues, whenever the number h and the potentials q_j on $(0,a_2)$ for $j=1,2$ are known a priori.

Proof of Theorem 3.1. Let us consider another differential pencil $\tilde{L}$ of the same form (1.1)-(1.3) but with different coefficients $(\tilde{q}_0,\tilde{q}_1,h,\tilde{H})$. Then both pencils have common eigenvalues $\{\rho_{n_k}\}_{\rho_{n_k}\in\sigma_0}$, and satisfy $\tilde{q}_0(x)=q_0(x)$ a.e. and $\tilde{q}_1(x)=q_1(x)$

for $x \in [0, a_2]$. By Lemma 2.1, we assume that $0 \notin \sigma$ (if not, then we can consider the modified pencil consisting of Eq. (2.8) and the boundary conditions (1.2)-(1.3)). Define the function $g_{\sigma_0}(\rho)$ by

$$g_{\sigma_0}(\rho) = \text{p.v.} \prod_{\rho_{n_k} \in \sigma_0} \left(1 - \frac{\rho}{\rho_{n_k}}\right) \tag{3.3}$$

and consider the function

$$F(\rho) = \frac{\langle \psi, \tilde{\psi} \rangle (a_2, \rho)}{g_{\sigma_0}(\rho)}, \tag{3.4}$$

where $\langle \psi, \tilde{\psi} \rangle (a_2, \rho) = (\psi \tilde{\psi}' - \psi' \tilde{\psi})(a_2, \rho)$. Since $h = \tilde{h}, q_j(x) = \tilde{q}_j(x)$ on $[0, a_2]$ for $j = 0, 1$, it follows from (1.1) and (2.4) that

$$\begin{aligned} &\langle \psi, \tilde{\psi} \rangle (a_2, \rho) \\ &= \int_0^{a_2} [(q_0 + 2\rho q_1) - (q_0 + 2\rho q_1)](x)(\psi \tilde{\psi})(x, \rho) \mathrm{d}x + \langle \psi, \tilde{\psi} \rangle (0, \rho) \\ &= \langle \psi, \tilde{\psi} \rangle (0, \rho) \\ &= \psi(0, \rho) \tilde{\Delta}(\rho) - \tilde{\psi}(0, \rho) \Delta(\rho). \end{aligned} \tag{3.5}$$

This means that $\langle \psi, \tilde{\psi} \rangle (a_2, \rho)$ vanishes at each point where $g_{\sigma_0}(\rho)$ vanishes. Furthermore, it should be noted that $\{\rho_{n_k}\}_{\rho_{n_k} \in \sigma_0}$ are common eigenvalues of L and $\tilde{L}$ and therefore the multiplicity of each eigenvalue ρ_{n_k} is not exceeding that of $\rho_{n_k} \in \sigma(L)$ and $\rho_{n_k} \in \tilde{\sigma}(\tilde{L})$. This together with (3.4) shows that $F(\rho)$ is an entire function.

Our first purpose here is to prove that the entire function $F(\rho)$ is of zero exponential type. We estimate the numerator of $F(\rho)$ using (2.2) and (2.6). Because $Q(a_2) = \tilde{Q}(a_2)$, where $Q(x)$ is defined by (2.3) for $x \in [0, 1]$, we have

$$\begin{aligned} \langle \psi, \tilde{\psi} \rangle (a_2, \rho) = &\rho \cos(\rho(1 - a_2) - \omega_0 + Q(a_2)) \sin(\rho(1 - a_2) - \tilde{\omega}_0 + \tilde{Q}(a_2)) \\ &- \rho \sin(\rho(1 - a_2) - \omega_0 + Q(a_2)) \cos(\rho(1 - a_2) - \tilde{\omega}_0 + \tilde{Q}(a_2)) \\ &+ O(\exp(2(1 - a_2)|\mathrm{Im}\rho|)) \\ = &\rho \sin(\omega_0 - \tilde{\omega}_0) + O(\exp(2(1 - a_2)|\mathrm{Im}\rho|)). \end{aligned}$$

According to (2.6) the specification of the spectrum $\{\rho_n\}$ determines ω_0 up to a constant $k\pi,\ k \in \mathbb{Z}$. Thus we deduce $\omega_0 - \tilde{\omega}_0 = k\pi$, and as $|\rho| \to \infty$

$$\langle \psi, \tilde{\psi} \rangle (a_2, \rho) = O(\exp(2(1 - a_2)|\mathrm{Im}\rho|)). \tag{3.6}$$

On the other hand, we estimate the denominator of $F(\rho)$ in virtue of Lemma 2.2. If $|\rho - \rho_{n_k}| \geq \delta$ and $|\rho - n_k\pi - \omega_0| \geq \delta$ for all $\rho_{n_k} \in \sigma_0$, then from (2.13) we have

$$|g_{\sigma_0}(\rho)| \asymp |\hat{g}_{\sigma_0}(\rho)|, \tag{3.7}$$

where

$$\hat{g}_{\sigma_0}(\rho) = \text{p.v.} \prod_{\lambda_{n_k} \in \sigma_0} \left(1 - \frac{\rho}{n_k\pi + \omega_0}\right).$$

In calculating $\hat{g}_{\sigma_0}(\rho + \omega_0)$ we use

$$1 - \frac{\rho + \omega_0}{n_k\pi + \omega_0} = \left(1 - \frac{\rho}{n_k\pi}\right)\left(1 - \frac{\omega_0}{n_k\pi + \omega_0}\right) =: c_{n_k}\left(1 - \frac{\rho}{n_k\pi}\right)$$

to obtain

$$\hat{g}_{\sigma_0}(\rho + \omega_0) = c\bar{g}_{\sigma_0}(\rho), \qquad \text{where } \bar{g}_{\sigma_0}(\rho) = \text{p.v.} \prod_{\lambda_{n_k} \in \sigma_0} \left(1 - \frac{\rho}{n_k\pi}\right). \tag{3.8}$$

Arrange the values $\{n_k\}$ in an increasing sequence $\{z_k\}$. Since $N_{\sigma_0}(z_k) = k/\pi + O(1)$, we have from (3.1) that

$$\frac{k}{z_k\pi} = \frac{N_{\sigma_0}(z_k)}{z_k} + o(1) \to \gamma$$

as $k \to \infty$. Now the almost symmetric property of σ_0 implies a lower estimate by Lemma 2.8 in [10]: for every $\varepsilon > 0$ there exists a $c > 0$ such that if $|\rho - n_k\pi| \geq \delta$ for all $\rho_{n_k} \in \sigma_0$

$$|\bar{g}_{\sigma_0}(\rho)| \geq c\exp(\pi\gamma|\text{Im}\rho| - \varepsilon|\rho|).$$

By the above considerations, one infers that

$$|g_{\sigma_0}(\rho)| \asymp |\hat{g}_{\sigma_0}(\rho)| \asymp |\bar{g}_{\sigma_0}(\rho - \omega_0)| \geq c\exp(\pi\gamma|\text{Im}\rho| - 2\varepsilon|\rho|) \tag{3.9}$$

for $|\rho|$ large enough. If both $|\rho - \rho_{n_k}| \geq \delta$ and $|\rho - n_k\pi - \omega_0| \geq \delta$ hold for $\rho_{n_k} \in \sigma_0$, then it follows from (3.9) that the whole denominator of $F(\rho)$ has a lower estimate

$$|g_{\sigma_0}(\rho)| \geq c\exp(2(1 - a_2)|\text{Im}\rho| - 2\varepsilon|\rho|). \tag{3.10}$$

Combined with (3.6), there exists a positive number C such that if $|\rho - \rho_{n_k}| \geq \delta$ and $|\rho - n_k\pi - \omega_0| \geq \delta$ hold for $\rho_{n_k} \in \sigma_0$, then

$$|F(\rho)| = \left|\frac{\langle\psi, \tilde{\psi}\rangle(a_2, \rho)}{g_{\sigma_0}(\rho)}\right| \leq C\exp(2\varepsilon|\rho|) \tag{3.11}$$

for $|\rho|$ large enough. Consequently, the maximum modulus principle [25] yields that $|F(\rho)| \leq C\exp(2\varepsilon|\rho|)$ for all $\rho \in \mathbb{C}$, which means that $F(\rho)$ is of zero exponential type according to Lemma 2.4.

Our second purpose here is to prove that $F(iy) \to 0$ for y (real) $\to \pm\infty$. Since the characteristic function $\Delta(\rho)$ defined by (2.4) is entire in ρ and is of exponential type, it follows by Hadamard′s factorization theorem [25] that $\Delta(\rho)$ is uniquely determined up to a multiplicative constant by its zeros, that is, there exists a constant C_0 such that

$$\Delta(\rho) = C_0 \text{ p.v.} \prod_{n\in\mathbb{Z}} \left(1 - \frac{\rho}{\rho_n}\right). \tag{3.12}$$

This together with (2.5) implies that

$$\ln|\Delta(iy)| \geq \ln|y| + |y| + O(1). \tag{3.13}$$

In virtue of Lemma 2.3, by replacing $W_\Lambda(z)$ and $W_{\Lambda_0}(z)$ by $\Delta(\rho)$ and $g_{\sigma_0}(\rho)$ respectively, it follows from condition (2.14) and (3.2) that

$$A = 2(1 - a_2), \qquad B_+ = \mu + \varepsilon - (1 - a_2), \qquad B_- = -\mu - (1 - a_2), \tag{3.14}$$

and therefore we deduce

$$|g_{\sigma_0}(iy)| \geq C|y|^\varepsilon \exp(2(1 - a_2)|y|), \tag{3.15}$$

where C is a positive constant. In view of (3.6), we have $|\langle\psi, \tilde{\psi}\rangle(a_2, iy)| \leq C\exp(2(1-a_2)|y|)$, which combined with (3.4) and (3.15) turns out that for $|y|$ sufficiently large

$$\begin{aligned}|F(iy)| &\leq C\frac{\exp(2(1-a_2)|y|)}{|y|^\varepsilon\exp(2(1-a_2)|y|)} \\ &= O(|y|^{-\varepsilon}).\end{aligned}$$

This yields that $|F(iy)| \to 0$ as y (real) $\to \pm\infty$.

With the above arguments, using Lemma 2.4 one derives that $F \equiv 0$. Therefore we obtain $\langle\psi, \tilde{\psi}\rangle(a_2, \rho) = 0$ and

$$\frac{\psi(a_2, \rho)}{\psi'(a_2, \rho)} = \frac{\tilde{\psi}(a_2, \rho)}{\tilde{\psi}'(a_2, \rho)} \tag{3.16}$$

for all $\rho \in \mathbb{C}$. According to the known uniqueness theorem in [14], we get $H = \tilde{H}$ and $q_j(x) = \tilde{q}_j(x)$ on $[a_2, 1]$ for $j = 0, 1$. The proof is complete. ■

As a typical example, knowing slightly more than half the spectrum, h and $q_j(x)$ for $j = 0, 1$ on $[0, 3/4]$ one can determine H and $q_j(x)$ uniquely on $[0, 1]$ for $j = 0, 1$. This fact is formulated as the following corollary.

Corollary 3.3. *Let $a_2 = 3/4$ and $n_0 \in \mathbb{Z}$ be given. Then the specification of the spectrum $\{\rho_{2n}\}_{n\in\mathbb{Z}} \cup \{\rho_{2n_0+1}\}$ (resp., $\{\rho_{2n+1}\}_{n\in\mathbb{Z}} \cup \{\rho_{2n_0}\}$) together with h and q_j on $[0, 3/4]$ for $j = 0, 1$ determines q_j on $[3/4, 1]$ for $j = 0, 1$ and H uniquely.*

Proof. Let $a_2 = 3/4, \varepsilon = 1/4$ and $\mu = 0$ in Theorem 3.1. Then it is easy to check that the above result holds and the proof is omitted. ∎

4 The Case Where $a_1 > 0$

In this section, we mainly consider the uniqueness problem of the inverse problem for differential pencil (1.1)-(1.3) with the potentials q_0 and q_1 known on an interior subinterval $[a_1, a_2] \subset [0, 1]$ with $1/2 \in [a_1, a_2]$. Without loss of generality, we always assume that $a_1 \le 1 - a_2$. Let us mention that in this situation the knowledge of the spectrum of problem (1.1)-(1.3) is insufficient to recover the potentials q_0 and q_1 uniquely (see [27]). Therefore, we need additional spectral information to deal with this uniqueness problem. We shall employ the so-called interior spectral data

$$\{m(a_1, \rho_{n_k})\}_{\rho_{n_k} \in \sigma_1} := \left\{ \frac{\varphi(a_1, \rho_{n_k})}{\varphi'(a_1, \rho_{n_k})} \right\}_{\rho_{n_k} \in \sigma_1} \tag{4.1}$$

(with the possibility of the values being infinite) corresponding to the known eigenvalues ρ_{n_k} as the additional spectral data for our uniqueness results, where $\varphi(x, \rho_{n_k})$ is the eigenfunction corresponding to the eigenvalue ρ_{n_k} and σ_1 is a subset of the spectrum of differential pencil L, which is almost symmetric. It should be noted that this type of spectral data together with eigenvalues was first posed by Mochizuki and Trooshin [20] to solve the inverse spectral problems of the classical Sturm-Liouville operators. This data has been further used to treat with different settings including differential pencils (see [4, 23] and references therein).

Denote by m_n the multiplicity of the eigenvalue ρ_n $(\rho_n = \rho_{n+1} = \cdots = \rho_{n+m_n-1})$ and put

$$\hat{\sigma} = \{\rho_k \in \sigma : \rho_{k-1} \ne \rho_k \ne \rho_{k+1}\}. \tag{4.2}$$

Note that by virtue of (2.6) for sufficiently large $|n|$ we have $m_n = 1$. We give our uniqueness results for the known potentials on interior subinterval $[a_1, a_2]$ through the following two cases: $a_1 < 1 - a_2$ and $a_1 = 1 - a_2$.

Theorem 4.1. *Let $1/2 \in [a_1, a_2]$ with $a_1 < 1 - a_2$ and $a \in [a_1, a_2]$ be given. Let both $\sigma_1 \subset \sigma$ and $\sigma_2 \subset \sigma$ be almost symmetric with respect to the origin satisfying $\sigma_1 \subset \sigma_2$ as well as $\sigma_1 \subset \hat{\sigma}$. Suppose that the limits*

$$\lim_{t \to \infty} \frac{N_{\sigma_j}(t)}{t} = \gamma_j \tag{4.3}$$

exist for $j = 1, 2$ and there are the constants $\varepsilon > 0$, $t_0 > 0$ and $\mu_j \in \mathbb{R}$ for $j = 1, 2$ such that

$$N_{\sigma_1}(t) \begin{cases} \ge 2a_1 N_\sigma(t) + \mu_1 - a_1 + \varepsilon & \text{if} \quad t \ge t_0, \\ \le 2a_1 N_\sigma(t) + \mu_1 + a_1 & \text{if} \quad t \le -t_0, \end{cases} \tag{4.4}$$

and

$$N_{\sigma_2}(t)\begin{cases}\geq 2(1-a_2)N_\sigma(t)+\mu_2-(1-a_2)+\varepsilon & \textit{if} \quad t\geq t_0,\\ \leq 2(1-a_2)N_\sigma(t)+\mu_2+(1-a_2) & \textit{if} \quad t\leq -t_0,\end{cases} \tag{4.5}$$

where $N_\sigma(t)$ and $N_{\sigma_j}(t)$ for $j=1,2$ are defined by (2.10) with replacing Λ_0 by σ and σ_j respectively.

Then q_j on $[a_1,a_2]$, $\{\rho_n\}_{\rho_n\in\sigma_2}$ and $\{m(a,\rho_n)\}_{\rho_n\in\sigma_1}$ uniquely determine h, H and q_j on $[0,1]$ for $j=0,1$.

For the case $a_1=1-a_2$, we have the following theorem.

Theorem 4.2. *Let $a_1=1-a_2$ with $1/2\in(a_1,a_2)$ and $a\in[a_1,a_2]$ be given. Let $\sigma_1\subset\sigma$ be almost symmetric with respect to the origin satisfying $\sigma_1\subset\hat{\sigma}$. Assume that the limit*

$$\lim_{t\to\infty}\frac{N_{\sigma_1}(t)}{t}=\gamma \tag{4.6}$$

exists and there are the constants $\varepsilon>0$, $t_0>0$ and $\mu\in\mathbb{R}$ such that

$$N_{\sigma_1}(t)\begin{cases}\geq 2a_1N_\sigma(t)+\mu-a_1+\varepsilon & \textit{if} \quad t\geq t_0,\\ \leq 2a_1N_\sigma(t)+\mu+a_1 & \textit{if} \quad t\leq -t_0,\end{cases} \tag{4.7}$$

where $N_\sigma(t)$ and $N_{\sigma_1}(t)$ are defined by (2.10) with replacing Λ_0 by σ and σ_1 respectively.

Then q_j on $[a_1,a_2]$, $\{\rho_n\}_{\rho_n\in\sigma_1}$ and $\{m(a,\rho_n)\}_{\rho_n\in\sigma_1}$ uniquely determine h, H and q_j on $[0,1]$ for $j=0,1$.

Remark 4.3. Note that if $a_1\geq 1-a_2$ then the similar results remain true provided replacing the interval $[0,a_1]$ by $[a_2,1]$ with $a_2>1/2$. Moreover, we require the knowledge of part of the second spectrum if we want to prove the uniqueness on a more than half interval. However, it is outside the scope of this paper, the interested reader may consult [2, 7, 10, 11] for recent developments.

Remark 4.4. In [23] the authors showed that if the function either q_0 or q_1, but not both, is assumed to be known a priori, then the pencil can be uniquely determined by the given interior spectral data and spectrum. From this point of view, the result obtained here is new and a natural generalization of the well-known ones.

Proof of Theorem 4.1. Let us consider another differential pencil $\tilde{L}$ of the same form (1.1)-(1.3) but with different coefficients $(\tilde{q}_0,\tilde{q}_1,\tilde{h},\tilde{H})$. Then both pencils have common eigenvalues $\{\rho_n\}_{\rho_n\in\sigma_2}$ and common interior spectral data $\{m(a,\rho_n)\}_{\rho_n\in\sigma_1}$.

Moreover, $\tilde{q}_0(x) = q_0(x)$ a.e. and $\tilde{q}_1(x) = q_1(x)$ for $x \in [a_1, a_2]$. Under the hypothesis of Theorem 4.1, we will prove $L = \tilde{L}$ through the following two steps.

(1) We first show that $h = \tilde{h}$ and $q_j(x) = \tilde{q}_j(x)$ on $[0, a_1]$ for $j = 0, 1$. The proof of this step is similar to that of Theorem 3.1. Thus, we shall only give a sketch here. Define functions

$$g_{\sigma_1}(\rho) = \text{p.v.} \prod_{\rho_n \in \sigma_1} \left(1 - \frac{\rho}{\rho_n}\right) \tag{4.8}$$

and

$$F(\rho) = \frac{\langle \varphi, \tilde{\varphi} \rangle (a_1, \rho)}{g_{\sigma_1}(\rho)}. \tag{4.9}$$

From the condition $\sigma_1 \subset \hat{\sigma}$, we infer that $g_{\sigma_1}(\rho)$ only has simple zeros. Note that (3.5) and conditions $\tilde{q}_0(x) = q_0(x)$ a.e. and $\tilde{q}_1(x) = q_1(x)$ for $x \in [a_1, a_2]$ yield that $\langle \varphi, \tilde{\varphi} \rangle (a_1, \rho) = \langle \varphi, \tilde{\varphi} \rangle (x, \rho)$ for all $x \in [a_1, a_2]$. This follows from $m(a, \rho_n) = \tilde{m}(a, \rho_n)$ for $\rho_n \in \sigma_1$ that $m(a_1, \rho_n) = \tilde{m}(a_1, \rho_n)$. Therefore, the function $F(\rho)$ is entire.

If we take into account the asymptotics for eigenvalues $\{\rho_n\}_{n \in \mathbb{Z}}$ (see (2.6)), then it follows from (2.2) that

$$\langle \varphi, \tilde{\varphi} \rangle (a_1, \rho) = O(\exp(2a_1 |\text{Im}\rho|)) \tag{4.10}$$

as $|\rho| \to \infty$. Repeating the arguments of the proof of Theorem 3.1 we deduce

$$|g_{\sigma_1}(\rho)| \asymp |\hat{g}_{\sigma_1}(\rho)| \asymp |\bar{g}_{\sigma_1}(\rho - \omega_0)|.$$

Using Lemma 2.8 in [10], we obtain for every $\varepsilon > 0$ that

$$|g_{\sigma_1}(\rho)| \geq c \exp(2a_1 |\text{Im}\rho| - 2\varepsilon |\rho|)$$

for $|\rho - \rho_n| \geq \delta$ and $\rho_n \in \sigma_1$, where $c > 0$. This together with (4.9) and (4.10) yields that there exists a positive number C satisfying

$$|F(\rho)| \leq C \exp(2\varepsilon |\rho|) \tag{4.11}$$

for $|\rho - \rho_n| \geq \delta$, $|\rho - n\pi - \omega_0| \geq \delta$, $\rho_n \in \sigma_1$. By the maximum modulus principle the above inequality (4.11) also holds for all $\rho \in \mathbb{C}$, thus one easily checks that $F(\rho)$ is of zero exponential type according to Lemma 2.4.

We next show that $F(iy) \to 0$ as y (real) $\to \pm\infty$. From Lemma 2.1 we can assume $\rho_n \neq 0$ for all $n \in \mathbb{Z}$, without loss of generality, also assume that $N_\sigma(t) = 0$ for $-1 \leq t \leq 1$. By similar arguments as in the proof of Theorem 3.1, we obtain from (2.14) and (4.4) that

$$A = 2a_1, \qquad B_+ = \mu + \varepsilon - a_1, \qquad B_- = -\mu - a_1,$$

and therefore

$$|g_{\sigma_1}(iy)| \geq C|y|^{\varepsilon} \exp(2a_1|y|). \tag{4.12}$$

Using (4.10), we obtain $|\langle \varphi, \tilde{\varphi} \rangle (a_1, iy)| \leq C(\exp(2a_1|y|))$, which together with (4.9) and (4.12) yields that for $|y|$ sufficiently large

$$|F(iy)| \leq C \frac{\exp(2a_1|y|)}{|y|^{\varepsilon} \exp(2a_1|y|)} = O(|y|^{-\varepsilon}). \tag{4.13}$$

This implies that $|F(iy)| \to 0$ as $y \to \pm\infty$ (y real). By virtue of Lemma 2.4 one derives that $F \equiv 0$. Therefore, we obtain $\langle \varphi, \tilde{\varphi} \rangle (a_1, \rho) = 0$ and

$$\frac{\varphi(a_1, \rho)}{\varphi'(a_1, \rho)} = m(a_1, \rho) = \tilde{m}(a_1, \rho) = \frac{\tilde{\varphi}(a_1, \rho)}{\tilde{\varphi}'(a_1, \rho)} \tag{4.14}$$

for all $\rho \in \mathbb{C}$. According to the uniqueness theorem in [14], we get $h = \tilde{h}$ and $q_j(x) = \tilde{q}_j(x)$ on $[0, a_1]$ for $j = 0, 1$.

(2) We next show that $H = \tilde{H}$ and $q_j(x) = \tilde{q}_j(x)$ on $[a_2, 1]$ for $j = 0, 1$. Notice that here we have known $h = \tilde{h}$ and $q_j(x) = \tilde{q}_j(x)$ on $[0, a_2]$. In this situation, the uniqueness of determining q_j and H needs to be in virtue of the set σ_2 of common eigenvalues. This can be immediately derived from Theorem 3.1. Hence the proof of this theorem is complete. ∎

Proof of Theorem 4.2. The proof of this theorem is analogous to that of Theorem 4.1 and therefore is omitted. ∎

As a special case of Theorem 4.2, we have the following corollary.

Corollary 4.5. *Let $a_1 = a_2 = 1/2$. If $\sigma(L)$ consists of simple eigenvalues. Then $\sigma(L)$ and $\{m(1/2, \rho_n)\}_{n\in\mathbb{Z}}$ uniquely determine h, H and q_j on $[0, 1]$ for $j = 0, 1$.*

ACKNOWLEDGMENT

The authors would like to thank the referee for careful reading of the manuscript and helping us to improve the presentation of this paper. The research was supported in part by the NNSF (No. 11171198) and Fundamental Research Funds for the Central Universities (No. GK201304001 and GK201401004) of China.

REFERENCES

1. D. Borisov and P. Freitasb, Eigenvalue asymptotics, inverse problems and a trace formula for the linear damped wave equation, *J. Differential Equations*, **247** (2009), 3028-3039.

2. S. A. Buterin and C.-T. Shieh, Incomplete inverse spectral and nodal problems for differential pencils, *Results. Math.*, **62** (2012), 167-179.

3. S. A. Buterin and C.-T. Shieh, Inverse nodal problem for differential pencils, *Appl. Math. Lett.*, **22** (2009), 1240-1247.

4. S. A. Buterin and V. A. Yurko, Inverse spectral problem for pencils of differential operators on a finite interval, *Vestnik Bashkirsk University* (*Russian*), **4** (2006), 8-12.

5. M. G. Gasymov and G. Sh. Gusejnov, Determination of a diffusion operator from spectral data, *Akad. Nauk Azerb. SSR Dokl.*, **37** (1981), 19-23.

6. F. Gesztesy and B. Simon, Inverse spectral analysis with partial information on the potential: II. The case of discrete spectrum, *Trans. Am. Math. Soc.*, **352** (2000), 2765-2787.

7. G. S. Guseĭnov, Inverse spectral problems for a quadratic pencil of Sturm-Liouville operators on a finite interval, *Spectral Theory of Operators and Its Applications*, **7** (1986), 51-101.

8. I. M. Guseĭnov and I. M. Nabiev, An inverse spectral problem for pencils of differential operators, *Mat. Sb.*, **198** (2007), 47-66.

9. H. Hochstadt and B. Lieberman, An inverse Sturm-Liouville problem with mixed given data, *SIAM J. Appl. Math.*, **34** (1978), 676-680.

10. M. Horváth, On the inverse spectral theory of Schrödinger and Dirac operators, *Trans. Amer. Math. Soc.*, **353** (2001), 4155-4171.

11. R. Hryniv and N. Pronska, Inverse spectral problems for energy-dependent Sturm-Liouville equations, *Inverse Problems*, **28** (2012), 085008, 21 pp.

12. P. Jonas, On the spectral theory of operators associated with perturbed Klein-Gordon and wave type equations, *J. Operator Theory*, **29** (1993), 207-224.

13. A. G. Kostyuchenko and A. A. Shkalikov, Selfadjoint quadratic operator pencils and elliptic problems, *Funct. Anal. Appl.*, **17** (1983), 109-128.

14. H. Koyunbakan, A new inverse problem for the diffusion operator, *Appl. Math. Lett.*, **19** (2006), 995-999.

15. H. Koyunbakan and E. S. Panakhov, Half-inverse problem for diffusion operators on the finite interval, *J. Math. Anal. Appl.*, **326** (2007), 1024-1030.

16. H. Koyunbakan, Inverse problem for a quadratic pencil of Sturm-Liouville operator, *J. Math. Anal. Appl.*, **378** (2011), 549-554.

17. B. J. Levin, *Distribution of Zeros of Entire Functions*, AMS Transl., Vol. **5**, AMS, Providence, 1964.

18. B. M. Levitan and I. S. Sargsjan, *Sturm-Liouville and Dirac Operators*, Kluwer, Dordrecht, 1991.

19. A. S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, Shtinitsa, Kishinev, 1986; English transl., AMS, Providence, 1988.

20. K. Mochizuki and I. Trooshin, Inverse problem for interior spectral data of Sturm-Liouville operator, *J. Inverse Ill-Posed Probl.*, **9** (2001), 425-433.

21. C. Van der Mee and V. Pivovarchik, Inverse scattering for a Schrödinger equation with energy dependent potential, *J. Math. Phys.*, **42** (2001), 158-181.

22. C.-F. Yang and A. Zettl, Half inverse problems for quadratic pencils of Sturm-Liouville operators, *Taiwan. J. Math.*, **16** (2012), 1829-1846.

23. C.-F. Yang and Y. Guo, Determination of a differential pencil from interior spectral data, *J. Math. Anal. Appl.*, **375** (2011), 284-293.

24. M. Yamamoto, Inverse eigenvalue problem for a vibration of a string with viscous drag, *J. Math. Anal. Appl.*, **152** (1990), 20-34.

25. E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1939.

26. V. A. Yurko, An inverse problem for pencils of differential operators, *Matem. Sbornik*, **191** (2000), 137-160 (Russian); English transl. in *Sbornik: Mathematics*, **191** (2000), 1561-1586.

27. V. A. Yurko, *Inverse Spectral Problems for Linear Differential Operators and Their Applications*, Gordon and Breach, New York, 2000.

Yongxia Guo
College of Mathematics and Information Science
Shaanxi Normal University
Xi'an 710062
P. R. China
E-mail: hailang615@126.com

Guangsheng Wei
College of Mathematics and Information Science
Shaanxi Normal University
Xi'an 710062
P. R. China
E-mail: weimath@vip.sina.com