MORDUKHOVICH SUBGRADIENTS OF THE VALUE FUNCTION IN A PARAMETRIC OPTIMAL CONTROL PROBLEM

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Abstract. This paper is devoted to the study the first-order behavior of the value function of a parametric optimal control problem with linear constraints and a nonconvex cost function. By establishing an abstract result on the Mordukhovich subdifferential of the value function of a parametric mathematical programming problem, we derive a formula for computing the Mordukhovich subdifferential of the value function to a parametric optimal control problem.

1. INTRODUCTION

A wide variety of problems in optimal control problem can be posed in the following form.

Determine a control vector $u \in L^p([0,1], \mathbb{R}^m)$ and a trajectory $x \in W^{1,p}([0,1], \mathbb{R}^n)$, $1 < p < \infty$, which minimize the cost

$$(1) \qquad g\big(x(1)\big) + \int_0^1 L\big(t, x(t), u(t), \theta(t)\big)dt$$

with the state equation

$$(2) \qquad \dot{x}(t) = A(t)x(t) + B(t)u(t) + T(t)\theta(t) \text{ a.e. } t \in [0,1],$$

and the initial value

$$(3) \qquad x(0) = \alpha.$$

Here $W^{1,p}([0,1], \mathbb{R}^n)$ is the Sobolev space, which consists of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$ such that $\dot{x} \in L^p([0,1], \mathbb{R}^n)$. Its norm is given by

$$\|x\|_{1,p} = |x(0)| + \|\dot{x}\|_p.$$

The notations in (1)–(3) have the following meanings:

x, u are the state variable and the control variable, respectively,

$(\alpha, \theta) \in \mathbb{R}^n \times L^p([0,1], \mathbb{R}^k)$ are parameters,

$g : \mathbb{R}^n \to \bar{\mathbb{R}}$, $L : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \bar{\mathbb{R}}$ are given functions,

$A(t) = \big(a_{ij}(t)\big)_{n \times n}$, $B(t) = \big(b_{ij}(t)\big)_{n \times m}$ and $T(t) = \big(c_{ij}(t)\big)_{n \times k}$ are matrix-valued functions. Put

$$X = W^{1,p}([0,1], \mathbb{R}^n), \quad U = L^p([0,1], \mathbb{R}^m), \quad Z = X \times U,$$

$$\Theta = L^p([0,1], \mathbb{R}^k), \quad W = \mathbb{R}^n \times \Theta.$$

It is well known that X, U, Z, Θ and W are Asplund spaces. For each $w = (\alpha, \theta) \in W$, the optimal value and the solution set of the problem (1)-(3) corresponding to parameter $w \in W$ are denoted by $V(w)$ and $S(w)$, respectively. Thus,

$$V : W \to \bar{\mathbb{R}}$$

is an extended real-valued function, which is called the value function or the marginal function of the problem (1)–(3). It is assumed that V is finite at $\bar{w}$ and $\bar{z} = (\bar{x}, \bar{u})$ is a solution of the problem corresponding to a parameter $\bar{w}$, that is $\bar{z} = (\bar{x}, \bar{u}) \in S(\bar{w})$.

The study of the first-order behavior of value functions is important in variational analysis and optimization. An example is the study of distance functions and its applications to optimal control problems (see [2, 10, 28]). There are many papers dealing with differentiability properties and the Fréchet subdifferential of value functions (see [8, 9, 15, 16, 17, 20]). By considering a set of assumptions, which involves a kind of coherence property, Penot [20] showed that the value functions are Fréchet differentiable. The results of Penot gave sufficient conditions under which the value functions are Fréchet differentiable rather than formulas for computing their derivatives. In [17], Mordukhovich, Nam and Yen derived formulas for computing and estimating the Fréchet subdifferential and Mordukhovich subdifferential of value functions of parametric mathematical programming problems in Banach spaces.

Besides the study of the first-order behavior of value functions in parametric mathematical programming, the study of the first-order behavior of value functions in optimal control problems has attracted attention of many researchers (see [4, 6, 7, 12, 18, 19, 21, 22, 23, 24, 25, 26] and [27]). Recently, Toan and Kien [25] have derived a formula for an upper evaluation of the Fréchet subdifferential of the value function V for the case where g and L were not assumed to be convex. Under some assumptions which are weaker than the ones in [25], Chieu, Kien and Toan in [6] have obtained a formula for an upper evaluation of the Fréchet subdifferential of the value function V, which complements the results in [25].

However, we still could not find a suitable upper evaluation for the Mordukhovich subdifferential of the value function V in the case g and L not assumed to be convex.

It is known that Mordukhovich subdifferential is the outer limit of a family of the ε-Fréchet subdifferentials. Hence, upper estimates for the Mordukhovich subdifferential of the value function V are more complicated than the corresponding results in [6] and [25].

The aim of this paper is to derive some new formulas for computing the Mordukhovich subdifferential of V at $\bar{w}$ via the tool of generalized differentiation. In order to obtain the result, we first establish a formula for computing and estimating the Mordukhovich differential of value functions for a special class of parametric mathematical programming problems. We then apply the obtained result to our problem together with using some techniques of functional analysis. We note that upper estimates for the Fréchet subdifferential in [6] are at the same time upper estimates for the Mordukhovich subdifferential of the value function V if the solution map S is V-inner semicontinuous at $(\bar{w}, \bar{z})$. Besides, the Mordukhovich subdifferential of the value function V at $\bar{w}$ contains the Fréchet subdifferential of V at $\bar{w}$. So, the upper-evaluation in this paper is better than the corresponding evaluation in [6].

Let us recall some notions on generalized differentiation, which are related to our problem. The notions and results of generalized differentiation can be found in [3], [13] and [14]. Let Z be an Asplund space, $\varphi : Z \to \bar{R}$ be an extended real-valued function and $\bar{z} \in Z$ be such that $\varphi(\bar{z})$ is finite. For each $\varepsilon \geq 0$, the set

$$\hat{\partial}_\varepsilon \varphi(\bar{z}) := \left\{ z^* \in Z^* | \liminf_{z \to \bar{z}} \frac{\varphi(z) - \varphi(\bar{z}) - \langle z^*, z - \bar{z} \rangle}{\|z - \bar{z}\|} \geq -\varepsilon \right\}$$

is called the ε-*Fréchet subdifferential* of φ at $\bar{z}$. A given vector $z^* \in \hat{\partial}_\varepsilon \varphi(\bar{z})$ is called an ε-*Fréchet subgradient* of φ at $\bar{z}$. The set $\hat{\partial}\varphi(\bar{z}) = \hat{\partial}_0 \varphi(\bar{z})$ is called the *Fréchet subdifferential* of φ at $\bar{z}$ and the set

(4)
$$\partial \varphi(\bar{z}) := \operatorname{Limsup}_{\substack{z \xrightarrow{\varphi} \bar{z} \\ \varepsilon \downarrow 0}} \hat{\partial}_\varepsilon \varphi(z)$$

is called the *Mordukhovich subdifferential* of φ at $\bar{z}$, where the notation $z \xrightarrow{\varphi} \bar{z}$ means $z \to \bar{z}$ and $\varphi(z) \to \varphi(\bar{z})$. Hence

$$z^* \in \partial \varphi(\bar{z}) \iff \text{there exist sequences } z_k \xrightarrow{\varphi} \bar{z}, \varepsilon_k \to 0^+, \text{ and } z_k^* \in \hat{\partial}_{\varepsilon_k} \varphi(z_k)$$

such that $z_k^* \xrightarrow{w^*} z^*$. If φ is lower semicontinuous around $\bar{z}$, then we can equivalently put $\varepsilon = 0$ in (4). Moreover, we have $\partial \varphi(\bar{z}) \neq \emptyset$ for every locally Lipschitzian function. It is known that the Mordukhovich subdifferential reduces to the classical Fréchet derivative for strictly differentiable functions and to subdifferential of convex analysis for convex functions. The set

$$\partial^\infty \varphi(\bar{z}) := \operatorname{Limsup}_{\substack{z \xrightarrow{\varphi} \bar{z} \\ \lambda, \varepsilon \downarrow 0}} \lambda \hat{\partial}_\varepsilon \varphi(z)$$

is called the *singular subdifferential* of φ at $\bar{z}$. Hence

$$z^* \in \partial^\infty \varphi(\bar{z}) \Longleftrightarrow \text{there exist sequences } z_k \xrightarrow{\varphi} \bar{z}, \varepsilon_k \to 0^+, \lambda_k \to 0^+,$$

$$\text{and } z_k^* \in \lambda_k \widehat{\partial}_{\varepsilon_k} \varphi(z_k) \text{ such that } z_k^* \xrightarrow{w^*} z^*.$$

Let Ω be a nonempty set in Z and $z_0 \in \Omega$. The set

$$\widehat{N}_\varepsilon(z_0; \Omega) := \left\{ z^* \in Z^* \mid \limsup_{z \xrightarrow{\Omega} z_0} \frac{\langle z^*, z - z_0 \rangle}{\|z - z_0\|} \leq \varepsilon \right\}$$

is called the ε-*Fréchet normal cone* to Ω at z_0 and the set

$$N(z_0; \Omega) := \lim_{\substack{z \xrightarrow{\Omega} z_0 \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(z; \Omega)$$

is called the *Mordukhovich normal cone* to Ω at z_0. It is also known that if Ω is a convex set, then the Mordukhovich normal cone coincides with the Fréchet normal cone and coincides with normal cone of convex analysis for convex sets.

A set Ω is called *sequentially normally compact* (SNC) at $\bar{z}$ if for any sequences $\varepsilon_k \downarrow 0, z_k \xrightarrow{\Omega} \bar{z}$, and $z_k^* \in \widehat{N}_{\varepsilon_k}(z_k; \Omega)$ one has

$$\left[z_k^* \xrightarrow{w^*} 0 \right] \Longrightarrow \left[\|z_k^*\| \to 0 \right] \text{ as } k \to \infty,$$

where ε_k can be omitted if Ω is locally closed around $\bar{z}$. An extended real-valued function φ on Z is called *sequentially normally epi-compact* (SNEC) at $\bar{z}$ if its epigraph is SNC at $(\bar{z}, \varphi(\bar{z}))$.

We say that a set-valued map $F : Z \rightrightarrows E$, where Z and E are Asplund spaces, admits a *locally upper Lipschitzian selection* at $(\bar{z}, \bar{v}) \in \text{gph} F := \{(z, v) \in Z \times E \mid v \in F(z)\}$ if there is a single-valued mapping $\phi : Z \to E$, which is locally upper Lipschitzian at $\bar{z}$, that is there exist numbers $\eta > 0$ and $l > 0$ such that

$$\|\phi(z) - \phi(\bar{z})\| \leq l \|z - \bar{z}\| \quad \text{whenever} \quad z \in B(\bar{z}, \eta),$$

which satisfies $\phi(\bar{z}) = \bar{v}$ and $\phi(z) \in F(z)$ for all z in a neighborhood of $\bar{z}$.

We now return to the problem (1)–(3). For each $w = (\alpha, \theta) \in W$, we put

$$J(x, u, w) = g(x(1)) + \int_0^1 L(t, x(t), u(t), \theta(t)) dt$$

and

(5) $$G(w) = \{z = (x, u) \in X \times U \mid (2) \text{ and } (3) \text{ are satisfied}\}.$$

Then the problem (1)–(3) can be formulated in the following form:

$$(6) \qquad V(w) = \inf_{z \in G(w)} J(z, w).$$

We say that the solution map $S(w)$ is *V-inner semicontinuous* at $(\bar{w}, \bar{z})$ if for every sequence $w_k \xrightarrow{V} \bar{w}$ $\big($i.e., $w_k \to \bar{w}$ and $V(w_k) \to V(\bar{w})\big)$, there is a sequence $\{z_k\}$ with $z_k \in S(w_k)$ for all k, which contains a subsequence converging to $\bar{z}$.

To deal with our problem, we impose the following assumptions:

$(H1)$ The functions $L : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \bar{\mathbb{R}}$ and $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ have properties that $L(\cdot, x, u, v)$ is measurable for all $(x, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$, $L(t, \cdot, \cdot, \cdot)$ and $g(\cdot)$ are continuously differentiable for almost every $t \in [0, 1]$, and there exist constants $c_1 > 0, c_2 > 0, r \geq 0$, a nonnegative function $\omega_1 \in L^p([0, 1], \mathbb{R})$, constants $0 \leq p_1 \leq p$, $0 \leq p_2 \leq p - 1$ such that

$$|L(t, x, u, v)| \leq c_1\big(\omega_1(t) + |x|^{p_1} + |u|^{p_1} + |v|^{p_1}\big),$$
$$\max\big\{|L_x(t, x, u, v)|, |L_u(t, x, u, v)|, |L_v(t, x, u, v)|\big\} \leq c_2\big(|x|^{p_2} + |u|^{p_2} + |v|^{p_2}\big) + r$$

for all $(t, x, u, v) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$.

$(H2)$ The matrix-valued functions $A : [0, 1] \to M_{n,n}(\mathbb{R})$, $B : [0, 1] \to M_{n,m}(\mathbb{R})$ and $T : [0, 1] \to M_{n,k}(\mathbb{R})$ are measurable and essentially bounded.

$(H3)$ There exists a constant $c_3 > 0$ such that

$$|T^T(t)v| \geq c_3|v| \quad \forall\, v \in \mathbb{R}^n, \text{ a.e. } t \in [0, 1].$$

We are now ready to state our main result.

Theorem 1.1. *Suppose that the solution map S is V-inner semicontinuous at $(\bar{w}, \bar{z}) \in \mathrm{gph}S$, and assumptions $(H1)$–$(H3)$ are fulfilled. Then for a vector $(\alpha^*, \theta^*) \in \mathbb{R}^n \times L^q([0, 1], \mathbb{R}^k)$ to be a Mordukhovich subgradient of V at $(\bar{\alpha}, \bar{\theta})$, it is necessary that there exists a function $y \in W^{1,q}([0, 1], \mathbb{R}^n)$ such that the following conditions are satisfied:*

$$(7) \qquad \alpha^* = g'\big(\bar{x}(1)\big) + \int_0^1 L_x\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)dt - \int_0^1 A^T(t)y(t)dt,$$

$$(8) \qquad y(1) = -g'\big(\bar{x}(1)\big)$$

and

$$\Big(\dot{y}(t) + A^T(t)y(t), B^T(t)y(t), \theta^*(t) + T^T(t)y(t)\Big)$$
$$(9) \qquad = \nabla L\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big) \text{ a.e. } t \in [0, 1].$$

The above conditions are also sufficient for $(\alpha^*, \theta^*) \in \partial V(\bar\alpha, \bar\theta)$ if the solution map S has a locally upper Lipschitzian selection at $(\bar w, \bar x, \bar u)$. Here, A^T stands for the transpose of A, $\nabla L\big(t, \bar x(t), \bar u(t), \bar\theta(t)\big)$ stands for the gradient of $L(t, \cdot, \cdot, \cdot)$ at $\big(\bar x(t), \bar u(t), \bar\theta(t)\big)$ and q is the conjugate number of p, that is, $1 < q < +\infty$ and $1/p + 1/q = 1$.

In order to prove Theorem 1.1 we first reduce the problem (1)–(3) to a mathematical programming problem and then establish some formulas for computing the Mordukhovich subdifferential of its value function. This procedure is presented in Section 2. A complete proof for Theorem 1.1 will be provided in Section 3.

2. THE OPTIMAL CONTROL PROBLEM AS A PROGRAMMING PROBLEM

In this section, we suppose that X, W and Z are Asplund spaces with the dual spaces X^*, W^* and Z^*, respectively. Assume that $M : Z \to X$ and $T : W \to X$ are continuous linear mappings. Let $M^* : X^* \to Z^*$ and $T^* : X^* \to W^*$ be adjoint mappings of M and T, respectively. Let function $f : W \times Z \to \bar R$ be lower semicontinuous around $(\bar w, \bar z)$. For each $w \in W$, we put

$$H(w) := \big\{ z \in Z| \; Mz = Tw \big\}.$$

Consider the problem of computing the Mordukhovich subdifferential of the value function

(10)
$$h(w) := \inf_{z \in H(w)} f(w, z).$$

This is an abstract model for (6).

We denote by $\hat S(w)$ the solution set of (10) corresponding to parameter $w \in W$. Assume that the value function h is finite at $\bar w$ and $\bar z$ is a solution of the problem corresponding to a parameter $\bar w$, that is $\bar z \in \hat S(\bar w)$.

We now derive a formula for computing the Mordukhovich subdifferential and the singular subdifferential of the value function to the parametric programming problem (10). Note that the structure of the formulas for evaluation of the Mordukhovich subdifferential $\partial h(\bar w)$ and singular subdifferential $\partial^\infty h(\bar w)$ is different from (although having some similarities) those established in [6]. The main differences are that we do not assume $\partial^+ f(\bar w, \bar z) \neq \emptyset$ and instead of using the intersection over $(w^*, z^*) \in \partial^+ f(\bar w, \bar z)$ as in [6, Theorem 3.1], we use the union over $(w^*, z^*) \in \partial f(\bar w, \bar z)$ or $(w^*, z^*) \in \partial^\infty f(\bar w, \bar z)$ to evaluate $\partial h(\bar w)$ and $\partial^\infty h(\bar w)$. On the other hand, we need additional assumptions that the objective function f is SNEC at $(\bar w, \bar z)$ and the solution map $\hat S$ is h-inner semicontinuous at $(\bar w, \bar z)$.

Theorem 2.1. *Suppose the solution map $\hat S$ is h-inner semicontinuous at $(\bar w, \bar z) \in$ gph$\hat S$, f is SNEC at $(\bar w, \bar z)$ and*

(i) *the following qualification condition is satisfied*

$$\left[\partial^\infty f(\bar{w}, \bar{z})\right] \cap \left[\left\{(T^*x^*, -M^*x^*) : x^* \in X^*\right\}\right] = \{0\},$$

(ii) *there exists a constant $c > 0$ such that*

$$\|T^*x^*\| \geq c\|x^*\|, \quad \forall x^* \in X^*.$$

Then one has

(11)
$$\partial h(\bar{w}) \subseteq \bigcup_{(w^*, z^*) \in \partial f(\bar{w}, \bar{z})} \left[w^* + T^*\left((M^*)^{-1}(z^*)\right)\right].$$

(12)
$$\partial^\infty h(\bar{w}) \subseteq \bigcup_{(w^*, z^*) \in \partial^\infty f(\bar{w}, \bar{z})} \left[w^* + T^*\left((M^*)^{-1}(z^*)\right)\right].$$

Moreover, assume that f is strictly differentiable at $(\bar{w}, \bar{z})$ and the solution map $\hat{S}$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{z})$. Then

(13)
$$\partial h(\bar{w}) = \nabla_w f(\bar{w}, \bar{z}) + T^*\left((M^*)^{-1}(\nabla_z f(\bar{w}, \bar{z}))\right).$$

For the proof of this theorem, we need the following lemma from [6]. Here we put $Q = \text{gph} H$.

Lemma 2.2. *Suppose that assumptions of Theorem 2.1 are satisfied. Then for each $(\bar{w}, \bar{z}) \in Q$, one has*

$$N\left((\bar{w}, \bar{z}); Q\right) = \left\{(-T^*x^*, M^*x^*)|x^* \in X^*\right\}.$$

We now consider the mapping $\vartheta : W \times Z \to \bar{\mathbb{R}}$ defined by setting

$$\vartheta(w, z) = f(w, z) + \delta\left((w, z); Q\right),$$

where $\delta\left(\cdot; Q\right)$ is the indicator function of Q, that is $\delta\left((w, z); Q\right) = 0$ if $(w, z) \in Q$ and $\delta\left((w, z); Q\right) = +\infty$ otherwise.

Proof of Theorem 2.1. Take any $w^* \in \partial h(\bar{w})$. Then there exist sequences $w_k \xrightarrow{h} \bar{w}, \varepsilon_k \to 0^+$, and $w_k^* \in \hat{\partial}_{\varepsilon_k} h(w_k)$ such that $w_k^* \xrightarrow{w^*} w^*$ as $k \to \infty$. Hence, there is a sequence $\eta_k \downarrow 0$ such that

$$\langle w_k^*, w - w_k \rangle \leq h(w) - h(w_k) + 2\varepsilon_k \| w - w_k \|, \quad \forall w \in w_k + \eta_k B_W.$$

So

$$\langle w_k^*, w - w_k \rangle + \langle 0, z - z_k \rangle \leq \vartheta(w, z) - \vartheta(w_k, z_k) + 2\varepsilon_k(\| w - w_k \| + \| z - z_k \|),$$

for all $z_k \in \hat{S}(w_k), (w, z) \in (w_k, z_k) + \eta_k B_{W \times Z}, \forall k \in \mathbb{N}$. This gives $(w_k^*, 0) \in \hat{\partial}_{\widetilde{\varepsilon}_k} \vartheta(w_k, z_k)$ with $\widetilde{\varepsilon}_k = 2\varepsilon_k$. Since $\hat{S}$ is h-inner semicontinuous at $(\bar{w}, \bar{z})$, we can select a sequence of $z_k \in \hat{S}(w_k)$ converging to $\bar{z}$. Observe that $\vartheta(w_k, z_k) \to \vartheta(\bar{w}, \bar{z})$ due to $h(w_k) \to h(\bar{w})$. Hence

$$\partial h(\bar{w}) \subset \big\{ w^* \in W^* : (w^*, 0) \in \partial \vartheta(\bar{w}, \bar{z}) \big\}$$

$$= \big\{ w^* \in W^* : (w^*, 0) \in \partial \big[f(w, z) + \delta\big((w, z); Q\big) \big] (\bar{w}, \bar{z}) \big\}.$$

From the assumed SNEC property of f at $(\bar{w}, \bar{z})$, condition (i) and [13, Theorem 3.36], it follows that

$$\partial h(\bar{w}) \subset \big\{ w^* \in W^* : (w^*, 0) \in \partial f(\bar{w}, \bar{z}) + N\big((\bar{w}, \bar{z}); Q\big) \big\},$$

which is equivalent to

$$\partial h(\bar{w}) \subset \Big\{ w^* \in W^* : (w^*, 0) \in \bigcup_{(w_1^*, z_1^*) \in \partial f(\bar{w}, \bar{z})} \big[(w_1^*, z_1^*) + N\big((\bar{w}, \bar{z}); Q\big) \big] \Big\}.$$

By Lemma 2.2, there exists $x^* \in X^*$ such that

$$w^* - w_1^* = -T^*(x^*) \text{ and } -z_1^* = M^*(x^*),$$

for some $(w_1^*, z_1^*) \in \partial f(\bar{w}, \bar{z})$. Hence

$$w^* = w_1^* + T^*(-x^*) \text{ and } -x^* = (M^*)^{-1}(z_1^*).$$

Consequently
$$w^* \in w_1^* + T^*\big[(M^*)^{-1}(z_1^*) \big],$$

for some $(w_1^*, z_1^*) \in \partial f(\bar{w}, \bar{z})$. We obtain the inclusion (11).

To prove the inclusion (12), take any $w^* \in \partial^\infty h(\bar{w})$. Then there exist sequences $w_k \xrightarrow{h} \bar{w}, \varepsilon_k \to 0^+, \xi_k \to 0^+$, and $w_k^* \in \xi_k \hat{\partial}_{\varepsilon_k} h(w_k)$ such that $w_k^* \xrightarrow{w^*} w^*$ as $k \to \infty$. So $\frac{1}{\xi_k} w_k^* \in \hat{\partial}_{\varepsilon_k} h(w_k), \forall k \in \mathbb{N}$. Hence, there is a sequence $\eta_k \downarrow 0$ such that

$$\langle \frac{1}{\xi_k} w_k^*, w - w_k \rangle \leq h(w) - h(w_k) + 2\varepsilon_k \| w - w_k \|, \quad \forall w \in w_k + \eta_k B_W.$$

So

$$\langle \frac{1}{\xi_k} w_k^*, w - w_k \rangle + \langle 0, z - z_k \rangle \leq \vartheta(w, z) - \vartheta(w_k, z_k) + 2\varepsilon_k(\| w - w_k \| + \| z - z_k \|),$$

for all $z_k \in \hat{S}(w_k), (w, z) \in (w_k, z_k) + \eta_k B_{W \times Z}, \forall k \in \mathbb{N}$. This gives $\frac{1}{\xi_k}(w_k^*, 0) \in \hat{\partial}_{\widetilde{\varepsilon}_k} \vartheta(w_k, z_k)$ with $\widetilde{\varepsilon}_k = 2\varepsilon_k$. So $(w_k^*, 0) \in \xi_k \hat{\partial}_{\widetilde{\varepsilon}_k} \vartheta(w_k, z_k)$. Since $\hat{S}$ is h-inner semicontinuous at $(\bar{w}, \bar{z})$, we can select a sequence of $z_k \in \hat{S}(w_k)$ converging to $\bar{z}$. Observe that $\vartheta(w_k, z_k) \to \vartheta(\bar{w}, \bar{z})$ due to $h(w_k) \to h(\bar{w})$. Hence

$$\partial^\infty h(\bar{w}) \subset \left\{ w^* \in W^* : (w^*, 0) \in \partial^\infty \vartheta(\bar{w}, \bar{z}) \right\}$$

$$= \left\{ w^* \in W^* : (w^*, 0) \in \partial^\infty \left[f(w, z) + \delta\big((w, z); Q\big) \right](\bar{w}, \bar{z}) \right\}.$$

From the assumed SNEC property of f at $(\bar{w}, \bar{z})$, condition (i), [13, Theorem 3.36] and [13, Prosition 1.79], it follows that

$$\partial^\infty h(\bar{w}) \subset \left\{ w^* \in W^* : (w^*, 0) \in \partial^\infty f(\bar{w}, \bar{z}) + N\big((\bar{w}, \bar{z}); Q\big) \right\}.$$

By using similar arguments, we can show that

$$w^* \in w_2^* + T^* \left[(M^*)^{-1}(z_2^*) \right],$$

for some $(w_2^*, z_2^*) \in \partial^\infty f(\bar{w}, \bar{z})$. Hence, we obtain the first assertion.

In order to prove the second assertion, we will prove

$$\partial h(\bar{w}) \supset \left[\nabla_w f(\bar{w}, \bar{z}) + T^* \big((M^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})) \big) \right].$$

Since $\hat{\partial} h(\bar{w}) \subset \partial h(\bar{w})$, it is sufficient to show that

$$\hat{\partial} h(\bar{w}) \supset \left[\nabla_w f(\bar{w}, \bar{z}) + T^* \big((M^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})) \big) \right].$$

On the contrary, suppose that there exists $w^* \in W^*$ such that

$$(14) \qquad w^* \in \left[\nabla_w f(\bar{w}, \bar{z}) + T^* \big((M^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})) \big) \right] \backslash \hat{\partial} h(\bar{w}).$$

Then we can find $\bar{\gamma} > 0$ and a sequence $w_k \longrightarrow \bar{\omega}$ such that

$$(15) \qquad \langle w^*, w_k - \bar{w} \rangle > h(w_k) - h(\bar{w}) + \bar{\gamma} ||w_k - \bar{w}||, \quad \forall k \in \mathbb{N}.$$

Let ϕ be an upper Lipschitzian selection of the solution map $\hat{S}$. Putting $z_k = \phi(w_k)$, we have $z_k \in \hat{S}(w_k)$ and $||z_k - \bar{z}|| \leqslant l ||w_k - \bar{w}||$ for $k > 0$ sufficiently large. Hence, (15) implies

$$\langle w^*, w_k - \bar{w} \rangle > h(w_k) - h(\bar{w}) + \bar{\gamma} ||w_k - \bar{w}||$$

$$= f(w_k, z_k) - f(\bar{w}, \bar{z}) + \bar{\gamma} ||w_k - \bar{w}||$$

$$= \langle \nabla_z f(\bar{w}, \bar{z}), z_k - \bar{z} \rangle + \langle \nabla_w f(\bar{w}, \bar{z}), w_k - \bar{w} \rangle$$

$$+ 0(||z_k - \bar{z}|| + ||w_k - \bar{w}||) + \bar{\gamma} ||w_k - \bar{w}||$$

$$\geqslant \langle \nabla_z f(\bar{w}, \bar{z}), z_k - \bar{z} \rangle + \langle \nabla_w f(\bar{w}, \bar{z}), w_k - \bar{w} \rangle$$

$$+ 0(||z_k - \bar{z}|| + ||w_k - \bar{w}||) + \frac{\bar{\gamma}}{2} ||w_k - \bar{w}|| + \frac{\bar{\gamma}}{2l} ||z_k - \bar{z}||.$$

Putting $\hat{\gamma} = \min\{\frac{\bar{\gamma}}{2}, \frac{\bar{\gamma}}{2l}\}$, we get from the above inequality that

$$\langle (w^* - \nabla_w f(\bar{w}, \bar{z}), -\nabla_z f(\bar{w}, \bar{z})), (w_k - \bar{w}, z_k - \bar{z})\rangle$$
$$> 0(||z_k - \bar{z}|| + ||w_k - \bar{w}||) + \hat{\gamma}(||z_k - \bar{z}|| + ||w_k - \bar{w}||).$$

Consequently,

$$\limsup_{(w,z) \xrightarrow{Q} (\bar{w}, \bar{z})} \frac{\langle (w^* - \nabla_w f(\bar{w}, \bar{z}), -\nabla_z f(\bar{w}, \bar{z})), (w - \bar{w}, z - \bar{z})\rangle}{||z - \bar{z}|| + ||w - \bar{w}||} \geqslant \hat{\gamma},$$

which means that

$$\left(w^* - \nabla_w f(\bar{w}, \bar{z}), -\nabla_z f(\bar{w}, \bar{z})\right) \notin N\left((\bar{w}, \bar{z}); Q\right).$$

By Lemma 2.2, we have

(16) $\left(w^* - \nabla_w f(\bar{w}, \bar{z}), -\nabla_z f(\bar{w}, \bar{z})\right) \notin N\left((\bar{w}, \bar{z}); Q\right) = \left\{(-T^* x^*, M^* x^*) | x^* \in X^*\right\}.$

From (14),
$$w^* - \nabla_w f(\bar{w}, \bar{z}) \in T^*\left((M^*)^{-1}(\nabla_z f(\bar{w}, \bar{z}))\right).$$

So there exists $x^* \in (M^*)^{-1}\left(\nabla_z f(\bar{w}, \bar{z})\right)$ such that

$$w^* - \nabla_w f(\bar{w}, \bar{z}) = T^*(x^*).$$

Hence

$$w^* - \nabla_w f(\bar{w}, \bar{z}) = -T^*(-x^*) \quad \text{and} \quad -\nabla_z f(\bar{w}, \bar{z}) = M^*(-x^*).$$

Consequently,

$$\left(w^* - \nabla_w f(\bar{w}, \bar{z}), -\nabla_z f(\bar{w}, \bar{z})\right) \in N\left((\bar{w}, \bar{z}); Q\right) = \left\{(-T^* v^*, M^* v^*) | v^* \in X^*\right\},$$

which contradicts (16). Thus, the second assertion is valid and the proof of the theorem is complete. ∎

The next example demonstrates that the inclusion (11) of the Theorem 2.1 may hold as equality with no strict differentiability assumption on the objective function f as in (13).

Example 2.1 Let $Z = \mathbb{R}^3$, $W = \mathbb{R}^2$,

$$f(w, z) = \sqrt{z_1^3 + 3|w_1|z_2} - w_1^3 - |w_2|$$

and $H(w) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 + z_2 = 2w_1, z_3 = 2w_2\}$. Assume that $\bar{w} = (1, 0)$. Then assumptions of Theorem 2.1 are satisfied and $\partial h(\bar{w}) = \{(-\frac{3}{4}, -1); (-\frac{3}{4}, 1)\}$. Besides, $\hat{\partial} h(\bar{w}) = \emptyset$.

Indeed, for $\bar{w} = (1, 0)$ we have

$$h(\bar{w}) = \inf_{(z_1, z_2, z_3) \in H(\bar{w})} \{\sqrt{z_1^3 + 3z_2} - 1\},$$

where $H(\bar{w}) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 + z_2 = 2, z_3 = 0\}$. It is easy to check that $\bar{z} = (1, 1, 0)$ is the unique solution of the problem corresponding to $\bar{w}$ and therefore $h(\bar{w}) = 1$.

By a direct computation, we see that

$$\hat{S}(w) = \left\{ \left(\sqrt{|w_1|}, 2w_1 - \sqrt{|w_1|}, 2w_2 \right) \right\},$$

where $w = (w_1, w_2)$. Thus, the solution map $\hat{S} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^3$ is h-inner simicontinuous at $(\bar{w}, \bar{z})$. It is easy to check that f is locally Lipschitzian around $(\bar{w}, \bar{z})$. By [13, Corollary 1.81], we have $\partial^\infty f(\bar{w}, \bar{z}) = \{0\}$. So, condition (i) of Theorem 2.1 is satisfied. Note that

$$M^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathrm{T}^* = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

from which it follows that condition (ii) of Theorem 2.1 is satisfied. It is known that if $g(w) = -|w|$, then $\partial g(0) = \{-1, 1\}$ and $\hat{\partial} g(0) = \emptyset$. Hence, (11) becomes

$$(17) \quad \partial h(\bar{w}) \subset \bigcup_{(w^*, z^*) \in \partial f(\bar{w}, \bar{x})} \left[w^* + T^*\left((M^*)^{-1}(z^*) \right) \right]$$

$$= \left\{ \left(-\frac{9}{4}, -1 \right) + T^*\left((M^*)^{-1}\left(\frac{3}{4}, \frac{3}{4}, 0 \right) \right); \left(-\frac{9}{4}, 1 \right) + T^*\left((M^*)^{-1}\left(\frac{3}{4}, \frac{3}{4}, 0 \right) \right) \right\}.$$

We have $(M^*)^{-1}(\frac{3}{4}, \frac{3}{4}, 0) = (\frac{3}{4}, 0)$ and $T^*\left((M^*)^{-1}(\frac{3}{4}, \frac{3}{4}, 0) \right) = (\frac{3}{2}, 0)$. Combining this with (17), we get $\partial h(\bar{w}) \subset \{(-\frac{3}{4}, -1); (-\frac{3}{4}, 1)\}$. By computing directly, we see that $\partial h(\bar{w}) = \{(-\frac{3}{4}, -1); (-\frac{3}{4}, 1)\}$ and $\hat{\partial} h(\bar{w}) = \emptyset$.

The next example demonstrates that the upper Lipschitzian assumption of Theorem 2.1 is sufficient but not necessary to ensure the equality in the Mordukhovich subgradient inclusion (13) for value functions.

Example 2.2 Let $Z = \mathbb{R}^3$, $W = \mathbb{R}^2$,

$$f(w, z) = (w_1 - z_1^2)^2$$

and $H(w) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2 = 2w_1, z_3 = 2w_2\}$. Assume that $\bar{w} = (0,0)$. Then assumptions of Theorem 2.1 are satisfied and $\partial h(\bar{w}) = \{(0,0)\}$ but the solution map $\hat{S}(\cdot)$ does not admit an upper Lipschitzian selection.

Indeed, for $\bar{w} = (0,0)$ we have

$$h(\bar{w}) = \inf_{(z_1, z_2, z_3) \in H(\bar{w})} z_1^4,$$

where $H(\bar{w}) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2 = 0, z_3 = 0\}$. It is easy to check that $\bar{z} = (0,0,0)$ is the unique solution of the problem corresponding to $\bar{w}$ and therefore $h(\bar{w}) = 0$.

By a direct computation, we see that

$$\hat{S}(w) = \begin{cases} \{(-\sqrt{w_1}, 2w_1, 2w_2), (\sqrt{w_1}, 2w_1, 2w_2)\} & \text{if } w_1 > 0 \\ \{(0, 2w_1, 2w_2)\} & \text{if } w_1 \leq 0, \end{cases}$$

where $w = (w_1, w_2)$. Thus, the solution map $\hat{S} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^3$ is h-inner simicontinuous at $(\bar{w}, \bar{z})$. But $\hat{S}$ has no locally upper Lipschitzian selection at $(\bar{w}, \bar{z})$. It is easy to check that f is locally Lipschitzian around $(\bar{w}, \bar{z})$. By [13, Corollary 1.81], we have $\partial^\infty f(\bar{w}, \bar{z}) = \{0\}$. So, condition (i) of Theorem 2.1 is satisfied. Note that

$$M^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathrm{T}^* = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

from which it follows that condition (ii) of Theorem 2.1 is satisfied. Hence, (11) becomes

$$\partial h(\bar{w}) \subset \left[\nabla_w f(\bar{z}, \bar{w}) + T^* \big((M^*)^{-1} (\nabla_z f(\bar{z}, \bar{w})) \big) \right]$$

$$\subset \left[(0,0) + T^* \big((M^*)^{-1} (0,0,0) \big) \right].$$

Thus, we get $\partial h(\bar{w}) \subset \{(0,0)\}$. By computing directly, we see that

$$h(w) = \begin{cases} w_1^2 & \text{if } w_1 \leq 0 \\ 0 & \text{if } w_1 > 0, \end{cases}$$

Hence, we have $\partial h(\bar{w}) = \{(0,0)\}$.

3. PROOF OF THE MAIN RESULT

To prove Theorem 1.1, we first formulate problem (1)–(3) in the form to which Theorem 2.1 can be applied to. We now consider the following linear mappings:

$\mathcal{A} : X \to X$ defined by

$$\mathcal{A}x := x - \int_0^{(\cdot)} A(\tau)x(\tau)d\tau,$$

$\mathcal{B} : U \to X$ defined by

$$\mathcal{B}u := - \int_0^{(\cdot)} B(\tau)u(\tau)d\tau,$$

$\mathcal{M} : X \times U \to X$ defined by

(18)
$$\mathcal{M}(x, u) := \mathcal{A}x + \mathcal{B}u$$

and $\mathcal{T} : W \to X$ defined by

(19)
$$\mathcal{T}(\alpha, \theta) := \alpha + \int_0^{(\cdot)} T(\tau)\theta(\tau)d\tau.$$

Under the hypotheses (H_2) and (H_3), (5) can be written in the form

$$G(w) = \left\{ (x, u) \in X \times U \mid x = \alpha + \int_0^{(\cdot)} Ax d\tau + \int_0^{(\cdot)} Bu d\tau + \int_0^{(\cdot)} T\theta d\tau \right\}$$
$$= \left\{ (x, u) \in X \times U \mid x - \int_0^{(\cdot)} Ax d\tau - \int_0^{(\cdot)} Bu d\tau = \alpha + \int_0^{(\cdot)} T\theta d\tau \right\}$$
$$= \left\{ (x, u) \in X \times U \mid \mathcal{M}(x, u) = \mathcal{T}(w) \right\}.$$

Recall that for $1 < p < \infty$, we have $L^p([0, 1], \mathbb{R}^n)^* = L^q([0, 1], \mathbb{R}^n)$, where

$$1 < q < +\infty, \quad 1/p + 1/q = 1.$$

Besides, $L^p([0, 1], \mathbb{R}^n)$ is pared with $L^q([0, 1], \mathbb{R}^n)$ by the formula

$$\langle x^*, x \rangle = \int_0^1 x^*(t)x(t)dt,$$

for all $x^* \in L^q([0, 1], \mathbb{R}^n)$ and $x \in L^p([0, 1], \mathbb{R}^n)$.

Also, we have $W^{1,p}([0, 1], \mathbb{R}^n)^* = \mathbb{R}^n \times L^q([0, 1], \mathbb{R}^n)$ and $W^{1,p}([0, 1], \mathbb{R}^n)$ is pared with $\mathbb{R}^n \times L^q([0, 1], \mathbb{R}^n)$ by the formula

$$\langle (a, u), x \rangle = \langle a, x(0) \rangle + \int_0^1 u(t)\dot{x}(t)dt,$$

for all $(a, u) \in \mathbb{R}^n \times L^q([0, 1], \mathbb{R}^n)$ and $x \in W^{1,p}([0, 1], \mathbb{R}^n)$ (see [11, p. 21]).

In the case of $p = 2$, $W^{1,2}([0, 1], \mathbb{R}^n)$ becomes a Hilbert space with the inner product given by

$$\langle x, y \rangle = \langle x(0), y(0) \rangle + \int_0^1 \dot{x}(t)\dot{y}(t)dt,$$

for all $x, y \in W^{1,2}([0, 1], \mathbb{R}^n)$.

In the sequel, we shall need the following lemmas.

Lemma 3.1. ([25, Lemma 2.3]). *Suppose that $\mathcal{M}^*$ and $\mathcal{T}^*$ are adjoint mappings of $\mathcal{M}$ and $\mathcal{T}$, respectively. Then the following assertions are valid:*

(a) *The mappings $\mathcal{M}$ and $\mathcal{T}$ are continuous.*

(b) *$\mathcal{T}^*(a,u) = \left(a, T^T u\right)$ for all $(a,u) \in \mathbb{R}^n \times L^q([0,1], \mathbb{R}^n)$.*

(c) *$\mathcal{M}^*(a,u) = \left(\mathcal{A}^*(a,u), \mathcal{B}^*(a,u)\right)$, where $\mathcal{B}^*(a,u) = -B^T u$ and*

$$\mathcal{A}^*(a,u) = \left(a - \int_0^1 A^T(t)u(t)dt; u + \int_0^{(\cdot)} A^T(\tau)u(\tau)d\tau - \int_0^1 A^T(t)u(t)dt\right),$$

for all $(a,u) \in \mathbb{R}^n \times L^q([0,1], \mathbb{R}^n)$.

Recall that

$$Z = X \times U$$

and

$$G(w) = \left\{(x,u) \in X \times U \mid \mathcal{M}(x,u) = \mathcal{T}(w)\right\}.$$

Then our problem can be written in the form

$$V(w) := \inf_{z \in G(w)} J(z,w)$$

with $z = (x,u) \in Z, w = (\alpha, \theta) \in W$ and

$$G(w) = \left\{z \in Z : \mathcal{M}(z) = \mathcal{T}(w)\right\},$$

where $\mathcal{M} : Z \to X$ and $\mathcal{T} : W \to X$ are defined by (18) and (19), respectively.

Lemma 3.2. ([25, Lemma 3.1]) *Suppose that assumptions $(H_1), (H_2)$ and (H_3) are valid. Then the following assertions are fulfilled:*

(a) *There exists a constant $c > 0$ such that*

$$||\mathcal{T}^* x^*|| \geqslant c||x^*||, \quad \forall x^* \in X^*.$$

(b) *The functional J is strictly differentiable at $(\bar{z}, \bar{w})$ and $\nabla J(\bar{z}, \bar{w})$ is given by*

$$\nabla_w J(\bar{z}, \bar{w}) = \left(0, L_\theta\left(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\right)\right),$$

$$\nabla_z J(\bar{z}, \bar{w}) = \left(J_x(\bar{x}, \bar{u}, \bar{\theta}), J_u(\bar{x}, \bar{u}, \bar{\theta})\right)$$

with

$$J_u(\bar{x}, \bar{u}, \bar{\theta}) = L_u(\cdot, \bar{x}, \bar{u}, \bar{\theta})$$

and

$$J_x(\bar{x}, \bar{u}, \bar{\theta}) = \left(g'\left(x(1)\right) + \int_0^1 L_x\left(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\right)dt,\right.$$

$$\left. g'\left(x(1)\right) + \int_{(\cdot)}^1 L_x\left(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\right)dt\right).$$

We now return to the proof of Theorem 1.1, our main result.

Since $J(x, u, w)$ is strictly differentiable at $(\bar{x}, \bar{u}, \bar{w})$, it is also locally Lipschitzian around $(\bar{x}, \bar{u}, \bar{w})$ (see [13, p. 19]). By [13, Corollary 1.81], we have $\partial^\infty J(\bar{x}, \bar{u}, \bar{w}) = \{0\}$. So qualification condition (i) of Theorem 2.1 is satisfied. From [13, p. 121], it follows that J is SNEC at $(\bar{x}, \bar{u}, \bar{w})$. By Lemma 3.2, all conditions of Theorem 2.1 are fulfilled. According to Theorem 2.1, we obtain

$$(20) \qquad \partial V(\bar{w}) \subset \nabla_w J(\bar{z}, \bar{w}) + \mathcal{T}^*\big((\mathcal{M}^*)^{-1}(\nabla_z J(\bar{z}, \bar{w}))\big).$$

We now take $(\alpha^*, \theta^*) \in \partial V(\bar{w})$. By (20), it follows that

$$(\alpha^*, \theta^*) - \nabla_w J(\bar{z}, \bar{w}) \in \mathcal{T}^*\big((\mathcal{M}^*)^{-1}(\nabla_z J(\bar{z}, \bar{w}))\big),$$

which is equivalent to

$$\big(\alpha^*, \theta^* - J_\theta(\bar{z}, \bar{w})\big) \in \mathcal{T}^*\big((\mathcal{M}^*)^{-1}(\nabla_z J(\bar{z}, \bar{w}))\big).$$

Hence, there exists $(a, v) \in \mathbb{R}^n \times L^q([0, 1], \mathbb{R}^n)$ such that

$$(21) \qquad \big(\alpha^*, \theta^* - J_\theta(\bar{z}, \bar{w})\big) = \mathcal{T}^*(a, v) \text{ and } \nabla_z J(\bar{z}, \bar{w}) = \mathcal{M}^*(a, v).$$

By Lemma 3.1, we get

$$(21) \Leftrightarrow \begin{cases} \alpha^* = a; \quad \theta^* - J_\theta(\bar{z}, \bar{w}) = T^T(\cdot)v(\cdot) \\ \big(J_x(\bar{x}, \bar{u}, \bar{w}), J_u(\bar{x}, \bar{u}, \bar{w})\big) = \big(\mathcal{A}^*(a, v), \mathcal{B}^*(a, v)\big). \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha^* = a \\ \theta^* = L_\theta\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big) + T^T(\cdot)v(\cdot) \\ g'\big(\bar{x}(1)\big) + \int_0^1 L_x\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)dt = a - \int_0^1 A^T(t)v(t)dt \\ g'\big(\bar{x}(1)\big) + \int_{(\cdot)}^1 L_x\big(\tau, \bar{x}(\tau), \bar{u}(\tau), \bar{\theta}(\tau)\big)d\tau \\ \qquad = v(\cdot) + \int_0^{(\cdot)} A^T(\tau)v(\tau)d\tau - \int_0^1 A^T(t)v(t)dt \\ L_u\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big) = -B^T(\cdot)v(\cdot). \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha^* = a \\ \theta^* = L_\theta\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big) + T^T(\cdot)v(\cdot) \\ g'\big(\bar{x}(1)\big) + \int_0^1 L_x\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)dt = a - \int_0^1 A^T(t)v(t)dt \\ g'\big(\bar{x}(1)\big) - \int_1^{(\cdot)} L_x\big(\tau, \bar{x}(\tau), \bar{u}(\tau), \bar{\theta}(\tau)\big)d\tau = v(\cdot) + \int_1^{(\cdot)} A^T(\tau)v(\tau)d\tau \\ L_u\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big) = -B^T(\cdot)v(\cdot). \end{cases}$$

$$(22) \quad \Leftrightarrow \quad \begin{cases} v \in W^{1,q}([0,1], R^n) \\ \theta^* - T^T(\cdot)v(\cdot) = L_\theta\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big) \\ \alpha^* = g'\big(\bar{x}(1)\big) + \int_0^1 L_x\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)dt + \int_0^1 A^T(t)v(t)dt \\ v(1) = g'\big(\bar{x}(1)\big) \\ -\dot{v}(\cdot) - A^T(\cdot)v(\cdot) = L_x\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big) \\ -B^T(\cdot)v(\cdot) = L_u\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big). \end{cases}$$

Putting $y = -v$, we have

$$(22) \Leftrightarrow \begin{cases} \alpha^* = g'\big(\bar{x}(1)\big) + \int_0^1 L_x\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)dt - \int_0^1 A^T(t)y(t)dt \\ y(1) = -g'\big(\bar{x}(1)\big) \\ \big(\dot{y}(t) + A^T(t)y(t), B^T(t)y(t), \theta^*(t) + T^T(t)y(t)\big) \\ \quad = \nabla L\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big), \end{cases}$$

for a.e. $t \in [0,1]$. This is the first assertion of the theorem. Using the second conclusion of Theorem 2.1, we also obtain the second assertion of the theorem. The proof is complete. ∎

Example 3.1. We will illustrate the obtained result by a concrete problem. Put

$$X = W^{1,2}([0,1], \mathbb{R}^2), \quad U = L^2([0,1], \mathbb{R}^2),$$

$$\Theta = L^2([0,1], \mathbb{R}^2), \quad W = \mathbb{R}^2 \times \Theta.$$

Consider the problem

$$J(x, u, \theta) = -x_2(1) + \int_0^1 \Big(u_1^2 + \frac{1}{1+u_1^2} + u_2^2 + \theta_1^2 + \theta_2^2\Big)dt \longrightarrow \inf$$

$$(23) \qquad \text{subject to} \quad \begin{cases} \dot{x}_1 = 2x_1 + u_1 + \theta_1 \\ \dot{x}_2 = x_2 + u_2 + \theta_2 \\ x_1(0) = \alpha_1 \\ x_2(0) = \alpha_2. \end{cases}$$

Let $(\bar{\alpha}, \bar{\theta}) = \big((1,1), (0,0)\big)$. The following assertions are valid:

(i) The pair $(\bar{x}, \bar{u})$, where

$$\bar{x} = \Big(e^{2t}, (1 + \frac{e}{4})e^t - \frac{e}{4}e^{-t}\Big), \quad \bar{u} = \Big(0, \frac{1}{2}e^{-t+1}\Big)$$

is a solution of the problem corresponding to $(\bar{\alpha}, \bar{\theta})$.

(ii) If $(\alpha^*, \theta^*) \in \partial V(\bar{\alpha}, \bar{\theta})$, then $\alpha^* = (0, -e)$ and $\theta^* = (0, -e^{1-t})$.

Indeed, for $(\bar{\alpha}, \bar{\theta}) = \big((1, 1), (0, 0)\big)$ the problem becomes

$$J_0(x, u) = -x_2(1) + \int_0^1 \Big(u_1^2 + \frac{1}{1+u_1^2} + u_2^2\Big)dt \longrightarrow \inf$$

(24)
$$\text{subject to} \quad \begin{cases} \dot{x}_1 = 2x_1 + u_1 \\ \dot{x}_2 = x_2 + u_2 \\ x_1(0) = 1 \\ x_2(0) = 1. \end{cases}$$

By a direct computation, we see that the pair $(\bar{x}, \bar{u})$ satisfies (24). Besides,

$$J_0(\bar{x}, \bar{u}) = -\frac{e^2}{8} - e + \frac{9}{8}.$$

Note that $(\bar{x}, \bar{u})$ is a solution of the problem. In fact, for all (x, u) satisfying (24) we have

(25)
$$\begin{aligned} J_0(x, u) &= -x_2(1) + \int_0^1 \Big(u_1^2 + \frac{1}{1+u_1^2} + u_2^2\Big)dt \\ &\geq -x_2(1) + \int_0^1 \big(1 + u_2^2\big)dt \\ &= -x_2(1) + \int_0^1 \big(1 + (\dot{x}_2 - x_2)^2\big)dt \\ &= \int_0^1 \big((\dot{x}_2 - x_2)^2 - \dot{x}_2\big)dt. \end{aligned}$$

We now consider the variational problem

(26)
$$\hat{J}(x_2) := \int_0^1 \big((\dot{x}_2 - x_2)^2 - \dot{x}_2\big)dt \longrightarrow \inf.$$

By solving the Euler equation with noting that $\hat{J}$ is a convex function, we obtain that $\hat{x}_2(t) = ce^t + (1-c)e^{-t}$ is a solution of (26), where c is determined by $c = \frac{ae-1}{e^2-1}$ and $a = x_2(1)$. Hence

(27) $\hat{J}(x_2) \geqslant \hat{J}(\hat{x}_2) = 1 + 2\dfrac{e^2-1}{e^2}(c-1)^2 + \dfrac{1}{e}(c-1) - e(c-1) - e.$

Combining (25) with (27) and putting $r = c - 1$, we obtain

$$J_0(x, u) \geqslant 1 + 2\frac{e^2 - 1}{e^2}r^2 + \frac{1}{e}r - er - e$$

$$\geqslant -\frac{e^2}{8} - e + \frac{9}{8} = J_0(\bar{x}, \bar{u}).$$

Hence, $(\bar{x}, \bar{u})$ is a solution of the problem corresponding to $(\bar{\alpha}, \bar{\theta})$. Assertion (i) is proved.

It remains to prove (ii). We first prove that the solution map S is V-inner semi-continuous at $(\bar{w}, \bar{z})$ with $\bar{w} = (\bar{\alpha}, \bar{\theta})$ and $\bar{z} = (\bar{x}, \bar{u})$. Indeed, for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $\theta = (\theta_1, \theta_2) \in L^2([0, 1], \mathbb{R}^2)$, we recall the problem

$$J(x, u, \theta) = -x_2(1) + \int_0^1 \left(u_1^2 + \frac{1}{1 + u_1^2} + u_2^2 + \theta_1^2 + \theta_2^2\right)dt \longrightarrow \inf$$

$$\text{subject to} \quad \begin{cases} \dot{x}_1 = 2x_1 + u_1 + \theta_1 \\ \dot{x}_2 = x_2 + u_2 + \theta_2 \\ x_1(0) = \alpha_1 \\ x_2(0) = \alpha_2. \end{cases}$$

Let

$$\mathcal{L} = \lambda_0\left(u_1^2 + \frac{1}{1 + u_1^2} + u_2^2 + \theta_1^2 + \theta_2^2\right)$$
$$+ p_1\left(\dot{x}_1 - 2x_1 - u_1 - \theta_1\right) + p_2\left(\dot{x}_2 - x_2 - u_2 - \theta_2\right)$$

and

$$l\big(x(0), x(1)\big) = \lambda_0\big(-x_2(1)\big) + \lambda_1\big(x_1(0) - \alpha_1\big) + \lambda_2\big(x_2(0) - \alpha_2\big)$$

be the Lagrangian and endpoint functional, respectively. By the Pontryagin maximum principle (see [1] and [5]), there exist Lagrange multipliers $\lambda_0 > 0, \lambda_1, \lambda_2$ and absolutely continuous functions p_1, p_2, not all zero, such that

$$(28) \qquad \frac{d}{dt}\mathcal{L}_{\dot{x}} = \mathcal{L}_x \Leftrightarrow \begin{cases} \dot{p}_1 = -2p_1 \\ \dot{p}_2 = -p_2, \end{cases}$$

$$(29) \qquad \begin{cases} \mathcal{L}_{\dot{x}}(0) = l_{x(0)} \\ \mathcal{L}_{\dot{x}}(1) = -l_{x(1)} \end{cases} \Leftrightarrow \begin{cases} p_1(0) = \lambda_1 \\ p_2(0) = \lambda_2 \end{cases} \text{ and } \begin{cases} p_1(1) = 0 \\ p_2(1) = \lambda_0 \end{cases}$$

and

$$(30) \qquad \mathcal{L}_u = 0 \Leftrightarrow \begin{cases} 2\lambda_0\big(u_1 - \frac{u_1}{(1+u_1^2)^2}\big) - p_1 = 0 \\ 2\lambda_0 u_2 - p_2 = 0. \end{cases}$$

By a simple computation, from (28)-(30) we obtain

$$u(t) = (u_1(t), u_2(t)) = \left(0, \frac{1}{2}e^{1-t}\right).$$

From the equations

$$\dot{x}_2(t) = x_2(t) + \frac{1}{2}e^{1-t} + \theta_2(t), \ a.e. \ t \in [0, 1]$$

and

$$\dot{\bar{x}}_2(t) = \bar{x}_2(t) + \frac{1}{2}e^{1-t}, \ a.e. \ t \in [0, 1],$$

we have

$$\dot{x}_2(t) - \dot{\bar{x}}_2(t) = \big(x_2(t) - \bar{x}_2(t)\big) + \theta_2(t), \ a.e. \ t \in [0, 1].$$

It follows that

(31)
$$|\dot{x}_2(t) - \dot{\bar{x}}_2(t)| \le |x_2(t) - \bar{x}_2(t)| + |\theta_2(t)|, \ a.e. \ t \in [0, 1].$$

Since $x_2(t) - \bar{x}_2(t) = (\alpha_2 - 1) + \int_0^t \big(\dot{x}_2(s) - \dot{\bar{x}}_2(s)\big)ds$, we get from the above inequality that

$$
\begin{aligned}
|x_2(t) - \bar{x}_2(t)| &\le |\alpha_2 - 1| + \int_0^t |x_2(s) - \bar{x}_2(s)|ds + \int_0^t |\theta_2(s)|ds \\
&\le |\alpha_2 - 1| + \int_0^t |x_2(s) - \bar{x}_2(s)|ds + \int_0^1 |\theta_2(s)|ds \\
&\le |\alpha_2 - 1| + \int_0^t |x_2(s) - \bar{x}_2(s)|ds + \| \theta_2 \| \ .
\end{aligned}
$$

By the Gronwall inequality (see [5, Lemma 18.1.i]), we get

$$|x_2(t) - \bar{x}_2(t)| \le \Big(|\alpha_2 - 1| + \| \theta_2 \|\Big)e.$$

Combining this with (31), we obtain

$$|\dot{x}_2(t) - \dot{\bar{x}}_2(t)| \le e\Big(|\alpha_2 - 1| + \| \theta_2 \|_2\Big) + |\theta_2(t)|, \ a.e. \ t \in [0, 1].$$

We note that $\| x_2 - \bar{x}_2 \|_{1,2} = |\alpha_2 - 1| + \| \dot{x}_2 - \dot{\bar{x}}_2 \|_2$ and

$$
\begin{aligned}
\| \dot{x}_2 - \dot{\bar{x}}_2 \|_2 &= \Big(\int_0^1 |\dot{x}_2(t) - \dot{\bar{x}}_2(t)|^2 dt \Big)^{\frac{1}{2}} \\
&\le e\Big(|\alpha_2 - 1| + \| \theta_2 \|_2 \Big) + \| \theta_2 \|_2 \ .
\end{aligned}
$$

So

$$\| x_2 - \bar{x}_2 \|_{1,2} \le (e+1)\Big(|\alpha_2 - 1| + \| \theta_2 \|_2\Big).$$

Hence, if

$$\big((\alpha_1^k, \alpha_2^k), (\theta_1^k, \theta_2^k)\big) \xrightarrow{V} \big((1, 1), (0, 0)\big)$$

and

$$\big((x_1^k, x_2^k), (u_1^k, u_2^k)\big) \in S\big((\alpha_1^k, \alpha_2^k), (\theta_1^k, \theta_2^k)\big), \forall k \in \mathbb{N}$$

then $x_2^k \to \bar{x}_2$. By using similar arguments, we also show that $x_1^k \to \bar{x}_1$. It is easy to see that $(u_1^k, u_2^k) \to (\bar{u}_1, \bar{u}_2)$. Thus, the solution map S is V-inner semicontinuous at $\big((\bar{\alpha}, \bar{\theta}), (\bar{x}, \bar{u})\big)$.

From (23), we have

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that conditions (H_2) and (H_3) are fulfilled. Since

$$L(t, x, u, \theta) = u_1^2 + \frac{1}{1 + u_1^2} + u_2^2 + \theta_1^2 + \theta_2^2,$$

we have

$$|L(t, x, u, \theta)| \leqslant |u|^2 + |\theta|^2 + 1,$$

$$|L_u(t, x, u, \theta)| \leqslant 2(|u| + 1); \quad |L_\theta(t, x, u, \theta)| \leqslant 2|\theta|$$

for all $(x, u, \theta) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ and a.e. $t \in [0, 1]$. Thus, condition (H_1) is also valid. Hence, all conditions of Theorem 1.1 are fulfilled. Let $(\alpha^*, \theta^*) \in \partial V(\bar{\alpha}, \bar{\theta})$. By Theorem 1.1, there exists $y = (y_1, y_2) \in W^{1,2}([0, 1], \mathbb{R}^2)$ such that

$$\begin{cases} \dot{y} + A^T y = \nabla_x L(t, \bar{x}, \bar{u}, \bar{\theta}) \\ y(1) = -g'(x(1)). \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{y}_1 = -2y_1 \\ \dot{y}_2 = -y_2 \\ y_1(1) = 0 \\ y_2(1) = 1. \end{cases}$$

This implies that $(y_1, y_2) = (\bar{y}_1, \bar{y}_2) = (0, e^{1-t})$. By (9), we have

$$\theta^* = -T^T y + \nabla_\theta L(t, \bar{x}, \bar{u}, \bar{\theta}) = -T^T y.$$

It follows that $\theta^*(t) = (0, -e^{1-t})$. On the other hand, from (7) we get

$$(32) \qquad \alpha^* = g'\big(\bar{x}(1)\big) + \int_0^1 L_x\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)dt - \int_0^1 A^T(t)y(t)dt.$$

Substituting $y = \bar{y}$ into (32), we obtain $\alpha^* = (0, -e)$.

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