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DECOMPOSITIONS OF MULTICROWNS INTO CYCLES AND STARS

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Abstract. Let C_k (respectively, S_k) denote a cycle (respectively, a star) with k edges. For graphs F, G and H, a (G, H)-decomposition of F is a partition of the edge set of F into copies of G and copies of H with at least one copy of G and at least one copy of H. In this paper, necessary and sufficient conditions for the existence of the (C_k, S_k)-decomposition of multicrowns are given.

1. Introduction

Let F, G and H be graphs. A *G-decomposition* of F is a partition of the edge set of F into copies of G. If F has a G-decomposition, we say that F is *G-decomposable*. A *(G, H)-decomposition* of F is a partition of the edge set of F into copies of G and copies of H with at least one copy of G and at least one copy of H. If F has a (G, H)-decomposition, we say that F is *(G, H)-decomposable*.

For positive integers m and n, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. A *k-star*, denoted by S_k, is the complete bipartite graph $K_{1,k}$. For $k \geq 2$, the vertex of degree k in S_k is called the *center* of S_k. A *k-cycle*, denoted by C_k, is a cycle of length k. Let $(v_1 v_2 \ldots v_k)$ denote the k-cycle with edges $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1$. A *$k$-path*, denoted by P_k, is a path with k edges. A *k-matching*, denoted by M_k, is a matching with k edges. A *spanning subgraph* H of a graph G is a subgraph of G with $V(H) = V(G)$. A *1-factor* of G is a spanning subgraph of G with each vertex incident with exactly one edge. For positive integers ℓ and n with $1 \leq \ell \leq n$, the *crown* $C_{n,\ell}$ is a bipartite graph with bipartition (A, B) where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$, and edge set $\{a_ib_j : i = 0, 1, \ldots, n-1,\ j \equiv i+1, i+2, \ldots, i+\ell \pmod{n}\}$. Hereafter (A, B) always means the bipartition of $C_{n,\ell}$ defined here. Note that $C_{n,n}$ is isomorphic to $K_{n,n}$, and $C_{n,n-1}$ is the graph obtained from the complete bipartite graph $K_{n,n}$ with a 1-factor removed. For a graph G and a positive integer $\lambda \geq 2$, we use λG to denote

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the multigraph obtained from G by replacing each edge e by λ edges, each of which has the same ends as e.

The problem of k-star decomposition of graphs has been extensively studied; see [8, 16, 17, 18, 19]. There are several works about decompositions of graphs into k-cycles; see [4, 5, 9, 10, 15]. Around 2003, Abueida and Daven [1] proposed the problem of the (G, H)-decomposition. Shortly after, they [2] investigated the problem of the (K_k, S_k)-decomposition of the complete graph K_n. Abueida and O'Neil [3] focused on the existence problem of the (C_k, S_{k-1})-decomposition of the complete multigraph λK_n for $k = 3, 4$ and 5. As for the existence of the (G, H)-decomposition of λK_n where $G, H \in \{C_n, P_{n-1}, S_{n-1}\}$, it was studied by Priyadharsini and Muthusamy [11]. Recently, Shyu [12] investigated the problem of decomposing K_n into paths and stars with k edges, giving a necessary and sufficient condition for $k = 3$. In [13], Shyu considered the existence of a decomposition of K_n into paths and cycles with k edges, giving a necessary and sufficient condition for $k = 4$. Shyu [14] investigated the problem of decomposing K_n into cycles and stars with k edges, settling the case $k = 4$. Lee [6] established necessary and sufficient conditions for the existence of the (C_k, S_k)-decomposition of a complete bipartite graph. Lee and Lin [7] investigated the problems of the (C_k, S_k)-decomposition of crowns $C_{n,n-1}$. It is natural to consider the problem of the (C_k, S_k)-decomposition of multicrowns $\lambda C_{n,n-1}$ for $\lambda \geq 2$. In this paper, the necessary and sufficient conditions for the existence of such decomposition are given.

2. Preliminaries

Let G be a multigraph, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For sets $A \subseteq V(G)$ and $B \subseteq E(G)$, we use $G[A]$ to denote the subgraph of G induced by A and $G - B$ to denote the subgraph obtained from G by deleting the edges in B. Suppose that $G_1, G_2, \ldots, G_t$ are edge disjoint subgraphs of a graph G. Then $G_1 + G_2 + \cdots + G_t$, or $\sum_{i=1}^{t} G_i$, denotes the graph G with vertex set $\bigcup_{i=1}^{t} V(G_i)$, and edge set $\bigcup_{i=1}^{t} E(G_i)$. Thus if a multigraph G can be decomposed into subgraphs $G_1, G_2, \ldots, G_t$, we write $G = G_1 + G_2 + \cdots + G_t$, or $G = \sum_{i=1}^{t} G_i$. For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer not less than x and $\lfloor x \rfloor$ denotes the largest integer not greater than x. Let H be a subgraph of $C_{n,n-1}$ and let r be a positive integer. We use H_{+r} to denote the graph with vertex set $\{a_i : a_i \in V(H)\} \bigcup \{b_{j+r} : b_j \in V(H)\}$ and edge set $\{a_i b_{j+r} : a_i b_j \in E(H)\}$ where the subscripts of b are taken modulo n. For any vertex x of a digraph G, the *outdegree* $\deg_G^+ x$ (respectively, *indegree* $\deg_G^- x$) of x is the number of arcs incident from (respectively, to) x. A *multistar* is a star with multiple edges allowed. We use $\overline{S}_k$ to denote a multistar with k edges. Let G be a multigraph. The *edge-multiplicity* of an edge in G is the number of edges joining the vertices of the edge. The *multiplicity* of G, denoted by $m(G)$, is the maximum edge-multiplicity of G. We list some results we need in this paper.

Proposition 2.1. ([19]). *Let $m \geq n \geq 1$ be integers. Then $K_{m,n}$ is S_k-decomposable if and only if $m \geq k$ and*

$$\begin{cases} m \equiv 0 \pmod{k} & \text{if } n < k \\ mn \equiv 0 \pmod{k} & \text{if } n \geq k. \end{cases}$$

Proposition 2.2. ([8]). *$\lambda C_{n,\ell}$ is S_k-decomposable if and only if $k \leq \ell$ and $\lambda n \ell \equiv 0 \pmod{k}$.*

Proposition 2.3. ([10]). *Let k and n be positive integers and let I be a 1-factor of $K_{n,n}$. Then $K_{n,n} - I$ is C_k-decomposable if and only if $n \equiv 1 \pmod{2}$, $k \equiv 0 \pmod{2}$, $4 \leq k \leq 2n$ and $n(n-1) \equiv 0 \pmod{k}$.*

Proposition 2.4. ([7]). *$C_{n,n-1}$ is (C_k, S_k)-decomposable if and only if $4 \leq k < n-1$, $k \equiv 0 \pmod{2}$ and $n(n-1) \equiv 0 \pmod{k}$.*

Lemma 2.5. ([7]). *Suppose that $k \geq 4$ is an even integer. Let G be the subgraph of $C_{n,n-1}$ induced by $\{a_0, a_1, \ldots, a_{k/2-1}\} \cup \{b_0, b_1, \ldots, b_{k-1}\}$. Then there exist $k/2-1$ edge-disjoint k-cycles in G.*

The following corollary follows from Lemma 2.5.

Corollary 2.6. *Suppose that $k \geq 4$ is an even integer. Let G be the subgraph of $\lambda C_{n,n-1}$ induced by $\{a_0, a_1, \ldots, a_{k/2-1}\} \cup \{b_0, b_1, \ldots, b_{k-1}\}$. Then there exist $\lambda(k/2-1)$ edge-disjoint k-cycles in G.*

Lemma 2.7. ([8]). *Suppose that $m(\overline{S}_{\lambda k}) \leq \lambda$. Then $\overline{S}_{\lambda k}$ is S_k-decomposable.*

3. Main Result

The goal of this paper is to settle the (C_k, S_k)-decomposition problem for $\lambda C_{n,n-1}$. We prove the following theorem.

Main Theorem. *For an integer $\lambda \geq 2$, $\lambda C_{n,n-1}$ is (C_k, S_k)-decomposable if and only if $k \equiv 0 \pmod{2}$, $4 \leq k \leq n-1$ and $\lambda n(n-1) \equiv 0 \pmod{k}$.*

We first give the necessary conditions for the (C_k, S_k)-decomposition of $\lambda C_{n,n-1}$.

Lemma 3.1. *Let λ be a positive integer with $\lambda \geq 2$. If $\lambda C_{n,n-1}$ is (C_k, S_k)-decomposable, then $k \equiv 0 \pmod{2}$, $4 \leq k \leq n-1$ and $\lambda n(n-1) \equiv 0 \pmod{k}$.*

Proof. Since bipartite graphs contain no odd cycle, $k \equiv 0 \pmod{2}$. In addition, since the minimum length of a cycle and the maximum size of a star in $\lambda C_{n,n-1}$ are 4 and $n-1$, respectively, we have $4 \leq k \leq n-1$. Finally, the size of each member in the decomposition is k and $|E(\lambda C_{n,n-1})| = \lambda n(n-1)$; thus $\lambda n(n-1) \equiv 0 \pmod{k}$. ∎

We now show that the necessary conditions are also sufficient. The proof is divided into cases $n \leq 2k$ and $n > 2k$, and consists of Lemmas 3.2 and 3.3, respectively.

Lemma 3.2. *Let λ, k and n be integers with $\lambda \geq 2$, $k \equiv 0 \pmod 2$, $k \geq 4$, $n/2 \leq k \leq n-1$. If $\lambda n(n-1) \equiv 0 \pmod k$, then $\lambda C_{n,n-1}$ is (C_k, S_k)-decomposable.*

Proof. Suppose that $n-1 = k+r$. Then $0 \leq r \leq k-1$ from the assumption $n/2 \leq k \leq n-1$. If $r = 0$, then $n = k+1$. By Propositions 2.2 and 2.3, we have that $C_{k+1,k}$ has S_k-decomposition and C_k-decomposition. Hence $\lambda C_{k+1,k}$ is (C_k, S_k)-decomposable for $\lambda \geq 2$.

Consider the case $r \neq 0$. Since $k \mid \lambda n(n-1)$, it follows that $k \mid \lambda r(r+1)$, which implies $\lambda r(r+1)/k$ is a positive integer. Let $t = \lambda r(r+1)/k$. The proof is divided into three cases according to the values of t and λ.

Case 1. $t < \lambda$**.**

Let $F_1 = \lfloor \lambda/2 \rfloor C_{k+1,k}$ with bipartition $(\{a_0, a_1, \ldots, a_k\}, \{b_0, b_1, \ldots, b_k\})$, $F_2 = \lceil \lambda/2 \rceil C_{k+1,k}$ with bipartition $(\{a_0, a_1, \ldots, a_k\}, \{b_0, b_1, \ldots, b_k\})$, $H_1 = \lambda K_{k,r}$ with bipartition $(\{a_0, a_1, \ldots, a_{k-1}\}, \{b_{k+1}, b_{k+2}, \ldots, b_{k+r}\})$, $H_2 = \lambda K_{r,k}$ with bipartition $(\{a_{k+1}, a_{k+2}, \ldots, a_{k+r}\}, \{b_0, b_1, \ldots, b_{k-1}\})$, $G^{(1)}$ (respectively, $G^{(2)}$) be the bipartite graph with bipartition $(\{a_k, a_{k+1}, \ldots, a_{k+r}\}, \{b_k, b_{k+1}, \ldots, b_{k+r}\})$ and edge set $\{a_i b_j : i < j\}$ (respectively, $\{a_i b_j : i > j\}$). Then $\lambda C_{n,n-1} = F_1 + F_2 + H_1 + H_2 + \lambda G^{(1)} + \lambda G^{(2)}$.

Claim 1. Let $M' = \{a_i b_{i+\frac{k}{2}} : 0 \leq i \leq k/2 - 1\}$ and $M'' = \{a_{i+\frac{k}{2}} b_i : 0 \leq i \leq k/2 - 1\}$, which are matchings in F_1. Then $F_1 + H_1$ can be decomposed into $\lceil t/2 \rceil$ copies of M', $\lfloor t/2 \rfloor$ copies of M'', several copies of S_k, and the following multistars: $\overline{S}_{\lambda(2k-j)}$ (with center at b_j), $j = k+1, k+2, \ldots, k+r$.

Check. Since $t < \lambda$, we have $\lceil t/2 \rceil \leq \lfloor \lambda/2 \rfloor$. This assures us that there exist $\lceil t/2 \rceil$ copies of M' and $\lfloor t/2 \rfloor$ copies of M'' in F_1.

Let $\overline{F}_1$ be the graph obtained from F_1 by deleting the edges in the $\lceil t/2 \rceil$ copies of M' and $\lfloor t/2 \rfloor$ copies of M''. We obtain that

$$\deg_{\overline{F}_1} a_i = \begin{cases} \lfloor \lambda/2 \rfloor k - \lceil t/2 \rceil, & 0 \leq i \leq k/2 - 1, \\ \lfloor \lambda/2 \rfloor k - \lfloor t/2 \rfloor, & k/2 \leq i \leq k-1, \\ \lfloor \lambda/2 \rfloor k, & i = k. \end{cases}$$

Let $X_i = \overline{F}_1[\{a_i, b_0, b_1, \ldots, b_k\}]$ for $i = 0, 1, \ldots, k$. Then

$$X_i = \begin{cases} \overline{S}_{\lfloor \frac{\lambda}{2} \rfloor k - \lceil \frac{t}{2} \rceil}, & 0 \leq i \leq k/2 - 1, \\ \overline{S}_{\lfloor \frac{\lambda}{2} \rfloor k - \lfloor \frac{t}{2} \rfloor}, & k/2 \leq i \leq k-1, \\ \overline{S}_{\lfloor \frac{\lambda}{2} \rfloor k}, & i = k \end{cases}$$

with the center at a_i. Thus we decompose F_1 into $\lceil t/2 \rceil$ copies of M', $\lfloor t/2 \rfloor$ copies of M'' and $X_0, X_1, \ldots, X_k$.

Next we will show that H_1 can be decomposed into the following multistars: $\overline{S}_{\lambda(2k-j)}$ with b_j as the center for $j = k+1, k+2, \ldots, k+r$ and $\overline{S}_{\lceil t/2 \rceil}$ with center at each vertex in $\{a_0, a_1, \ldots, a_{k/2-1}\}$, $\overline{S}_{\lfloor t/2 \rfloor}$ with center at each vertex in $\{a_{k/2}, a_{k/2+1}, \ldots, a_{k-1}\}$. Equivalently, we show that there exists an orientation of H_1 such that

$$\deg^+_{H_1} b_j \;=\; \lambda(2k-j), \tag{1}$$

where $j = k+1, k+2, \ldots, k+r$ and

$$\deg^+_{H_1} a_i = \begin{cases} \lceil t/2 \rceil, & 0 \le i \le k/2-1, \\ \lfloor t/2 \rfloor, & k/2 \le i \le k-1. \end{cases} \tag{2}$$

We orient the edges in H_1 as follows. Let $m = j-k$, the edges $a_{\frac{\lambda m(m-1)}{2}} b_j$, $a_{\frac{\lambda m(m-1)}{2}+1} b_j, a_{\frac{\lambda m(m-1)}{2}+2} b_j, \ldots, a_{\frac{\lambda m(m-1)}{2}+\lambda m-1} b_j$ are all oriented to b_j for $j = k+1, k+2, \ldots, k+r$, where the subscripts of a are taken modulo k. Since $\lambda r < \lambda k$, this assures us that there are enough edges for the above orientation. The edges which are not oriented yet are all oriented to $\{a_0, a_1, \ldots, a_{k-1}\}$.

Since $\deg^+_{H_1} b_j + \deg^-_{H_1} b_j = \lambda k$, from the construction of the above orientation we have that $\deg^+_{H_1} b_j = \lambda k - \deg^-_{H_1} b_j = \lambda k - \lambda m = \lambda k - \lambda(j-k) = \lambda(2k-j)$. Hence (1) is satisfied. On the other hand, we have

$$\begin{aligned} \sum_{i=0}^{k-1} \deg^+_{H_1} a_i &= \lambda + 2\lambda + \cdots + r\lambda \\ &= \lambda r(r+1)/2 \\ &= tk/2 \\ &= k/2 \cdot \lceil t/2 \rceil + k/2 \cdot \lfloor t/2 \rfloor. \end{aligned}$$

Since $1 \ge \deg^+_{H_1} a_i - \deg^+_{H_1} a'_i \ge 0$ for $0 \le i \le i' \le k-1$, we obtain (2). Hence there exists the required decomposition $\mathscr{D}$ of H_1. Let X'_i be the multistar with center at a_i in $\mathscr{D}$ for $0 \le i \le k-1$. Then

$$X'_i = \begin{cases} \overline{S}_{\lceil \frac{t}{2} \rceil}, & 0 \le i \le k/2-1, \\ \overline{S}_{\lfloor \frac{t}{2} \rfloor}, & k/2 \le i \le k-1. \end{cases}$$

We have decomposed H_1 into $X'_0, X'_1, \ldots, X'_{k-1}$ and $\overline{S}_{\lambda(2k-j)}$ with center at b_j, $(j = k+1, k+2, \ldots, k+r)$. Treat the multistars $X_0, X_1, \ldots, X_k$ in the decomposition of F_1 and $X'_0, X'_1, \ldots, X'_{k-1}$ in the decomposition of H_1. For $0 \le i \le k-1$, $X_i + X'_i$ is an $\overline{S}_{\lfloor\lambda/2\rfloor k}$. We have $m(X_i) \le \lfloor\lambda/2\rfloor$ since X_i is contained in $F_1 = \lfloor\lambda/2\rfloor C_{k+1,k}$. We also have $m(X_i') \le |E(X_i')| \le \lceil t/2\rceil \le \lfloor\lambda/2\rfloor$. Thus $m(X_i + X_i') \le \lfloor\lambda/2\rfloor$. Then, by Lemma 2.7, $X_i + X_i'$ has an S_k-decomposition for $0 \le i \le k-1$. It is trivial that $X_k = \lfloor\lambda/2\rfloor S_k$ has an S_k-decomposition. Now we have the required decomposition for $F_1 + H_1$. This completes the check of Claim 1.

By similar arguments as in the check of Claim 1, we have the following.

Claim 2. $F_2 + H_2$ can be decomposed into $\lceil t/2\rceil$ copies of M'_{+1}, $\lfloor t/2\rfloor$ copies of M''_{+1}, several copies of S_k, and the multistars: $\overline{S}_{\lambda(2k-i)}$ (with center at a_i), $i = k+1, k+2, \ldots, k+r$, where the subscripts of b in M'_{+1} (respectively, M''_{+1}) are taken modulo $k/2$ in the set of numbers $\{k/2, k/2+1, \ldots, k-1\}$ (respectively, $\{0, 1, \ldots, k/2-1\}$).

Note that one copy of M' and one copy of M'_{+1} constitute a k-cycle; so do M'' and M''_{+1}. Thus by Claims 1 and 2, there exists a decomposition $\mathscr{T}$ of $F_1 + F_2 + H_1 + H_2$ into t copies of C_k, several copies of S_k, the multistar $\overline{S}_{\lambda(2k-j)}$ with b_j in $\{b_{k+1}, b_{k+2}, \ldots, b_{k+r}\}$ as the center and $\overline{S}_{\lambda(2k-i)}$ with a_i in $\{a_{k+1}, a_{k+2}, \ldots, a_{k+r}\}$ as the center for $k+1 \le i, j \le k+r$.

Let Y_i and Z_j be the $\lambda(2k-i)$-multistar and $\lambda(2k-j)$-multistar (in $\mathscr{T}$) with a_i and b_j as the center, respectively. Recall that $G^{(1)}$ (respectively, $G^{(2)}$) is the bipartite graph with bipartition $(\{a_k, a_{k+1}, \ldots, a_{k+r}\}, \{b_k, b_{k+1}, \ldots, b_{k+r}\})$ and edge set $\{a_ib_j : i < j\}$ (respectively, $\{a_ib_j : i > j\}$). Let $Y_i' = \lambda G^{(2)}[\{a_i, b_k, b_{k+1}, \ldots, b_{k+r}\}]$ and $Z_j' = \lambda G^{(1)}[\{a_k, a_{k+1}, \ldots, a_{k+r}, b_j\}]$ for $k+1 \le i, j \le k+r$. Then $Y_i' = \overline{S}_{\lambda(i-k)}$ and $Z_j' = \overline{S}_{\lambda(j-k)}$. It follows that $Y_i + Y_i' = \overline{S}_{\lambda k}$, and $Z_j + Z_j' = \overline{S}_{\lambda k}$. Since $m(Y_i + Y_i') \le \lambda$ and $m(Z_j + Z_j') \le \lambda$, by Lemma 2.7 we have that $Y_i + Y_i'$ and $Z_j + Z_j'$ are S_k-decomposable. Hence $\lambda C_{n,n-1}$ is (C_k, S_k)-decomposable.

Case 2. $t = \lambda$.

Since $t = \lambda r(r+1)/k$, it follows that $k = r(r+1)$. Then $n(n-1) = (k+r+1)(k+r) = k(k+2r+1) + r(r+1)$, which implies that $k|n(n-1)$. For the case $k = n-1$, by Propositions 2.2 and 2.3 $C_{n,n-1}$ has S_k-decomposition and C_k-decomposition. For the case $4 \le k < n-1$, by Proposition 2.4 $C_{n,n-1}$ is (C_k, S_k)-decomposable. Hence we have that $\lambda C_{n,n-1}$ is (C_k, S_k)-decomposable for $\lambda \ge 2$.

Case 3. $t > \lambda$.

Let $F = \lambda C_{k,k-1}$ with bipartition $(\{a_0, a_1, \ldots, a_{k-1}\}, \{b_0, b_1, \ldots, b_{k-1}\})$, $H = \lambda K_{k,r+1}$ with bipartition $(\{a_0, a_1, \ldots, a_{k-1}\}, \{b_k, b_{k+1}, \ldots, b_{k+r}\})$, $D_1 = \lambda K_{r+1,k}$ with bipartition $(\{a_k, a_{k+1}, \ldots, a_{k+r}\}, \{b_0, b_1, \ldots, b_{k-1}\})$ and $D_2 = \lambda C_{r+1,r}$ with bipartition $(\{a_k, a_{k+1}, \ldots, a_{k+r}\}, \{b_k, b_{k+1}, \ldots, b_{k+r}\})$. Then $\lambda C_{n,n-1} = F + H + D_1 + D_2$. It is trivial that D_1 is S_k-decomposable.

Claim 3. $F + H$ can be decomposed into $t - \lambda$ copies of C_k, λk copies of S_k and the multistars: $\overline{S}_{\lambda(k-r)}$ (with center at b_j), $j = k, k+1, \ldots, k+r$.

Check. Let $p_0 = \lceil (t-\lambda)/2 \rceil$, $p_1 = \lfloor (t-\lambda)/2 \rfloor$ and $G_i = F[A_i \cup \{b_0, b_1, \ldots, b_{k-1}\}]$, where $A_i = \{a_{ik/2}, a_{ik/2+1}, \ldots, a_{(i+1)k/2-1}\}$ for $i = 0, 1$. Thus F is decomposed into G_0 and G_1. In the following, we will show that, for each $i \in \{0, 1\}$, G_i can be decomposed into p_i copies of C_k and $k/2$ copies of $\overline{S}_{\lambda(k-1)-2p_i}$. Since $r \le k-1$, we have $r+1 \le k$ and in turn $t \le \lambda r$. Thus $p_0 = \lceil (t-\lambda)/2 \rceil \le (t-\lambda+1)/2 \le (\lambda r - \lambda + 1)/2 = (\lambda(r-1)+1)/2 \le (\lambda(k-2)+1)/2$. Since p_0 is an integer and $p_1 \le p_0$, it follows that $p_i \le \lambda(k-2)/2 = \lambda(k/2-1)$ for $i = 0, 1$. Hence for $i = 0, 1$ there exist p_i edge-disjoint k-cycles in G_i by Corollary 2.6. Note that $p_0 + p_1 = t - \lambda$. Suppose that $C_{i,0}, C_{i,1}, \ldots, C_{i,p_i-1}$ are edge-disjoint k-cycles in G_i. For $i = 0, 1$, $j = 0, 1, \ldots, k/2-1$, let $U_i = G_i - E(\sum_{r=0}^{p_i-1} C_{i,r})$ and $X_{i,j} = U_i[\{a_{ik/2+j}\} \cup \{b_0, b_1, \ldots, b_{k-1}\}]$. Since $\deg_{G_i} a_{ik/2+j} = \lambda(k-1)$ and each $C_{i,r}$ uses two edges incident with $a_{ik/2+j}$ for each i and j, we have $\deg_{U_i} a_{ik/2+j} = \lambda(k-1) - 2p_i$. Hence $X_{i,j}$ is a $(\lambda(k-1)-2p_i)$-multistar with its center at $a_{ik/2+j}$. We have decomposed F into $X_{i,j}$ $(i = 0, 1, j = 0, 1, \ldots, k/2-1)$, and $t - \lambda$ copies of C_k.

Next we will show that H can be decomposed into the following multistars: $\overline{S}_{\lambda+2p_i}$ (with center at $a_{ik/2+j}$) $i = 0, 1, j = 0, 1, \ldots, k/2-1$, and $\overline{S}_{\lambda(k-r)}$ (with center at b_w) $w = k, k+1, \ldots, k+r$. Equivalently, we need show that there exists an orientation of H such that, for $i = 0, 1$, $j = 0, 1, \ldots, k/2-1$ and $w = k, k+1, \ldots, k+r$,

$$\deg_H^+ a_{ik/2+j} = \lambda + 2p_i \tag{3}$$

$$\deg_H^+ b_w = \lambda(k-r). \tag{4}$$

We begin the orientation. Let $\alpha = 1 + \lambda/2 + p_0$. For $j = 0, 1, \ldots, k/2-1$, the edges

$$a_j b_{k+(\lambda+2p_0)j}, a_j b_{k+(\lambda+2p_0)j+1}, \ldots, a_j b_{k+(\lambda+2p_0)j+\lambda+2p_0-1}$$

and

$$a_{k/2+j} b_{\alpha k+(\lambda+2p_1)j}, a_{k/2+j} b_{\alpha k+(\lambda+2p_1)j+1}, \ldots, a_{k/2+j} b_{\alpha k+(\lambda+2p_1)j+\lambda+2p_1-1}$$

(where the subscripts of b are taken modulo $r+1$ in the set $\{k, k+1, \ldots, k+r\}$) are all oriented from $a_{ik/2+j}$. Note that from each $a_{ik/2+j}$, we orient $\lambda + 2p_i$ edges. Since $\lambda + 2p_1 \le \lambda + 2p_0 \le t+1 < \lambda(r+1)$, this assures us that there are enough edges for the above orientation. The edges which are not oriented yet are all oriented from $\{b_k, b_{k+1}, \ldots, b_{k+r}\}$ to $\{a_0, a_1, \ldots, a_{k-1}\}$.

From the construction of the orientation, it is easy to see that (3) is satisfied, and for all $w, w' \in \{k, k+1, \ldots, k+r\}$, we have

$$|\deg_H^- b_w - \deg_H^- b_{w'}| \le 1. \tag{5}$$

We check (4).

Since $\deg_H^+ b_w + \deg_H^- b_w = \lambda k$ for $w \in \{k, k+1, \ldots, k+r\}$, it follows from (5) that $|\deg_H^+ b_w - \deg_H^+ b_{w'}| \le 1$ for $w, w' \in \{k, k+1, \ldots, k+r\}$. Furthermore,

$$\begin{aligned}\sum_{w=k}^{k+r} \deg_H^+ b_w &= |E(\lambda(K_{k,r+1})| - \sum_{i=0}^{1}\sum_{j=0}^{k/2-1} \deg_H^+ a_{ik/2+j} \\ &= \lambda k(r+1) - (2p_0 + 2p_1 + 2\lambda)(k/2) \\ &= \lambda k(r+1) - tk \\ &= \lambda k(r+1) - \lambda r(r+1) \\ &= \lambda(k-r)(r+1).\end{aligned}$$

Thus $\deg_H^+ b_w = \lambda(k-r)$ for $w \in \{k, k+1, \ldots, k+r\}$. This proves (4). Hence there exists the required decomposition, say $\mathscr{G}$, of H. Let $X'_{i,j}$ be the multistar with center at $a_{ik/2+j}$ in $\mathscr{G}$ for $i = 0, 1$, $j = 0, 1, \ldots, k/2-1$. Hence we have decomposed H into X'_{ij} $i = 0, 1$, $j = 0, 1, \ldots, k/2-1$ and $\overline{S}_{\lambda(k-r)}$ (with center at b_w) $w = k, k+1 \ldots, k+r$. Note that $X'_{i,j} = \overline{S}_{\lambda+2p_i}$. Hence $X_{i,j} + X'_{i,j} = \overline{S}_{\lambda k}$. Since $m(X_{i,j} + X'_{i,j}) \le \lambda$, by Lemma 2.7 we obtain that $X_{i,j} + X'_{i,j}$ can be decomposed into λ copies of S_k for $i = 0, 1$, $j = 0, 1, \ldots, k/2-1$. This completes the check of Claim 3.

Let W_j be the $\lambda(k-r)$-multistar with center at b_j in $\mathscr{G}$ for $j = k, k+1, \ldots, k+r$. Let $W'_j = D_2[\{a_k, a_{k+1}, \ldots, a_{k+r}, b_j\}]$ for $k \le j \le k+r$. Then D_2 is decomposed into W'_k, $W'_{k+1}, \ldots, W'_{k+r}$, and each $W'_j = \overline{S}_{\lambda r}$. It follows that $W_j + W'_j = \overline{S}_{\lambda k}$. Since $m(W_j + W'_j) \le \lambda$, by Lemma 2.7 we obtain that $W_j + W'_j$ is S_k-decomposable. It follows from $\lambda C_{n,n-1} = F + H + D_1 + D_2$ and Claim 3, that $\lambda C_{n,n-1}$ is (C_k, S_k)-decomposable. ■

Lemma 3.3. *Let λ, k and n be positive integers with $\lambda \ge 2$, $k \equiv 0 \pmod 2$ and $4 \le k \le (n-1)/2$. If $\lambda n(n-1) \equiv 0 \pmod k$, then $\lambda C_{n,n-1}$ is (C_k, S_k)-decomposable.*

Proof. Let $n - 1 = qk + r$ where q and r are integers with $0 \le r \le k-1$. Then $q \ge 2$ from the assumption $k \le (n-1)/2$. Note that

$$\begin{aligned}\lambda C_{n,n-1} &= \lambda C_{qk+r+1,qk+r} \\ &= \lambda C_{(q-1)k+1,(q-1)k} + \lambda C_{k+r+1,k+r} + \lambda K_{(q-1)k,k+r} + \lambda K_{k+r,(q-1)k}.\end{aligned}$$

Trivially, $|E(\lambda C_{(q-1)k+1,(q-1)k})|$, $|E(\lambda K_{(q-1)k,k+r})|$ and $|E(\lambda K_{k+r,(q-1)k})|$ are multiples of k. Thus $\lambda(k+r+1)(k+r) \equiv 0 \pmod k$ from the assumption $\lambda n(n-1) \equiv 0 \pmod k$. It is trivial that $\lambda K_{(q-1)k,k+r}$, $\lambda K_{k+r,(q-1)k}$, $\lambda C_{(q-1)k+1,(q-1)k}$ are S_k-decomposable. In addition, by Lemma 3.2, $\lambda C_{k+r+1,k+r}$ is (C_k, S_k)-decomposable for $0 \le r \le k-1$. Hence $\lambda C_{n,n-1}$ is (C_k, S_k)-decomposable. ■

Now Lemmas 3.1, 3.2 and 3.3 serve to prove the Main Theorem.

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