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ON SOME STRUCTURAL PROPERTIES OF SPACES OF HOMOGENEOUS TYPE

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Abstract. We prove that every space of homogeneous type (X, ρ, μ) is either an LCH space and μ is a Radon measure, or X may be identified as a dense subset, with inherited quasi-distance and measure, of another space of homogeneous type which is LCH. We also take an opportunity to present a metamathematical principle which is useful in proving results for general quasi-metric measure spaces by reducing arguments to the case of metric measure spaces.

1. Introduction

A quasi-metric on a nonempty set X is a mapping $\rho\colon X \times X \to [0, \infty)$ which satisfies the conditions:

(i) for every $x, y \in X$, $\rho(x, y) = 0$ if and only if $x = y$;

(ii) for every $x, y \in X$, $\rho(x, y) = \rho(y, x)$;

(iii) there is a constant $K \geq 1$ such that for every $x, y, z \in X$,

$$\rho(x, y) \leq K\big(\rho(x, z) + \rho(z, y)\big).$$

The pair (X, ρ) is then called a *quasi-metric space*; if $K = 1$, then ρ is a metric and (X, ρ) is a metric space.

Given $r > 0$ and $x \in X$, let

$$B(x, r) = \{y \in X \colon \rho(x, y) < r\}$$

be the *(quasi-metric) ball* related to ρ of radius r and with center x. If (X, ρ) is a quasi-metric space, then $\mathcal{T}_\rho := \mathcal{T}(X, \rho)$, the topology in X induced by ρ, is canonically defined by declaring $G \subset X$ to be open, i.e. $G \in \mathcal{T}_\rho$, if and only if for every $x \in G$ there exists $r > 0$ such that $B(x, r) \subset G$ (at this point one easily checks directly that the topology axioms are satisfied for such a definition). Note that this definition enjoys the two features:

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- it is consistent with the definition of metric topology in case ρ is a genuine metric;
- the topology $\mathcal{T}_\rho$ is metrizable.

The second fact may be justified by a general topology argument or by a relatively simple construction included in the proof of Theorem 1.1, see [1, 13]. It is worth noting that balls themselves need not be open (unless ρ is a genuine metric). Originally, in [12] a topology in a quasi-metric space was defined by using the notion of a uniform structure, see [5, Chapter 8] for details concerning this way of introducing a topology. It is easily seen that the topology defined in this way coincides with $\mathcal{T}_\rho$; however the procedure of defining $\mathcal{T}_\rho$ as presented above seems to be more direct.

Two quasi-metrics ρ and ρ' on X are said to be equivalent if $c^{-1}\rho'(x,y) \le \rho(x,y) \le c\rho'(x,y)$ with some $c>0$ independent of $x,y\in X$. It is clear that for equivalent quasi-metrics induced topologies coincide. Moreover, for any $a>0$, ρ^a is a quasi-metric as well and $\mathcal{T}_\rho = \mathcal{T}_{\rho^a}$.

The following result is a refined version of a theorem proved by Aimar, Iaffei and Nitti [1]. In some sense it is fundamental in the theory of quasi-metric spaces and corresponds to an analogous result for quasi-normed spaces, known as the Aoki-Rolewicz theorem.

Theorem 1.1. [1, 13]. *Let (X,ρ) be a quasi-metric space and $0<q\le 1$ be given by $(2K)^q = 2$. Then d_q defined by*

$$d_q(x,y) = \inf\left\{\sum_{j=1}^{n} \rho(x_{j-1},x_j)^q\colon x = x_0, x_1, \dots, x_n = y, \quad n\ge 1\right\} \tag{1.1}$$

is a metric on X equivalent to ρ^q; more precisely, $d_q \le \rho^q \le 4\,d_q$.

Notions of convergent and Cauchy sequences and completeness carry over from metric to quasi-metric spaces. Also, the classic construction of completion of a metric space which is not complete (see [5, Chapter 4.4 (F)]) may be repeated in the framework of a quasi-metric space. Thus, given (X,ρ) such that ρ is not complete, by $(\widetilde{X},\tilde{\rho})$ we denote the complete quasi-metric space obtained by this construction. Then, by $i\colon X\to\widetilde{X}$ we mean the isometric embedding (that maps x into the equivalence class represented by $(x,x,\dots)$ and $\rho(x) = \tilde{\rho}([(x,x,\dots)])$) such that $i(X)$ is dense in $\widetilde{X}$; clearly, we can identify X with $i(X)$. It is worth noting that in general X need not be a Borel subset in $\widetilde{X}$ (take X to be a dense subset of $\mathbb{R}$ equipped with the Euclidean metric which is not Borel; then $\mathbb{R}$ with the Euclidean metric is the completion). However, if (X,ρ) is explicitely given we can directly identify $\widetilde{X}$ and then verify whether or not X is Borel in $\widetilde{X}$.

It is easily seen that equivalent quasi-metrics on X lead to identical completion (that is, the resulting *tilde* spaces and *tilde* quasi-metrics coincide). Moreover, the same is true for the pair of quasi-metrics ρ and ρ^a, where $a > 0$. Thus, if ρ is given and d_q is the corresponding metric as in Theorem 1.1, then completeness of ρ is equivalent to completeness of d_q and completion of (X, ρ) gives the identical result as that of (X, d_q).

We take an opportunity to present here an example (we were not able to find a similar one in the literature), rather pathological, of a quasi-normed space such that each ball fails to be Borel. Note that if $\|\cdot\|$ is a quasi-norm on a real or complex vector space (that is $\|\cdot\|$ satisfies the axioms of a norm except the triangle inequality which is replaced by a modified inequality with a constant $K \geq 1$, as in (iii)), then $\rho(x, y) = \|x - y\|$ is a quasi-metric there.

Example 1.1. Let $X = \mathbb{R}^2$ and E be a symmetric with respect to $(0, 0)$ subset of $\Sigma^1 = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$, which is not Borel; here $\|\cdot\|_2$ denotes the Euclidean norm. Define

$$S = \left\{\frac{1}{2}x : x \in \Sigma^1 \setminus E\right\} \cup \{2x : x \in E\}.$$

Then $S \subset \{x \in \mathbb{R}^2 : \frac{1}{2} \leq \|x\|_2 \leq 2\}$, and for every $x \in X$, $x \neq (0, 0)$, the intersection $\{ax : a > 0\} \cap S$ contains precisely one element. It is easy to check that above properties imply, that $\|\cdot\|_S$, the quasi-functional of Minkowski type given on X by

$$\|x\|_S = a \quad \Longleftrightarrow \quad a^{-1}x \in S, \qquad (0, 0) \neq x \in X,$$

and $\|(0, 0)\|_S = 0$, defines a quasi-norm on X and the topology generated by $\|\cdot\|_S$ coincides with the Euclidean topology.

The declared properties of E imply that the unit ball $B_{\|\cdot\|_S}$, which corresponds to the quasi-norm $\|\cdot\|_S$ and centered at $(0, 0)$, is not Borel and the same can be said about any ball. Indeed, if $B_{\|\cdot\|_S}$ were Borel in $\mathbb{R}^2$, then also $B_{\|\cdot\|_S} \cap \Sigma^1 = E$ would be Borel, a contradiction.

The abbreviation LCH is used for '*locally compact Hausdorff*'. We follow [6, Chapter 7] for terminology concerning regular and Radon measures on LCH spaces.

2. Main Results

The following definition originated the theory of spaces of homogeneous type, see [4].

Definition 2.1. A quasi-metric space (X, ρ) is said to be *geometrically doubling* if there exists $N \in \mathbb{N}$ such that every ball with radius r can be covered by at most N balls of radii $\frac{1}{2}r$.

Equivalently, the parameter $\frac{1}{2}$ can be replaced by any $\delta \in (0, 1)$ with $N = N(\delta)$ depending on δ.

In what follows, if (X,ρ) is a given quasi-metric space, then X is considered as a topological space equipped with the (metrizable) topology $\mathcal{T}_\rho$ and $\mathcal{B}(X,\mathcal{T}_\rho)$ denotes the Borel σ-algebra generated by $\mathcal{T}_\rho$. If X is additionally equipped with a Borel measure μ, then we assume that all balls are Borel sets; from now on this is the *standing assumption*. We then say that (X,ρ,μ) is a *quasi-metric measure space*.

A Borel measure μ on X nontrivial in the sense that $\mu(X)>0$ and satisfying the *doubling condition*

$$\mu(B(x,2r)) \le C_\mu \mu(B(x,r)), \tag{2.1}$$

with a constant $C_\mu \ge 1$ independent of $x\in X$ and $r>0$, is called a *doubling measure*. Clearly (2.1) implies that $0<\mu(B(x,r))$ for every ball $B(x,r)$ and moreover

$$\mu(B(x,kr)) \le C_{\mu,k}\mu(B(x,r)), \qquad x\in X, \quad r>0, \tag{2.2}$$

where $k>1$ and $C_{\mu,k}=C_\mu^{1+\log_2 k}$.

It is well known, see [4, p. 67] and [8, Lemma 2.3], that if (X,ρ) admits a doubling measure, then (X,ρ) is geometrically doubling.

Definition 2.2. A space of homogeneous type is a triple (X,ρ,μ), where (X,ρ) is a quasi-metric space and μ is a Borel measure on X satisfying the doubling condition and such that $\mu(B(x,r))<\infty$ for every ball $B(x,r)$.

As already mentioned, originally the definition of space of homogeneous type was somewhat more general. Nowdays, the above definition seems to be commonly accepted, see [3] for example, though in the literature still some mutations appear.

Hytönen [8] enhanced the concept of doubling by introducing the following definition.

Definition 2.3. An upper doubling quasi-metric measure space is a quadruple (X,ρ,μ,λ), where (X,ρ,μ) is a quasi-metric measure space and $\lambda=\lambda_\rho$ is a *dominating function*, i.e. a function $\lambda\colon X\times\mathbb{R}_+\to\mathbb{R}_+$ such that for every $x\in X$ $r\to\lambda(x,r)$ is non-decreasing and

$$\mu(B(x,2r)) \le \lambda(x,2r) \le C_\lambda \lambda(x,r)$$

holds with a constant C_λ independent of $x\in X$ and $r>0$.

An argument analogous to that leading from (2.1) to (2.2) shows that given $k>1$ one has $\lambda(x,kr)\le C_{\lambda,k}\lambda(x,r)$ with $C_{\lambda,k}=C_\lambda^{1+\log_2 k}$.

The result that follows is a direct consequence of Theorem 1.1. We use ρ or d_q as subscripts to indicate that a ball is related to ρ or d_q, respectively.

Proposition 2.1. *Let (X,ρ) be a quasi-metric space and q and d_q be as in Theorem 1.1. Then:*

(*i*) (X, d_q) *is geometrically doubling whenever* (X, ρ) *is;*

(*ii*) (X, d_q, μ) *is a space of homogeneous type whenever* (X, ρ, μ) *is; in addition*

$$\mu(B_\rho(x, r^{1/q})) \simeq \mu(B_{d_q}(x, r)), \qquad x \in X, \quad r > 0; \tag{2.3}$$

(*iii*) (X, d_q, μ, λ_q) *is upper doubling whenever* (X, ρ, μ, λ) *is, where the dominating function* λ_q *is given by* $\lambda_q(x, r) = \lambda(x, (4r)^{1/q})$*; moreover, if* $\lambda(x, r) \le C\lambda(y, r)$ *whenever* $\rho(x, y) \le r$*, then also* $\lambda_q(x, r) \le C\lambda_q(y, r)$ *whenever* $d_q(x, y) \le r$ (*with the same constant* C).

Proof. Note that $d_q \le \rho^q \le 4d_q$ implies

$$B_\rho(x, r) \subset B_{d_q}(x, r^q) \subset B_\rho(x, 4^{1/q} r), \tag{2.4}$$

and hence i) follows. For ii) first of all note that d_q is a genuine metric and therefore every ball related to d_q is open hence Borel. To check that μ is also doubling for balls related to d_q we use (2.4) and write

$$\mu(B_{d_q}(x, 2r)) \le \mu(B_\rho(x, (8r)^{1/q})) \le C_{\mu, 8^{1/q}} \mu(B_\rho(x, r^{1/q})) \le C_{\mu, 8^{1/q}} \mu(B_{d_q}(x, r)),$$

where $C_{\mu, 8^{1/q}}$ is the constant from (2.2). In addition it follows that $\mu(B_{d_q}(x, r)) < \infty$ for every ball $B_{d_q}(x, r)$. Finally, (2.3) is a consequence of (2.4) and (2.2). For iii) checking that λ_q has the desired properties is, by (2.4) and (2.3), immediate. ∎

The doubling measure structure (or even geometrically doubling property) imposed on a quasi-metric space heavily influences its topological properties. This will be seen in the two results that follow. The first one is well known, see for instance [8, Lemma 2.5]. We include its proof only for the sake of completeness.

Proposition 2.2. *Every geometrically doubling space is separable; in particular, each space of homogeneous type is separable.*

Proof. Fix a reference point $x_0 \in X$ and for any $j \ge 2$ consider the ball $B(x_0, j)$. For $\delta_j := j^{-2}$ find $N_j = N(\delta_j)$ as in the geometrically doubling condition and let $\{x_{j1}, \dots, x_{jN_j}\}$ be the set of centers of balls with radii j^{-1} that cover $B(x_0, j)$. Then the union of these sets forms a dense countable subset in X. Finally, recall that (X, ρ) being equipped with a doubling measure is geometrically doubling. ∎

Not every space of homogeneous type is locally compact. For instance, if $Y = \mathbb{R} \setminus \mathbb{Q}$ with metric and measure inherited from the Euclidean metric and Lebesgue measure on $\mathbb{R}$, then Y is not locally compact. This 'simplest' example is, in some sense, a special case of a more general one. Recall that if (X, ρ) is not complete, then $(\widetilde{X}, \tilde{\rho})$ denotes its completion, X is identified with a dense subset of $\widetilde{X}$ and $\rho = \tilde{\rho}|_X$.

Theorem 2.3. *Let (X, ρ, μ) be a space of homogeneous type. Then:*

(1) *if (X, ρ) is complete, then X is an LCH space and μ is a Radon measure;*

(2) *if (X, ρ) is not complete, then there exists a Borel measure $\tilde{\mu}$ on $\widetilde{X}$ such that $(\widetilde{X}, \tilde{\rho}, \tilde{\mu})$ becomes a space of homogeneous type and $\tilde{\mu}$ extends μ in the following sense: if X is Borel in $\widetilde{X}$, then $\tilde{\mu}$ is the extension of μ in the usual sense and $\tilde{\mu}(\widetilde{X} \setminus X) = 0$; if X is not Borel in $\widetilde{X}$, then $\tilde{\mu}$ extends μ only on the family of those Borel sets in X which are also Borel in $\widetilde{X}$.*

Note that, in particular, in the case of (2) when X is Borel in $\widetilde{X}$, X is a dense subset of an LCH space and μ is a restriction of a Radon measure.

Proof. In what follows we shall use the following well-known facts (see, for instance, [6, p.118, Theorem 7.8, Theorem 0.25]):

(i) every separable metric space is second countable;

(ii) every Borel measure on a second countable LCH space that is finite on compact sets is regular hence Radon;

(iii) in a complete metric space a subset is relatively compact if and only if it is totally bounded.

Assume (X, ρ) is complete. Since (X, ρ) is geometrically doubling, hence every ball $B(x, r)$ is a totally bounded set. But $\mathcal{T}_\rho$, the topology in X, is metrizable thus the closure of $B(x, r)$ is compact and consequently X is an LCH space. In addition, X is separable, hence μ is Radon.

Assume now that (X, ρ) is not complete, which is equivalent with the statement that (X, d_q) is not complete. Since $\mathcal{T}(X, \rho)$ and $\mathcal{T}(\tilde{X}, \tilde{\rho})$ are metric topologies and $\tilde{\rho}$ extends ρ, the topology on X induced by ρ coincides with the relative topology inherited from the topology on $\tilde{X}$ induced by $\tilde{\rho}$. This means that

$$\mathcal{T}(X, \rho) = \{\tilde{U} \cap X \colon \tilde{U} \in \mathcal{T}(\widetilde{X}, \tilde{\rho})\}.$$

Moreover,

$$\mathcal{B}(X, \mathcal{T}_\rho) = \{\tilde{B} \cap X \colon \tilde{B} \in \mathcal{B}(\widetilde{X}, \mathcal{T}_{\tilde{\rho}})\}.$$

Indeed, the inclusion $\subset$ is clear. To justify the opposite inclusion let

$$\mathcal{A} = \{A \subset \widetilde{X} \colon A \cap X \in \mathcal{B}(X, \rho)\}.$$

$\mathcal{A}$ is a σ-algebra in $\widetilde{X}$ and $\mathcal{T}(\widetilde{X}, \tilde{\rho}) \subset \mathcal{A}$. Therefore $\mathcal{B}(\widetilde{X}, \mathcal{T}_{\tilde{\rho}}) \subset \mathcal{A}$ and the required inclusion follows. Note that the assumption 'X is Borel in $\widetilde{X}$' was not used here.

Therefore we may define the Borel measure $\tilde{\mu}$ in $\widetilde{X}$ by setting

$$\tilde{\mu}(\tilde{A}) = \mu(\tilde{A} \cap X), \qquad \tilde{A} \in \mathcal{B}(\widetilde{X}, \mathcal{T}_{\tilde{\rho}}).$$

Clearly, $\tilde{\mu}$ extends μ in the usual sense if X is Borel in $\widetilde{X}$, and in the sense described above, otherwise. It remains to check that for every ball $B_{\tilde{\rho}}(\tilde{x},r)$ in $\widetilde{X}$ we have $\tilde{\mu}(B_{\tilde{\rho}}(\tilde{x},r)) < \infty$ and that $\tilde{\mu}$ is doubling, i.e.,

$$\tilde{\mu}(B_{\tilde{\rho}}(\tilde{x},2r)) \le C\tilde{\mu}(B_{\tilde{\rho}}(\tilde{x},r)), \qquad \tilde{x} \in \widetilde{X}, \quad r > 0. \tag{2.5}$$

To check the first property note that if $\tilde{x} \in X$, then $B_{\tilde{\rho}}(\tilde{x},r) \cap X = B_{\rho}(\tilde{x},r)$, and hence

$$\tilde{\mu}(B_{\tilde{\rho}}(\tilde{x},r)) = \mu(B_{\tilde{\rho}}(\tilde{x},r) \cap X) = \mu(B_{\rho}(\tilde{x},r)) < \infty.$$

If $\tilde{x} \notin X$, then there exists $x \in X$ such that $\tilde{\rho}(x,\tilde{x}) < r$ and $B_{\tilde{\rho}}(\tilde{x},r) \cap X \subset B_{\rho}(x,2Kr)$, hence $\tilde{\mu}(B_{\tilde{\rho}}(\tilde{x},r)) < \infty$ again follows.

To check (2.5) we also distinguish the cases, $\tilde{x} \in X$ and $\tilde{x} \notin X$. For $\tilde{x} \in X$ we proceed as in the step done above and write ($C_\mu(\rho)$ denotes the doubling constant related to ρ)

$$\tilde{\mu}(B_{\tilde{\rho}}(\tilde{x},2r)) = \mu(B_{\rho}(\tilde{x},2r)) \le C_\mu(\rho)\mu(B_{\rho}(\tilde{x},r)) = C_\mu(\rho)\tilde{\mu}(B_{\tilde{\rho}}(\tilde{x},r)). \tag{2.6}$$

Assume that $\tilde{x} \notin X$. In fact we now prove (2.5) with $\widetilde{d_q}$ replacing $\tilde{\rho}$; this is enough since then (2.5) follows by an argument similar to that used in the proof of Proposition 2.1 ii). To begin with, note that if $x \in X$ and $r' > 0$ are such that $B_{\tilde{d}_q}(x,r') \subset B_{\tilde{d}_q}(\tilde{x},r)$, then

$$\tilde{\mu}(B_{\tilde{d}_q}(x,2r')) \le C_\mu(d_q)\tilde{\mu}(B_{\tilde{d}_q}(x,r')) \le C_\mu(d_q)\tilde{\mu}(B_{\tilde{d}_q}(\tilde{x},r)) \tag{2.7}$$

(the first inequality is just (2.6) but with $\widetilde{d_q}$ replacing $\tilde{\rho}$; $C_\mu(d_q)$ denotes the doubling constant related to d_q). It is now possible to find a sequence $(x_n,r_n) \in X \times (0,\infty)$, $n \ge 2$, with the property

$$B_{\tilde{d}_q}\Big(\tilde{x},\frac{n-1}{n}r\Big) \subset B_{\tilde{d}_q}(x_n,r_n) \subset B_{\tilde{d}_q}\Big(\tilde{x},\frac{n}{n+1}r\Big)$$

(it suffices to take x_n such that $\tilde{d}_q(\tilde{x},x_n) \le \frac{r}{2n(n+1)}$ and $r_n = \frac{2n^2-1}{2n(n+1)}r$; here we use the fact that $\tilde{d}_q$ is a genuine metric). This implies that $\{B_{\tilde{d}_q}(x_n,r_n)\}_{n\ge2}$ is increasing and $\bigcup_{n\ge2} B_{\tilde{d}_q}(x_n,r_n) = B_{\tilde{d}_q}(\tilde{x},r)$. Hence, by continuity of the measure $\tilde{\mu}$ and (2.7),

$$\tilde{\mu}(B_{\tilde{d}_q}(\tilde{x},2r)) = \lim_{n\to\infty}\tilde{\mu}(B_{\tilde{d}_q}(x_n,2r_n)) \le \sup_{n\ge2}\tilde{\mu}(B_{\tilde{d}_q}(x_n,2r_n)) \le C_\mu(d_q)\tilde{\mu}(B_{\tilde{d}_q}(\tilde{x},r)). \blacksquare$$

3. Metamathematical Principle

The triangle inequality with a constant $K > 1$ in the definition of a quasi-metric causes some complications in reasonings related to objects in quasi-metric measure spaces, in particular in spaces of homogeneous type. Frequently, the authors working in the enviroment of quasi-metric measure spaces for simplicity consider the case of genuine metric only, and then say something like "*with minor modifications similar results hold for the quasi-metric measure spaces*". It happens, however, that *minor* sometimes means *tedious*. The principle we suggest seems to be useful in overcoming such difficulties. It is based on Proposition 2.1 and allows to reduce reasonings to the case $K = 1$ (i.e. to the situation of a metric). In fact, in numerous circumstances the following *metamathematical principle* works:

> *If a theorem holds for some 'objects' in the context of metric measure spaces from a given class, then the analogous theorem is satisfied in the framework of quasi-metric measure spaces from that class.*

We explain how it works on concrete examples considering successively: the class of separable upper doubling spaces, the class of quasi-metric measure spaces with the property $\mu(B_\rho(x,r)) \le Cr^\tau$ (or, the class of spaces of homogeneous type with the property $\mu(B_\rho(x,r)) \ge Cr^\tau$, respectively) for some $\tau > 0$, and the class of complete geometrically doubling quasi-metric spaces. The 'objects' then are: Calderón-Zygmund operators, fractional integral operators, and doubling measures, respectively. In what follows, if not specified otherwise, (X, ρ, μ) is a quasi-metric measure space and q, d_q and λ_q are as in Theorem 1.1 and in Proposition 2.1.

(A) *Theorems on boundedness of C-Z operators.*

A kernel $K\colon X \times X \setminus \Delta \to \mathbb{C}$, $\Delta = \{(x,x)\colon x \in X\}$, is said to be a *standard kernel* on (X, ρ, μ, λ), if there exist constants $C > 0, c > 1, \delta > 0$ such that:
(i) for every $x, y \in X$, $x \neq y$, the *growth condition*

$$|K(x,y)| \le C\frac{1}{\lambda(x, \rho(x,y))} \tag{3.1}$$

holds;
(ii) for every $x, x', y \in X$, if $\rho(x,y) > c\rho(x,x')$, then the *smoothness condition*

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le C\Big(\frac{\rho(x,x')}{\rho(x,y)}\Big)^\delta \frac{1}{\lambda(x,\rho(x,y))} \tag{3.2}$$

is satisfied. A Calderón-Zygmund operator with an associated standard kernel K is an operator T_K which is bounded on $L^2(X,\mu)$ and such that

$$T_K f(x) = \int_X K(x,y) f(y)\, d\mu(y), \qquad x \notin \operatorname{supp} f,$$

for any $f \in L^2(X,\mu)$ with compact support.

We now show that the standard kernel K satisfying (3.1) and (3.2), fulfills the same conditions with replacement of ρ onto d_q and λ onto λ_q, and with the new triple of constants, namely (some) C', $c' = 4c^q$ and $\delta' = \delta/q$. Indeed, by $d_q \le \rho^q \le 4d_q$ we have

$$\lambda_q\big(x, d_q(x,y))\big) = \lambda\big(x, (4d_q(x,y))^{1/q}\big) \le \lambda\big(x, 4^{1/q}\rho(x,y)\big) \le C_{\lambda,4^{1/q}}\lambda(x,\rho(x,y)),$$

and hence the new growth condition is obtained from (3.1). Moreover,

$$\frac{\rho(x,x')}{\rho(x,y)} \le 4^{1/q}\Big(\frac{d_q(x,x')}{d_q(x,y)}\Big)^{1/q},$$

and, in addition, the condition $d_q(x,y) > c'd_q(x,x')$ with c' as above implies

$$\rho(x,y) \ge d_q(x,y)^{1/q} > (c')^{1/q}d_q(x,x')^{1/q} \ge (c'/4)^{1/q}\rho(x,x') = c\rho(x,x'),$$

so that the new smoothness condition, under the assumption $d_q(x,y) > c'd_q(x,x')$, is also satisfied.

Thus, for instance, [9, Theorem 1.1] may be also framed into the context of separable quasi-metric measure spaces. This means that the estimate

$$\|T_K f\|_{L^p(d\mu)} \le C_p\|f\|_{L^p(d\mu)}, \qquad f \in L^p(d\mu), \tag{3.3}$$

$1 \le p < \infty$, with $\|\cdot\|_{L^{1,\infty}(d\mu)}$ replacing $\|\cdot\|_{L^1(d\mu)}$ on the left-hand side when $p = 1$, holds in the framework of *any* upper doubling quasi-metric measure space.

(B) *Theorems on boundedness of fractional integral operators.*

In the literature there are several notions of fractional integral operators appearing in the framework of quasi-metric measure spaces and the most representative seem to be

$$I_\alpha f(x) = \int_X \frac{f(y)}{\rho(x,y)^{\tau-\alpha}} d\mu(y),$$

and

$$\hat{I}_\alpha f(x) = \int_X f(y)\frac{\rho(x,y)^\alpha}{\mu(B_\rho(x,\rho(x,y)))} d\mu(y),$$

as an alternative version. Here τ in some sense represents the 'dimension' of (X,ρ,μ), and $0 < \alpha < \tau$. (In addition the constraint $1/p - 1/q = \alpha/\tau$ is also assumed but this is, in fact, immaterial for the argument we present.) Usually additional assumptions are imposed on the measure μ when I_α or $\hat{I}_\alpha$ are discussed, like $\mu(B_\rho(x,r)) \le Cr^\tau$ in the case of I_α, see for instance [7], or $\mu(B_\rho(x,r)) \ge Cr^\tau$ in the case of $\hat{I}_\alpha$, see for instance [2]. If one of these estimates holds for **some** $\tau > 0$, then we shall refer to μ as to satisfying a *power growth* or a *reverse power growth condition*, respectively.

We now show that the estimate

$$\|I_\alpha f\|_{L^s(d\mu)} \le C_{p,s}\|f\|_{L^p(d\mu)}, \qquad f \in L^p(d\mu), \tag{3.4}$$

$1 \le p, s < \infty$, with a possible extensions to weighted inequalities or weak type estimates, holds in the framework of *any* quasi-metric measure space satisfying the power growth condition provided it holds in *each* case of metric measure space with this condition. Analogously, if (3.4) with $\hat{I}_\alpha$ replacing I_α holds for *each* space of homogeneous type with measure satisfying the reverse power growth condition, then it holds for *any* space of homogeneous type with this condition. First of all note that by (2.4), if the measure μ satisfies $\mu(B_\rho(x,r)) \le Cr^\tau$ (or $\ge Cr^\tau$), then also $\mu(B_{d_q}(x,r)) \le Cr^{\tau/q}$ ($\ge Cr^{\tau/q}$, respectively). It is therefore sufficient to show, that the integral kernels of the operators I_α or $\hat{I}_\alpha$, are dominated by analogous integral kernels with replacement of ρ onto d_q; of course the parameter τ may also change. In the case of I_α this is evident since $d_q \simeq \rho^q$ and we have

$$\frac{1}{\rho(x,y)^{\tau-\alpha}} \le C \frac{1}{d_q(x,y)^{\frac{\tau}{p}-\frac{\alpha}{p}}}.$$

In the case of $\hat{I}_\alpha$, if μ satisfies $\mu(B_\rho(x,r)) \ge Cr^\tau$, then by $\rho \le d_q^{1/q}$ and $\mu(B_\rho(x,\rho(x,y))) \ge C\mu(B_{d_q}(x,d_q(x,y)))$ (doubling property is used here) we have

$$\frac{\rho(x,y)^\alpha}{\mu(B_\rho(x,\rho(x,y)))} \le C \frac{d_q(x,y)^{\frac{\alpha}{q}}}{\mu(B_{d_q}(x,d_q(x,y)))}.$$

Thus, for instance, [7, Theorem 3.2] and [2, Corollary 5.2] have their counterparts in the framework of relevant quasi-metric measure spaces or relevant spaces of homogeneous type.

(C) *Existence of doubling measures.*

As already mentioned, if (X,ρ) admits a doubling measure, then (X,ρ) is geometrically doubling. The question if the opposite implication is true found the following answer (see [11] or [10, Theorem 3.1]): if (X,ρ) is a complete geometrically doubling *metric* space, then X carries a doubling measure. (A simple example of $X = \mathbb{Q}$ with the usual distance shows that the assumption on completeness is necessary.)

The argument analogous to that used in the proof of Proposition 2.1 easily shows that [10, Theorem 3.1] can be extended to the setting of any complete geometrically doubling quasi-metric space. Indeed, if (X,ρ) is such a space, then (X,d_q) is a complete geometrically doubling metric space, hence there exists a Borel measure μ which is doubling (with respect to balls related to d_q) on X. But then (2.4) shows that μ is also doubling with respect to balls related to ρ since we have

$$\begin{aligned}\mu(B_\rho(x,2r)) &\le \mu(B_{d_q}(x,2^q r^q)) \\ &\le C_{\mu,2^{q+2}}(d_q)\mu(B_{d_q}(x,4^{-1}r^q)) \le C_{\mu,2^{q+2}}(d_q)\mu(B_\rho(x,r)),\end{aligned}$$

where $C_{\mu,2^{q+2}}(d_q)$ is the constant as in (2.2) but related to d_q.

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