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THE SOLVABILITY AND OPTIMAL CONTROLS OF IMPULSIVE FRACTIONAL SEMILINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we deal with the impulsive control systems of fractional order and their optimal controls in Banach spaces. We firstly show the existence and uniqueness of mild solutions for a broad class of impulsive fractional infinite dimensional control systems under suitable assumptions. Then by using generally mild conditions of cost functionals, we extend the existence result of optimal controls to the impulsive fractional control systems. Finally, a concrete application is given to illustrate the effectiveness of our main results.

1. INTRODUCTION

The purpose of this paper is to study the following impulsive fractional control system:

$$(1.1)\qquad \begin{cases} {}^C D_t^{\alpha} x(t) = Ax(t) + f\left(t, x(t), \int_0^t g(t,s,x(s))ds\right) + B(t)u(t), \\ \quad t \in J = [0,b],\ t \neq t_k,\ u \in U_{ad}, \\ \Delta x(t_k) = I_k(x(t_k^-)), k = 1, 2, \ldots, m, \\ x(0) = x_0 \in X, \end{cases}$$

where $0 < \alpha \leq 1$, ${}^C D_t^{\alpha}$ denotes the Caputo fractional derivative of order α with the lower limit zero. $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a

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C_0-semigroup $\{T(t), t \geq 0\}$ on a Banach space X. $f : J \times X \times X \to X$, $g : \Sigma \times X \to X$(where $\Sigma = \{(t,s) \in [0,b] \times [0,b], t \geq s\}$) are two given functions specified later. $I_k : X \to X$ are continuous functions, and $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$. $x(t_k^+)$ and $x(t_k^-)$ denote the right and the left limits of $x(t)$ at $t = t_k (k = 1, 2, \ldots, m)$ respectively. The control function u is given in a suitable admissible control set U_{ad}. B is a linear operator from a separable reflexive Banach space Y into X. The cost functional over the family of admissible state control pair (x, u) is given by

$$\mathcal{J}(x,u) = \int_J \mathcal{L}(t, x(t), u(t))dt.$$

Recently, many authors have studied fractional differential equations from two aspects. One is the theoretical aspects of existence and uniqueness of solutions. The other is the analytic and numerical methods for finding solutions. It is known that many physical system can be represented more accurately through fractional derivative formulation. There are many fields of applications where we can use the fractional calculus as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, viscoelasticity, heat conduction, electricity mechanics, control theory, and so forth. For more details on this topics, one can see for instance, [3, 2, 4, 5, 8, 7, 15, 16] and the references therein.

The impulsive differential equations arise from the real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occurs. Such processes are naturally seen in biology, physics, engineering, etc. Due to their significance, many authors have established the solvability of impulsive differential equations. For the general theory and applications of such equations, we refer the interested reader to see the papers [9, 10, 11, 12, 18] and references therein.

Fractional semilinear equations play a pivotal role in electric-circuit analysis and the activity of interacting inhibitory and excitatory neurons. For this reason, they have become an active area of investigation by many researchers and impressive progress have been made in recent years (cf. [3, 5, 18, 19, 20, 21, 22]). In [19], Wang et al. obtained the existence and continuous dependence of mild solutions and optimal controls for fractional integro-differential evolution systems with infinite delay. Wang and Zhou [20] discussed the optimal controls of a Lagrange problem for fractional evolution equations. In [21], Wei et al. presented the optimal controls for nonlinear impulsive integro-differential equations of mixed type on Banach spaces. Strongly motivated by these papers, we investigate the solvability and optimal controls for impulsive fractional semilinear differential equations with initial value boundary conditions. Comparing with the literatures [13, 19, 20, 21], some appropriate sufficient hypotheses are introduced and different techniques are used in our paper to get a priori estimation of mild solutions, the existence and uniqueness of the mild solutions is discussed under the cases of the C_0-semigroup $T(t)$ is compact or not compact. More details can be found in our proof. Furthermore, to the best of our knowledge, the optimal controls for impulsive fractional semilinear differential equations (1.1) with initial value conditions are untreated topics in the literature and this fact is the motivation for us to extend the existing ones and make the new development of the present work on this issue.

The rest of this paper is organized as follows: In section 2, we will present some basic definitions and preliminary facts, such as definitions, lemmas, theorems and so on, which will be used throughout the following sections. In section 3, by applying the well-known fixed point theorem, some sufficient conditions are established for the existence and uniqueness of mild solutions of the system (1.1). In section 4, we will study the optimal controls for impulsive fractional semilinear differential equations with initial value boundary conditions. Finally, we present an example to demonstrate our main results in section 5.

2. Preliminaries

In this section, we introduce some definitions and preliminaries which are used throughout the paper. The norm of a space X will be denoted by $\|\cdot\|_X$. $L_b(X,Y)$ denotes the space of bounded linear operators from X to Y and we abbreviate the notation to $L_b(X)$ when $X=Y$. Let $C(J,X)$ denote the Banach space of all continuous functions from $J=[0,b]$ into X with the norm $\|x\|_C=\sup_{t\in J}\|x(t)\|_X$. We also introduce the Banach space $PC(J,X)=\{x: J\to X: x\in C((t_k,t_{k+1}],X),\ k=0,1,\ldots,m,$ and there exist $x(t_k^-),\ x(t_k^+),\ k=1,\ldots,m,$ with $x(t_k^-)=x(t_k)\}$ endowed with the norm $\|x\|_{PC}=\max\{\sup_{t\in J}\|x(t+0)\|,\ \sup_{t\in J}\|x(t-0)\|\}$.

Firstly, let us recall the following definitions from fractional calculus. For more details, one can see [7, 15]:

Definition 2.1. The integral

$$I_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,\ \alpha>0,\ t>0, \tag{2.1}$$

is called Riemann-Liouville fractional integral of order α, where Γ is the gamma function.

Definition 2.2. For a function $f(t)$ given in the interval $[0,\infty)$, the expression

$${}^L D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t (t-s)^{n-\alpha-1}f(s)ds,\ t>0, \tag{2.2}$$

where $n=[\alpha]+1,\ [\alpha]$ denotes the integer part of number $\alpha>0$, is called the Riemann-Liouville fractional derivative of order α.

Definition 2.3. Caputo derivative for a function $f:[0,\infty)\to R$ can be written as

$${}^C D_t^\alpha f(t)= {}^L D_t^\alpha[f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)],\ n=[\alpha]+1,\ t>0. \tag{2.3}$$

Remark 2.4.

(i) The Caputo derivative of a constant is equal to zero.

(ii) If the function $f^{(n)}(t) \in C[0,\infty)$, then we can get

$$^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = I_t^{n-\alpha} f^{(n)}(t),\ n = [\alpha]+1.$$

(iii) If f is an abstract function with values in X, then integrals which appear in Definition 2.1 and 2.2 are taken in Bochner's sense.

Now, let us recall the useful nonlinear impulsive Gronwall inequality with Caputo singular kernel which will be used in the sequel:

Lemma 2.5. (Lemma 3.4 of [18]). *Let $x \in PC(J,X)$ satisfy the following inequality*

$$\|x(t)\| \le c_1 + c_2 \int_0^t (t-s)^{\beta-1} \|x(s)\| ds + \sum_{0<t_k<t} h_k \|x(t_k^-)\|,$$

where $c_1, c_2, h_k \ge 0$ are constants. Then

$$\|x(t)\| \le c_1(1 + H^* E_\beta(c_2\Gamma(\beta)t^\beta)^k E_\beta(c_2\Gamma(\beta)t^\beta) \textit{ for } \ t \in (t_k, t_{k+1}],$$

where $H^* = max\{h_k : k = 1,2,\cdots,m\}$.

Next, we collect two well-known theorems including Krasnoselskii's fixed point theorem and $PC-$type Arzela$-$Ascoli theorem as follows:

Theorem 2.6. (Krasnoselskii's fixed point theorem). *Let V be a bounded closed and convex subset of a Banach space X and let $\mathcal{A}$ and $\mathcal{B}$ be two operators of V into X such that*

(i) $\mathcal{A}x + \mathcal{B}y \in V$ whenever $x,\ y \in V$;

(ii) $\mathcal{A}$ is a contraction mapping;

(iii) $\mathcal{B}$ is a completely continuous.

Then there exists a $z \in V$ such that $z = \mathcal{A}z + \mathcal{B}z$.

Theorem 2.7. ($PC-$type Arzela$-$Ascoli theorem, Theorem 2.1 of [21]). *Let X be a Banach space and $\mathcal{W} \subset PC(J,X)$. If the following conditions are satisfied:*

(i) $\mathcal{W}$ is a uniformly bounded subset of $PC(J,X)$;

(ii) $\mathcal{W}$ is enquicontinuous in $(t_k, t_{k+1})(k = 0,1,\cdots,m)$ where $t_0 = 0,\ t_{m+1} = b$;

(iii) $\mathcal{W}(t) = \{x(t) : x \in \mathcal{W}, t \in J \setminus \{t_1,\cdots,t_m\}\}$, $\mathcal{W}(t_k^+) = \{x(t_k^+) : x \in \mathcal{W}\}$ and $\mathcal{W}(t_k^-) = \{x(t_k^-) : x \in \mathcal{W}\}$ are all relatively compact subsets of X.

Then $\mathcal{W}$ is a relatively compact subsets of $PC(J,X)$.

3. Existence of Mild Solutions

In this section, we mainly investigate the existence and uniqueness results for impulsive fractional semilinear differential equations with initial value boundary conditions.

In what follows, we will make the following hypotheses:

$H(1)$ $\{T(t), t>0\}$ is a compact semigroup such that $M := \sup_{t\in[0,\infty)}\|T(t)\|_{L_b(X)} < \infty$.

$H(2)$ The function $f : J\times X\times X \to X$ satisfies:

(i) $f(\cdot, v, w) : J \to X$ is measurable for all $(v,w)\in X\times X$ and $f(t,\cdot,\cdot) : X\times X \to X$ is continuous for a.e. $t \in J$.

(ii) There exist functions $\phi(t),\ \mu_1(\cdot) \in L^{\frac{1}{\beta}}(J,R^+),\ \beta\in(0,\alpha)$ and a continuous function $\mu_2(\cdot)$ such that

$$\|f(t,x,y)\| \le \phi(t) + \mu_1(t)\|x\| + \mu_2(t)\|y\|,\ \text{for a.e. } t\in J, \forall x,y\in X.$$

(iii) There exist constants $L_1,\ L_2>0$, such that

$$\|f(t,x_1,x_2) - f(t,y_1,y_2)\| \le L_1\|x_1-y_1\| + L_2\|x_2-y_2\|,$$
$$\text{for a.e. } t\in J, \forall x_j, y_j\in X, j=1,2.$$

$H(3)$ There exist constants $h_k>0 (k=1,2,\cdots,m)$ such that

$$\|I_k(x) - I_k(y)\| \le h_k\|x-y\|,\ \forall x,y\in X.$$

$H(4)$ For each $(t,s)\in\Sigma = \{(t,s)\in[0,b]\times[0,b], t\ge s\}$, the function $g(t,s,\cdot) : X\to X$ is continuous and for each $x\in X$, $g(\cdot,\cdot,x):\Sigma\to X$ is strongly measurable. Moreover, there exists a function $a:\Sigma\to R^+$ with $\int_0^t a(t,s)ds := a^*(t)\in L^\infty(J)$ such that

$$\|g(t,s,x)\| \le a(t,s)\|x\|,\ \text{for a.e. } (t,s)\in\Sigma,\ \forall x\in X;$$

and

$$\|g(t,s,x) - g(t,s,y)\| \le a(t,s)\|x-y\|,\quad \text{for a.e. } (t,s)\in\Sigma,\ \forall x,y\in X.$$

$H(5)$ The operator $B\in L_b(L^p(J,Y), L^p(J,X))$.

$H(6)$ The multivalued map $U : J\to P_f(Y)$(where $P_f(Y)$ is a class of nonempty closed and convex subsets of Y) is measurable and there exists a function $v(t)\in L^p(J,R^+),\ p>\frac{1}{\alpha}$, such that

$$\|U(t)\| = \sup\{\|u\| : u\in U(t)\} \le v(t),\ \text{for a.e. } t\in J.$$

$H(7)$ The following inequality holds

$$M^* := M\sum_{k=1}^{m} h_k + \frac{M}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta} b^{\alpha-\beta}\|\mu_1\|_{L^{\frac{1}{\beta}}} + \frac{M\mu_2^*\|a^*\|_{L^\infty} b^\alpha}{\Gamma(1+\alpha)} < 1, \text{ where } \mu_2^* = \sup_{t\in J}\mu_2(t).$$

Set the admissible control set

$$U_{ad} = S_U^p = \{u \in L^p(J, Y) : u(t) \in U(t) \text{ a.e.}\},\ 1 < p < \infty.$$

Then, $U_{ad} \neq \emptyset$(Proposition 2.1.7 and Lemma 2.3.2 of [6]). And it is not difficult to check that U_{ad} is a closed and convex subset of $L^p(J, Y)$.

According to Definitions 2.1-2.3 and [22], we shall define the concept of mild solution for problem (1.1) as follows:

Definition 3.1. A function $x \in PC(J, X)$ is said to be a mild solution of the problem (1.1) if there exists a $u \in U_{ad}$ such that

$$\begin{aligned} x(t) &= S_\alpha(t)x_0 + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(x(t_k^-)) \\ &+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B(s)u(s)ds \\ &+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau)ds, \end{aligned} \tag{3.1}$$

where

$$S_\alpha(t) = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta,\ T_\alpha(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

and

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\ \theta \in (0,\infty),$$

ξ_α is a probability density function defined on $(0,\infty)$, that is

$$\xi_\alpha(\theta) \geq 0,\ \theta \in (0,\infty),\ \text{and } \int_0^\infty \xi_\alpha(\theta)d\theta = 1.$$

A solution $x(\cdot) \in PC(J, X)$ of the system (1.1) is referred to as a state trajectory of the fractional semilinear differential equation corresponding to the initial state x_0 and the control $u(\cdot)$.

Lemma 3.2. ([22]). *If the C_0−semigroup $T(t)$ is uniformly bounded (i.e. $\sup_{t\in[0,\infty)}$ $\|T(t)\| \leq M < \infty$), then the operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties:*

(i) *For any fixed* $t \geq 0,\ S_\alpha(t)$ *and* $T_\alpha(t)$ *are linear and bounded operators, i.e., for any* $x \in X$,

$$\|S_\alpha(t)x\| \leq M\|x\|, \text{ and } \|T_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|.$$

(ii) $\{S_\alpha(t), t \geq 0\}$ *and* $\{T_\alpha(t), t \geq 0\}$ *are strongly continuous.*

(iii) *For any* $t > 0,\ S_\alpha(t)$ *and* $T_\alpha(t)$ *are also compact operators if* $T(t)(t > 0)$ *is compact.*

In order to discuss the optimal control of system (1.1), we need

Lemma 3.3. *Assume that* $H(1) - H(7)$ *hold. Then there exists a constant* $\omega > 0$ *such that*

$$\| x \|_{PC} \leq \omega, \text{ for any solution } x \text{ of (1.1).}$$

Proof. If x is a mild solution of system (1.1) with respect to $u \in U_{ad}$ on $[0, b]$, then

$$\begin{aligned} x(t) =\ & S_\alpha(t)x_0 + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(x(t_k^-)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B(s)u(s)ds \\ & + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau)ds. \end{aligned}$$

For $t \in J$, we obtain

$$\begin{aligned} \|x(t)\| \leq\ & \|S_\alpha(t)x_0\| + \|\sum_{0<t_k<t} S_\alpha(t-t_k)I_k(x(t_k^-))\| \\ & + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)B(s)u(s)\|ds \\ & + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)f(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau)\|ds \\ \leq\ & M\|x_0\| + M\sum_{k=1}^m h_k\|x(t_k^-)\| + Mm\|I_k(0)\| \\ & + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|(Bu)(s)\|ds \\ & + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\phi(s) + \mu_1(s)\|x(s)\| \\ & + \mu_2(s)\int_0^s a(s,\tau)\|x(\tau)\|d\tau]ds \\ \leq\ & M\|x_0\| + M\sum_{k=1}^m h_k\|x(t_k^-)\| + Mm\|I_k(0)\| \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&+\frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}b^{\alpha-\frac{1}{p}}\|Bu\|_{L^p}\\
&+\frac{M}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}b^{\alpha-\beta}[\|\phi\|_{L^{\frac{1}{\beta}}}+\|\mu_1\|_{L^{\frac{1}{\beta}}}\cdot\sup_{s\in[0,t]}\|x(s)\|]\\
&+\frac{M\mu_2^*\|a^*\|_{L^\infty}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sup_{\tau\in[0,s]}\|x(\tau)\|ds.
\end{aligned}
$$

Let

$$
\begin{aligned}
\lambda_0 &= M\|x_0\|+Mm\|I_k(0)\|+\frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}b^{\alpha-\frac{1}{p}}\|Bu\|_{L^p}\\
&\quad+\frac{M}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}b^{\alpha-\beta}\|\phi\|_{L^{\frac{1}{\beta}}},\\
\lambda_1 &= \frac{M}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}b^{\alpha-\beta}\|\mu_1\|_{L^{\frac{1}{\beta}}},\ \lambda_2=\frac{M\mu_2^*\|a^*\|_{L^\infty}}{\Gamma(\alpha)},\ W(t)=\sup_{s\in[0,t]}\|x(s)\|.
\end{aligned}
$$

Then by (3.2), we have

$$
\begin{aligned}
W(t) &\le \lambda_0+\lambda_1W(t)+\lambda_2\int_0^t(t-s)^{\alpha-1}W(s)ds+M\sum_{0<t_k<t}h_kW(t_k^-)\\
&\le \frac{\lambda_0}{1-\lambda_1}+\frac{\lambda_2}{1-\lambda_1}\int_0^t(t-s)^{\alpha-1}W(s)ds+\frac{M}{1-\lambda_1}\sum_{0<t_k<t}h_kW(t_k^-).
\end{aligned}
$$

It follows from Lemma 2.5 that

$$
W(t)\le\frac{\lambda_0}{1-\lambda_1}(1+\varrho^*E_\alpha(\frac{\lambda_2}{1-\lambda_1}\Gamma(\alpha)b^\alpha)^mE_\alpha(\frac{\lambda_2}{1-\lambda_1}\Gamma(\alpha)b^\alpha):=\omega,
$$

where

$$
\varrho^*=\max\{\frac{M}{1-\lambda_1}h_k:k=1,2,\cdots,m\}.
$$

Therefore $\sup_{t\in J}\|x(t)\|\le\omega$, which completes the proof.

Firstly, for a compact semigroup, we have the following.

Lemma 3.4. (Theorem 2.3.2 of [14]). *Let $T(t)$ be a C_0-semigroup. If $T(t)$ is a compact semigroup for $t>t_0$, then $T(t)$ is continuous in the uniform operator topology for $t>t_0$.*

Next, we are ready to state the existence of mild solution which is based on the Krasnoselskii's fixed point theorem:

Theorem 3.5. *If $H(1)-H(7)$ hold, then the problem (1.1) has at least one mild solution on J.*

Proof. Choose

$$r \geq \frac{M[\|x_0\| + m\|I_k(0)\| + \frac{b^{\alpha-\beta}}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}\|\phi\|_{L^{\frac{1}{\beta}}} + \frac{b^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}\|Bu\|_{L^p}]}{1-M^*}.$$

Let $B_r = \{x \in PC(J,X) : \|x\| \leq r\}$. It is obvious that B_r is a bounded, closed and convex subset of $PC(J,X)$. Define two operators $\mathcal{A}$ and $\mathcal{B}$ on B_r as

$$(\mathcal{A}x)(t) = S_\alpha(t)x_0 + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(x(t_k^-)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B(s)u(s)ds,$$

and

$$(\mathcal{B}x)(t) = \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds.$$

Clearly, the problem of finding mild solutions of (1.1) is reduced to find the fixed point of the $\mathcal{A}+\mathcal{B}$. Now we prove that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of the Theorem 2.6. For the sake of convenience, we divide the proof into three steps.

Step 1. We prove that $\mathcal{A}x + \mathcal{B}y \in B_r$, whenever $x,\ y \in B_r$.
For each pair $x, y \in B_r,\ t \in J$, it follows from $H(7)$ that

$$\begin{aligned}
&\|(\mathcal{A}x)(t) + (\mathcal{B}y)(t)\| \\
&\leq \|S_\alpha(t)x_0\| + \|\sum_{0<t_k<t} S_\alpha(t-t_k)I_k(x(t_k^-))\| + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)(Bu)(s)\|ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)f(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau)\|ds \\
&\leq M\|x_0\| + M\sum_{k=1}^m h_k\|x(t_k^-)\| + Mm\|I_k(0)\| + \frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}b^{\alpha-\frac{1}{p}}\|Bu\|_{L^p} \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\phi(s) + \mu_1(s)\|y(s)\| + \mu_2(s)\int_0^s a(s,\tau)\|y(\tau)\|d\tau]ds \\
&\leq M[\|x_0\| + m\|I_k(0)\| + \frac{b^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}\|Bu\|_{L^p} + \frac{b^{\alpha-\beta}}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}\|\phi\|_{L^{\frac{1}{\beta}}}] \\
&\quad + \left[M\sum_{i=1}^m h_k + \frac{M}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}b^{\alpha-\beta}\|\mu_1\|_{L^{\frac{1}{\beta}}} + \frac{M\mu_2^*\|a^*\|_{L^\infty}b^\alpha}{\Gamma(1+\alpha)}\right]r \leq r,
\end{aligned}$$

which implies that $\mathcal{A}x + \mathcal{B}y \in B_r$ whenever $x,\ y \in B_r$.

Step 2. We show that $\mathcal{A}$ is a contraction mapping on B_r.

For any $x, y \in B_r$, and $t \in J$, we obtain

$$\begin{aligned}&\|(\mathcal{A}x)(t) - (\mathcal{A}y)(t)\| \\ &\leq \|\sum_{0<t_k<t} S_\alpha(t-t_k)[I_k(x(t_k^-)) - I_k(y(t_k^-))]\| \leq M\sum_{k=1}^{m} h_k\|x-y\|.\end{aligned} \tag{3.3}$$

By $M\sum_{k=1}^{m} h_k < M^* < 1$, we obtain $\mathcal{A}$ is a contraction mapping on B_r.

Step 3. $\mathcal{B}$ is a completely continuous operator.

Claim 1. We show that $\mathcal{B}$ is continuous on B_r. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in B_r as $n \to \infty$. Then for each $t \in J$, we obtain

$$\begin{aligned}&\|(\mathcal{B}x_n)(t) - (\mathcal{B}x)(t)\| \\ &\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[L_1\|x_n(s) - x(s)\| + L_2\int_0^s a(s,\tau)\|x_n(\tau) - x(\tau)\|d\tau]ds \\ &\leq \frac{M(L_1 + L_2\|a^*\|_{L^\infty})b^\alpha}{\Gamma(1+\alpha)}\|x_n - x\| \to 0 \text{ as } m \to \infty,\end{aligned}$$

which implies that $\mathcal{B}$ is continuous.

Claim 2. We prove that $\mathcal{B}$ is equicontinuous on B_r.

Firstly, for any $\varepsilon > 0$, there exists $\delta_0 = \min\{\big(\frac{M\varepsilon}{2\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}[\|\phi\|_{L^{\frac{1}{\beta}}} + r\|\mu_1\|_{L^{\frac{1}{\beta}}}]\big)^{\beta-\alpha}$, $\big(\frac{\Gamma(1+\alpha)\varepsilon}{2M\mu_2^*\|a^*\|_{L^\infty}r}\big)^\alpha\} > 0$, such that, for any $x \in B_r$, $\tau_1 = 0$, $0 < \tau_2 \leq \delta_0$, one can obtain

$$\begin{aligned}&\|(\mathcal{B}x)(\tau_2) - (\mathcal{B}x)(\tau_1)\| \\ &\leq \frac{M}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}[\|\phi\|_{L^{\frac{1}{\beta}}} + r\|\mu_1\|_{L^{\frac{1}{\beta}}}]\delta_0^{\alpha-\beta} + \frac{M\mu_2^*\|a^*\|_{L^\infty}r}{\Gamma(1+\alpha)}\delta_0^\alpha \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence, by the definition of equicontinuity, we get $\mathcal{B}$ is equicontinuous on $[0, \delta_0]$.

Next, for any $x \in B_r$ and $\frac{\delta_0}{2} \leq \tau_1 < \tau_2 \leq b$, we obtain

$$
\begin{aligned}
&\|(\mathcal{B}x)(\tau_2)-(\mathcal{B}x)(\tau_1)\| \\
=&\ \|\int_0^{\tau_2}(\tau_2-s)^{\alpha-1}T_\alpha(\tau_2-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds \\
&-\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}T_\alpha(\tau_1-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds\| \\
\le&\ \|\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}T_\alpha(\tau_2-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds\| \\
&+\|\int_0^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]T_\alpha(\tau_2-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds\| \\
&+\|\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}[T_\alpha(\tau_2-s)-T_\alpha(\tau_1-s)]f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds\| \\
\le&\ Q_1+Q_2+Q_3.
\end{aligned}
$$

By the assumption $H(2)$, we have

$$
\begin{aligned}
Q_1 \le&\ \frac{M}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}(\tau_2-\tau_1)^{\alpha-\beta}[\|\phi\|_{L^{\frac{1}{\beta}}}+r\|\mu_1\|_{L^{\frac{1}{\beta}}}]+\frac{M\mu_2^*\|a^*\|_{L^\infty}r}{\Gamma(1+\alpha)}(\tau_2-\tau_1)^{\alpha}, \\
Q_2 \le&\ \frac{M}{\Gamma(\alpha)}\int_0^{\tau_1}[(\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1}][\phi(s)+\mu_1(s)\|x(s)\| \\
&+\mu_2(s)\int_0^s a(s,\tau)\|x(\tau)\|d\tau]ds \\
\le&\ \frac{2M}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}(\tau_2-\tau_1)^{\alpha-\beta}[\|\phi\|_{L^{\frac{1}{\beta}}}+r\|\mu_1\|_{L^{\frac{1}{\beta}}}]+\frac{2M\mu_2^*\|a^*\|_{L^\infty}r}{\Gamma(1+\alpha)}(\tau_2-\tau_1)^{\alpha}, \\
Q_3 \le&\ \|\int_0^{\tau_1-\varepsilon}(\tau_1-s)^{\alpha-1}[T_\alpha(\tau_2-s)-T_\alpha(\tau_1-s)]f(s,x(s),\int_0^s g(s,\tau,x(\tau)d\tau)ds\| \\
&+\|\int_{\tau_1-\varepsilon}^{\tau_1}(\tau_1-s)^{\alpha-1}[T_\alpha(\tau_2-s)-T_\alpha(\tau_1-s)]f(s,x(s),\int_0^s g(s,\tau,x(\tau)d\tau)ds\| \\
\le&\ \sup_{s\in[0,\tau_1-\varepsilon]}\|T_\alpha(\tau_2-s)-T_\alpha(\tau_1-s)\|\Bigg[(\frac{1-\beta}{\alpha-\beta})^{1-\beta}(\tau_1^{\frac{\alpha-\beta}{1-\beta}}-\varepsilon^{\frac{\alpha-\beta}{1-\beta}})^{1-\beta} \\
&\Big[\|\phi\|_{L^{\frac{1}{\beta}}}+r\|\mu_1\|_{L^{\frac{1}{\beta}}}\Big]+\frac{\mu_2^*\|a^*\|_{L^\infty}r}{\alpha}(\tau_1^\alpha-\varepsilon^\alpha)\Bigg] \\
&+\frac{2M}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}\varepsilon^{\alpha-\beta}+\frac{2M\mu_2^*\|a^*\|_{L^\infty}r}{\Gamma(1+\alpha)}\varepsilon^{\alpha}.
\end{aligned}
$$

Since the compactness of $T(t)(t>0)$ and Lemma 3.4 imply the continuous of $T(t)(t>0)$ in t in the uniform operator topology, it can be easily seen that Q_3 tends to zero independently of $x\in B_r$ as $\tau_2\to\tau_1,\ \varepsilon\to 0$. It is also clear that Q_1 and Q_2 tend to zero as $\tau_2\to\tau_1$ does not depend on particular choice of x. Thus, we get that $\|(\mathcal{B}x)(\tau_2)-(\mathcal{B}x)(\tau_1)\|$ tends to zero independently of $x\in B_r$ as $\tau_2\to\tau_1$, which implies $\mathcal{B}$ is equicontinuous on $\frac{\delta_0}{2}\le\tau_1<\tau_2\le b$.

Hence, by the above arguments, we obtain that $\{\mathcal{B}(x) : x \in B_r\}$ is an equicontinuous subset in $PC(J, X)$.

Claim 3. Now we remain to show that $\mathcal{B}$ is compact.

Let $t \in J$ be fixed. We show that the set $\Pi(t) = \{(\mathcal{B}x)(t) : x \in B_r\}$ is relatively compact in X. Clearly, $\Pi(0) = \{0\}$ is compact. So it is only necessary to consider $t > 0$. For each $\epsilon \in (0, t),\ t \in (0, b],\ x \in B_r$ and any $\delta > 0$, we define

$$\Pi_{\epsilon,\delta}(t) = \{(\mathcal{B}_{\epsilon,\delta}x)(t) : x \in B_r\},$$

where

$$\begin{aligned}
&(\mathcal{B}_{\epsilon,\delta}x)(t)\\
&= \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)d\theta ds.\\
&= \alpha T(\epsilon^\alpha\delta) \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta\\
&\quad -\epsilon^\alpha\delta)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)d\theta ds.
\end{aligned}$$

From the compactness of $T(\epsilon^\alpha\delta)$ $(\epsilon^\alpha\delta > 0)$, we obtain that the set $\Pi_{\epsilon,\delta}(t) = \{(\mathcal{B}_{\epsilon,\delta}x)(t) : x \in B_r\}$ is relatively compact set in X for each $\epsilon \in (0, t)$ and $\delta > 0$. Moreover, we have

$$\begin{aligned}
&\|(\mathcal{B}x)(t) - (\mathcal{B}_{\epsilon,\delta}x)(t)\|\\
=&\|\alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)d\theta ds\\
&-\alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)d\theta ds\|\\
\leq&\alpha\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)d\theta ds\|\\
&+\alpha\| \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)d\theta ds\|\\
\leq&\alpha M(\frac{1-\beta}{\alpha-\beta})^{1-\beta}[\|\phi\|_{L^{\frac{1}{\beta}}} + r\|\mu_1\|_{L^{\frac{1}{\beta}}}][b^{\alpha-\beta}\int_0^\delta \theta\xi_\alpha(\theta)d\theta + \frac{1}{\Gamma(1+\alpha)}\epsilon^{\alpha-\beta}]\\
&+\frac{M\mu_2^*\|a^*\|_{L^\infty}r}{\Gamma(\alpha)}[b^\alpha \int_0^\delta \theta\xi_\alpha(\theta)d\theta + \frac{1}{\Gamma(1+\alpha)}\epsilon^\alpha].
\end{aligned}$$

Since $\int_0^\infty \xi_\alpha(\theta) = 1$, the last inequality tends to zero when $\epsilon \to 0$ and $\delta \to 0$. Therefore, there are relatively compact sets arbitrarily close to the set $\Pi(t),\ t > 0$. Hence the set $\Pi(t),\ t > 0$ is also relatively compact in X.

From the above claims and Theorem 2.7, we know that $\mathcal{B}$ is a completely continuous operator. Hence, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.6. As a result, by Theorem 2.6, we obtain that $\mathcal{A} + \mathcal{B}$ has a fixed point x on B_r. Therefore system (1.1) has at least one mild solution on J. The proof is completed.

Now, according to the Banach contraction mapping principle, we shall show a uniqueness result of mild solution as follows:

Theorem 3.6. *Assume that the hypotheses $H(2) - H(7)$ are satisfied. Then the problem (1.1) has a unique mild solution on J provided that*

$$\sum_{k=1}^{m} h_k + \frac{(L_1 + L_2\|a^*\|_{L^\infty})b^\alpha}{\Gamma(1+\alpha)} < \frac{1}{M},\ \text{where } M := \sup_{t\in[0,\infty)}\|T(t)\|_{L_b(X)} < \infty.$$

Proof. Consider the operator $\mathcal{F} : PC(J,X) \to PC(J,X)$ defined by

$$(\mathcal{F}x)(t) = S_\alpha(t)x_0 + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(x(t_k^-)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B(s)u(s)ds$$
$$+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds.$$

Obviously, any mild solutions of the problem (1.1) are fixed points of the operator $\mathcal{F} : PC(J,X) \to PC(J,X)$.

Let $B_r = \{x \in PC(J,X) : \|x\| \le r\}$ be a closed ball in $PC(J,X)$, where

$$r \ge \frac{M[\|x_0\| + m\|I_k(0)\| + \frac{b^{\alpha-\beta}}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta}\|\phi\|_{L^{\frac{1}{\beta}}} + \frac{b^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}\|Bu\|_{L^p}]}{1 - M^*}.$$

Like the proof of step 1 in Theorem 3.5, we can easily obtain $\|\mathcal{F}x\| \le r$ for any $x \in B_r$, which means that $\mathcal{F}B_r \subseteq B_r$.

For $t \in J$, and $x, y \in PC(J,X)$, we obtain

$$\begin{aligned}
&\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \\
\le\ &\|\sum_{0<t_k<t} S_\alpha(t-t_k)[I_k(x(t_k^-)) - I_k(y(t_k^-))\| \\
&+ \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)[f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau) \\
&-f(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau)]\|ds
\end{aligned}$$

$$\le M\sum_{k=1}^{m} h_k\|x-y\| + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[L_1\|x(s)-y(s)\|$$
$$+L_2\int_0^s a(s,\tau)\|x(\tau)-y(\tau)\|d\tau]ds$$
$$\le [M\sum_{k=1}^{m} h_k + \frac{M(L_1+L_2\|a^*\|_{L^\infty})b^\alpha}{\Gamma(1+\alpha)}]\|x-y\|,$$

which implies F is a contradiction operator from the assumptions in the theorem. According to Banach's fixed point theorem, we obtain the problem (1.1) has a unique mild solution on J. The proof is completed.

4. Optimal Control Results

In this section, we are concerned with the following Lagrange problem (P).

Minimize a cost function of the form

$$\mathcal{J}(x,u) := \int_0^b \mathcal{L}(t,x(t),u(t))dt, \tag{4.1}$$

among all the admissible state control pairs of the impulsive fractional semilinear differential equations (1.1), i.e., find an admissible state control pair $(x^0,u^0)\in PC(J,X)\times U_{ad}$ such that

$$\mathcal{J}(x^0,u^0)\le \mathcal{J}(x,u),\ \text{for all } (x,u)\in PC(J,X)\times U_{ad},$$

where x denotes the mild solution of system (1.1) corresponding to the control $u\in U_{ad}$.

For the existence of solutions for problem (P), we shall introduce the following assumptions:

$H(8)$: The function $\mathcal{L}: J\times X\times Y\to R\cup\{\infty\}$ satisfies:

(i) the function $\mathcal{L}: J\times X\times Y\to R\cup\{\infty\}$ is Borel measurable;

(ii) $\mathcal{L}(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $X\times Y$ for almost all $t\in J$;

(iii) $\mathcal{L}(t,x,\cdot)$ is convex on Y for each $x\in X$ and almost all $t\in J$;

(iv) there exist constants $c\ge 0$, $d>0$, φ is nonnegative and $\varphi\in L^1(J,R)$ such that

$$\mathcal{L}(t,x,u)\ge\varphi(t)+c\|x\|_X+d\|u\|_Y.$$

Next, we can give the following result on existence of optimal controls for problem (P):

Theorem 4.1. *Let the assumptions of Theorem 3.5 and $H(8)$ hold. Then Lagrange problem (P) admits at least one optimal pair, that is, there exists an admissible control pair $(x^0,u^0)\in PC(J,X)\times U_{ad}$ such that*

$$\mathcal{J}(x^0,u^0)=\int_0^b \mathcal{L}(t,x^0(t),u^0(t))dt\le\mathcal{J}(x,u),\ \textit{for all } (x,u)\in PC(J,X)\times U_{ad},$$

where x denotes a mild solution of system (1.1) corresponding to the control $u \in U_{ad}$.

Proof. If $\inf\{\mathcal{J}(x,u) : (x,u) \in PC(J,X) \times U_{ad}\} = +\infty$, then it is clear that the Lagrange problem (P) has an optimal pair.

Without loss of generality, we assume that $\inf\{\mathcal{J}(x,u) : (x,u) \in PC(J,X) \times U_{ad}\} = \rho < +\infty$. Using $H(8)$(iv), we have $\rho > -\infty$. By definition of infimum there exists a minimizing sequence feasible pair $\{(x^m, u^m)\} \subset \mathcal{P}_{ad} \equiv \{(x,u) : x$ is a mild solution of system (1.1) corresponding to $u \in U_{ad}\}$, such that $\mathcal{J}(x^m, u^m) \to \rho$ as $m \to +\infty$. Since $\{u^m\} \subseteq U_{ad}(m = 1, 2, \cdots)$, $\{u^m\}$ is a bounded subset of the separable reflexive Banach space $L^p(J,Y)$, there exists a subsequence, relabeled as $\{u^m\}$, and $u^0 \in L^p(J,Y)$ such that

$$u^m \ \to \ u^0 \text{ weakly in } L^p(J,Y).$$

Since U_{ad} is closed and convex, then by Marzur lemma, $u^0 \in U_{ad}$.

Let $\{x^m\}$ denote the sequence of solutions of the system (1.1) corresponding to $\{u^m\}$, i.e.,

$$\begin{aligned}
& x^m(t) \\
&= S_\alpha(t)x_0 + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(x^m(t_k^-)) \\
&\quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B(s)u^m(s)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s, x^m(s), \int_0^s g(s,\tau,x^m(\tau))d\tau)ds.
\end{aligned} \tag{4.2}$$

Now, we prove that $\{x^m(t)\}$ is relatively compact on $PC(J,X)$ based on Theorem 2.7.

Firstly, it follows the boundedness of $\{u^m\}$ and Lemma 3.3, one can check that there exists a positive number ω such that $\|x^m\|_{PC} \le \omega$, which implies that $\|x^m\|_{PC}$ is uniformly bounded.

Next, for each interval $(t_k, t_{k+1}](k = 0, 1, \cdots, m)$, $t_k \le \tau_1 < \tau_2 \le t_{k+1}$, like the proof of Claim 2 in Theorem 3.5, it is not difficult to show that $\{x^m(t)\}$ is equicontinuous on $(t_k, t_{k+1}](k = 0, 1, \cdots, m)$.

Finally, similar to the proof of Claim 3 in Theorem 3.5, we can show that $\{x^m(t)\} = \{y(t) : y \in x^m, t \in J \setminus \{t_1, \cdots, t_m\}\}$, $\{x^m(t_k^+)\} = \{y(t_k^+) : y \in x^m\}$ and $\{x^m(t_k^-)\} = \{y(t_k^-) : y \in x^m\}$ are all relatively compact subsets of $PC(J,X)$.

Hence, by Theorem 2.7, we can deduce that $\{x^m\}$ is relatively compact on $PC(J,X)$. Therefore, there exists a function $x^0 \in PC(J,X)$ such that

$$x^m \to x^0 \text{ in } PC(J,X). \tag{4.3}$$

Moreover, by $H(3)$, we get

$$\|I_k(x^m(t_k^-)) - I_k(x^0(t_k^-))\| \le h_k \|x^m(t_k^-) - x^0(t_k^-)\|.$$

It follows from (4.3) that

$$I_k(x^m(t_k^-)) \to I_k(x^0(t_k^-)), \text{ as } m \to \infty.$$

Similarly, we have

$$g(t, \tau, x^m(\tau)) \to g(t, \tau, x^0(\tau)), \text{ a.e. } (t, \tau) \in \Sigma,$$

and by $H(4)$, we obtain

$$\|g(t, \tau, x^m(\tau))\| \le a(t, \tau)\|x^m(\tau)\| \le \omega a(t, \tau), \text{ a.e. } (\mathrm{t}, \tau) \in \Sigma.$$

Hence, from the dominated convergence theorem, one can deduce that

$$\int_0^t g(t, \tau, x^m(\tau))d\tau \to \int_0^t g(t, \tau, x^0(\tau))d\tau, \text{ a.e. } (t, \tau) \in \Sigma. \tag{4.4}$$

By $H(2)$(iii), we get

$$\begin{aligned} &\|f(t, x^m(t), \int_0^t g(t, \tau, x^m(\tau))d\tau) - f(t, x^0(t), \int_0^t g(t, \tau, x^0(\tau))d\tau)\| \\ \le\ & L_1\|x^m(t) - x^0(t)\| + L_2\| \int_0^t g(t, \tau, x^m(\tau))d\tau - \int_0^t g(t, \tau, x^0(\tau))d\tau\|. \end{aligned}$$

Then, in view of (4.3) and (4.4), we can obtain

$$f(t, x^m(t), \int_0^t g(t, \tau, x^m(\tau))d\tau) \to f(t, x^0(t), \int_0^t g(t, \tau, x^0(\tau))d\tau), \text{ a.e. } t \in J.$$

and by $H(2)$(ii), we get

$$\begin{aligned} &\|f(t, x^m(t), \int_0^t g(t, \tau, x^m(\tau))d\tau)\| \\ \le\ & \phi(t) + \mu_1(t)\|x^m(t)\| + \mu_2(t)\| \int_0^t g(t, \tau, x^m(\tau))d\tau\| \\ \le\ & \phi(t) + \omega\mu_1(t) + \omega\|a^*\|_{L^\infty}\mu_2(t). \end{aligned}$$

Thus, by means of the dominated convergence theorem, one can prove that

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x^m(s), \int_0^s g(s, \tau, x^m(\tau))d\tau) \\ \to\ & \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x^0(s), \int_0^s g(s, \tau, x^0(\tau))d\tau), \text{ a.e. } t \in J. \end{aligned}$$

Hence, it follows from (4.2) that

$$\begin{aligned} & x^0(t) \\ = \ & S_\alpha(t)x_0 + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(x^0(t_k^-)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B(s)u^0(s)ds \\ & + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x^0(s),\int_0^s g(s,\tau,x^0(\tau))d\tau)ds, \end{aligned}$$

i.e., x^0 denotes the sequence of solutions of the system (1.1) corresponding to u^0.

Note that $H(8)$ implies all of the assumptions of Balder (see Theorem 2.1, [1]) are satisfied. Hence, by Balder's theorem, we can conclude that $(x,u) \to \int_0^b \mathcal{L}(t,x(t),u(t))$ dt is sequentially lower semicontinuous in the strong topology of $L^1(J,X)\times L^1(J,Y)$. Since $L^p(J,X)\times L^p(J,Y) \subset L^1(J,X)\times L^1(J,Y)$, $\mathcal{J}$ is also sequentially lower semicontinuous in $L^p(J,X)\times L^p(J,Y)$. Hence, $\mathcal{J}$ is weakly lower semicontinuous on $L^p(J,X)\times L^p(J,Y)$, and by $H(8)$(iv), $\mathcal{J} > -\infty$, $\mathcal{J}$ attains its infimum at $(x^0,u^0)\in PC(J,X)\times U_{ad}$, that is,

$$\rho = \lim_{m\to\infty} \int_0^b \mathcal{L}(t,x^m(t),u^m(t))dt \geq \int_0^b \mathcal{L}(t,x^0(t),u^0(t))dt = J(x^0,u^0) \geq \rho.$$

The proof is completed.

5. An Example

Consider the following initial-boundary value problem of fractional impulsive parabolic control system

$$(5.1)\qquad \begin{cases} \dfrac{\partial^\alpha}{\partial t^\alpha}x(t,y) = \dfrac{\partial^2}{\partial y^2}x(t,y) + e^{-t} + \dfrac{1}{(t+6)^2}\sin(x(t,y)) \\ \quad +\dfrac{t^2}{5}\displaystyle\int_0^t s^2\cos\frac{x(s,y)}{t}ds + \int_0^1 q(y,\tau)u(\tau,t)d\tau, \\ \quad t,s\in J' = [0,1]\backslash\{\tfrac{1}{2}\},\ y\in[0,\pi],\ u\in U_{ad}, \\ \Delta x(\dfrac{1}{2},y) = \dfrac{|x(y)|}{3+|x(y)|},\ y\in[0,\pi], \\ x(t,0) = x(t,\pi) = 0,\ t\in J=[0,1], \\ x(t,y) = x_0(y),\ t\in[0,1],\ y\in[0,\pi]. \end{cases}$$

with the cost function

$$\mathcal{J}(x,u) = \int_0^1\int_0^\pi |x(t,y)|^2dydt + \int_0^1\int_0^\pi |u(t,y)|^2dydt,$$

where $\alpha = \frac{2}{3},\ q : [0,1] \times [0,1] \to R$ is continuous, $u \in L^2(J, [0,\pi])$.

Take $X = Y = L^2(J, [0,\pi])$ and the operator $A : D(A) \subset X \to X$ is defined by

$$A\omega = \omega'',$$

where the domain $D(A)$ is given by

$$\{\omega \in X : \omega,\ \omega' \text{ are absolutely continuous},\ \omega'' \in X,\ \omega(0) = \omega(\pi) = 0\}.$$

Then A can be written as

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n,\ \omega \in D(A),$$

where $\omega_n(x) = \sqrt{2/\pi}\sin nx (n = 1, 2, \cdots)$ is an orthonormal basis of X. It is well known that A is the infinitesimal generator of a compact semigroup $T(t)(t > 0)$ in X given by

$$T(t)x = \sum_{n=1}^{\infty} \exp^{-n^2 t}(x, x_n)x_n,\ x \in X,\ \text{and } \|T(t)\| \le e^{-1} < 1 = M.$$

We take the functions $u : \Phi x([0,\pi]) \to R$ as the controls, such that $u \in L^2(\Phi x([0, \pi]))$. It means that $t \to u(t)$ going from J into Y is measurable. Set $U(t) := \{u \in Y : \|u\|_Y \le \vartheta\}$, where $\vartheta \in L^2(J, R^+)$. We restrict the admissible controls sets U_{ad} to be all $u \in L^2(\Phi x([0,\pi]))$ such that $\|u(\cdot, t)\|_2 \le \vartheta(t)$, a.e. $t \in J$.

Denote that $x(t, y) = x(t)(y)$, then

$$\begin{aligned}
&f(t, x(t), \int_0^t g(t, s, x(s))ds)(y) \\
&= e^{-t} + [\frac{1}{(t+6)^2}\sin(x(t)) + \frac{t^2}{5}\int_0^t s^2 \cos\frac{x(s)}{t}ds](y), \\
&g(t, s, x(s))(y) = [s^2 \cos\frac{x(s)}{t}](y), \\
&I_k(x(t))(y) = \frac{|x(t)|}{3 + |x(t)|}(y), \\
&B(t)u(t)(y) = [\int_0^1 q(\tau)u(\tau, t)d\tau](y).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\|f(t, x(t), \int_0^t g(t, s, x(s))ds)\| &= e^{-t} + \frac{1}{(t+6)^2}\|x(t)\| + \frac{t^2}{5}\|\int_0^t s^2 \cos\frac{x(s)}{t}ds\| \\
&:= \phi(t) + \mu_1(t)\|x(t)\| + \mu_2(t)\|\int_0^t s^2 \cos\frac{x(s)}{t}ds\|,
\end{aligned}$$

and for any $x, y \in X$,

$$\begin{aligned}
&\|f(t, x(t), \int_0^t g(t, s, x(s))ds) - f(t, y(t), \int_0^t g(t, s, y(s))ds)\| \\
\leq\ & \frac{1}{36}\|x - y\| + \frac{1}{5}\| \int_0^t g(t, s, x(s))ds) - \int_0^t g(t, s, y(s))ds)\|. \\
&\|g(t, s, x(s))\| \leq \frac{s^2}{t}\|x(s)\| := a(t, s)\|x(s)\|, \\
&\|g(t, s, x(s)) - g(t, s, y(s))\| \leq a(t, s)\|\|x - y\|,
\end{aligned}$$

where $\int_0^t a(t, s)ds := a^*(t) \in L^\infty([0, 1])$, and

$$\begin{aligned}
\operatorname{ess}\sup_{t\in[0,1]} \int_0^t a(t, s)ds = \operatorname{ess}\sup_{t\in[0,1]} \int_0^t \frac{s^2}{t} ds = \frac{1}{3} := \|a^*\|_{L^\infty}. \\
\|I_k(x(t))\| \leq \frac{1}{3}, \\
\|I_k(x(t)) - I_k(y(t))\| \leq \frac{1}{3}\|x - y\|,
\end{aligned}$$

Moreover,

$$M \sum_{K=1}^{m} h_k + \frac{M}{\Gamma(\alpha)}(\frac{1-\beta}{\alpha-\beta})^{1-\beta} b^{\alpha-\beta} \|\mu_1\|_{L^{\frac{1}{\beta}}} + \frac{M\mu_2^* \|a^*\|_{L^\infty} b^\alpha}{\Gamma(1+\alpha)} < 1.$$

Hence, all the conditions of Theorem 4.1 are satisfied, the system (5.1) has at least one optimal pair solution.

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