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MULTI PURSUER DIFFERENTIAL GAME OF OPTIMAL APPROACH WITH INTEGRAL CONSTRAINTS ON CONTROLS OF PLAYERS

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Abstract. We study a differential game of optimal approach of finite or countable number of pursuers with one evader in the Hilbert space l_2. On control functions of the players integral constraints are imposed. Such constraints arise in modeling the constraint on energy. The duration of the game θ is fixed. The payoff functional is the greatest lower bound of distances between the pursuers and evader when the game is terminated. The pursuers try to minimize the payoff functional, and the evader tries to maximize it. In this paper, we find formula for the value of the game and construct explicitly optimal strategies of the players. Important point to note is that the energy resource of any pursuer needs not be greater than that of the evader.

1. Introduction and Preliminaries

Theory of Differential Games was initiated by the book of Rufus Isaacs [9]. Since then many works have been devoted to differential games (see, for example, [1-18]).

Constructing the player's optimal strategies and finding the value of the game are of specific interest in studying of differential games. Isaacs [9] obtained an equation, the main equation of differential games, to find the value of a differential game and based on his method solved a number of interesting examples. However, the main equation may have no differentiable solution or may have infinite number of generalized solutions [18]. Subbotin [18] obtained necessary and sufficient conditions in terms of partial differential inequalities which a function must satisfy to be the value function. In its turn, these inequalities are very complicated to solve even for simple motion differential games, not to mention the general linear differential games. Therefore different approaches were chosen by different authors to solve pursuit-evasion differential games with one or several pursuers.

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There are a few works on differential games of optimal approach of many pursuers with one evader. Ivanov and Ledyaev [10] studied simple motion pursuit-evasion differential game of several players with geometric constraints. They obtained sufficient conditions to find optimal pursuit time in $\mathbb{R}^n$. Their approach is based on auxiliary differential game with one pursuer and one evader under state constraints.

Levchenkov and Pashkov [12] investigated differential game of optimal approach of two identical inertial pursuers to a noninertial evader on fixed time interval, with control parameters subjected to geometric constraints. They constructed the value function of the game and used necessary and sufficient conditions [18] which a function must satisfy to be the value function.

Rikhsiev [16] studied simple motion differential game of optimal pursuit with many pursuers and one evader. He first obtained a sufficient condition for optimality of pursuit time when initial position of the evader belongs to the interior of the convex hull of initial positions of the pursuers.

Ibragimov [7] considered a simple motion differential game of many pursuers and one evader with geometric constraints on control parameters in the Hilbert space l_2. Half-space and fictitious players methods are used to prove the main theorem.

Ibragimov and Salimi [8] studied a pursuit-evasion differential game of infinitely many inertial players with integral constraints on control functions. The duration of the game θ is fixed. The payoff functional of the game is the greatest lower bound of the distances between the evader and the pursuer at θ. The pursuer's goal is to minimize the payoff, and the evader's goal is to maximize it. The problem was solved under assumption that energy of each pursuer is greater than that of the evader.

The present paper is close in spirit to [8]. Different from [8] here, we assume that the energy of any pursuer is not necessarily greater than that of the evader. We give a sufficient condition to find the value of the game and construct the optimal strategies of players. It should be noted that there are no conditions between energies of the pursuers and the evader in the theorem, for example, energy of a pursuer can be less than energy of the evader.

2. Formulation of the Problem

In the space l_2 consisting of elements $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k, ...)$, with $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$, and inner product $(\alpha, \beta) = \sum_{k=1}^{\infty} \alpha_k \beta_k$, the motions of the countably many pursuers P_i and the evader E are described by the equations

$$
\begin{aligned}
&P_i : \ddot{x}_i = u_i, \quad x_i(0) = x_i^0, \quad \dot{x}_i(0) = x_i^1, \\
&E : \ddot{y} = v, \qquad y(0) = y^0, \quad \dot{y}(0) = y^1,
\end{aligned}
\tag{2.1}
$$

where $x_i, x_i^0, x_i^1, u, y, y^0, y^1, v \in l_2$, $u_i = (u_{i1}, u_{i2}, ..., u_{ik}, ...)$ is the control parameter of the pursuer P_i, $i \in I$, I stands for either finite set $I = \{1, 2, ..., m\}$ or the set of positive integers $I = \{1, 2, ..., m, ...\}$, and $v = (v_1, v_2, ..., v_k, ...)$ is the control parameter of the evader E. Let θ, the duration of the game, be a given positive number.

A ball (respectively, sphere) of radius r and center at the point x_0 is denoted by $H(x_0, r) = \{x \in l_2 \,|\, ||x - x_0|| \leq r\}$ (respectively, by $S(x_0, r) = \{x \in l_2 \,|\, ||x - x_0|| = r\}$).

Denote by $B(\rho)$ the set of all functions $u(\cdot) = (u_1(\cdot), u_2(\cdot), ...)$, $u : [0, \theta] \to l_2$ such that $u_k : [0, \theta] \to R^1$, $k = 1, 2, ...$, are Borel measurable functions and

$$\left(\int_0^\theta \|u(s)\|^2 \, ds\right)^{1/2} \leq \rho, \; \|u\| = \left(\sum_{k=1}^{\infty} u_k^2\right)^{1/2},$$

where ρ is given positive number. Functions $u_i(\cdot) \in B(\rho_i)$ and $v(\cdot) \in B(\sigma)$ are called admissible controls of the ith pursuer and the evader, respectively.

Once the players admissible controls $u_i(\cdot)$ and $v(\cdot)$ are chosen, the corresponding motions $x_i(\cdot)$ and $y(\cdot)$ of the players are defined as

$$x_i(t) = (x_{i1}(t), x_{i2}(t), ..., x_{ik}(t), ...), \qquad y(t) = (y_1(t), y_2(t), ..., y_k(t), ...),$$

$$x_{ik}(t) = x_{ik}^0 + t x_{ik}^1 + \int_0^t \int_0^s u_{ik}(r) dr ds, \qquad y_k(t) = y_k^0 + t y_k^1 + \int_0^t \int_0^s v_k(r) dr ds,$$

$$i \in I, \; k = 1, \; 2, \; ....$$

One can readily see that $x_i(\cdot), \; y(\cdot) \in C(0, \; \theta; \; l_2)$, where $C(0, \; \theta; \; l_2)$is the space of functions

$$f(t) = (f_1(t), \; f_2(t), \; ..., \; f_k(t), \; ...) \in l_2, \qquad t \geq 0,$$

such that (i) $f_k(t)$, $0 \leq t \leq \theta$, $k = 1, 2, ...$, are absolutely continuous functions; (ii) $f(t)$, $0 \leq t \leq \theta$, is a continuous function in the norm of l_2.

Definition 1. A function $U_i(t, v)$, $U_i : [0, \infty) \times l_2 \to l_2$, of the form $U_i(t, v) = at + b + v$, where $a, b, v \in l_2$ and t is time variable, is called a strategy of the pursuer P_i.

Note that for any $v(\cdot) \in B(\sigma)$ solution $x_i(t)$, $0 \leq t \leq \theta$, of the initial value problem

$$\ddot{x}_i = at + b + v, \; x_i(0) = x_i^0, \; \dot{x}_i(0) = x_i^1,$$

belongs to $C(0, \theta; l_2)$.

Definition 2. Strategies U_{i0}, $i \in I$, of the pursuers P_i, $i \in I$, are said to be optimal if

$$\inf_{U_1, ..., U_m, ..} \Gamma_1(U_1, \; ..., \; U_m, \; ...) = \Gamma_1(U_{10}, \; ..., \; U_{m0}, \; ...),$$

where $\Gamma_1(U_1, ..., U_m, ...) = \sup_{v(\cdot)} \inf_{i \in I} \|x_i(\theta) - y(\theta)\|$, U_i are admissible strategies of the pursuer P_i, and $v(\cdot)$ is an admissible control of the evader E.

Definition 3. A function $V(t,\, x_1,\, ...,\, x_m,\, ...,\, y)$, $V : [0, \infty)\times l_2\times\cdots\times l_2\times\cdots\times l_2 \to l_2$, is called a strategy of the evader E if the system of equations

$$\ddot{x}_i = u_i,\, x_i(0) = x_i^0,\, \dot{x}_i(0) = x_i^1,\, i = 1, 2, ..., m, ...,$$
$$\ddot{y} = V(t,\, x_1,\, ...,\, x_m,\, ...,\, y),\, y(0) = y^0,\, \dot{y}(0) = y^1,$$

has a unique solution $(x_1(\cdot),\, ...,\, x_m(\cdot),\, ...,\, y(\cdot))$, with $x_i(\cdot),\, y(\cdot) \in C(0,\, \theta;\; l_2)$, for arbitrary admissible controls $u_i = u_i(t)$, $0 \le t \le \theta$, of the pursuers P_i, $i \in I$. If each control formed by the strategy $V$is admissible, then the strategy V is said to be admissible.

Definition 4. A strategy V_0 of the evader E is said to be optimal if $\sup_v \Gamma_2(V) = \Gamma_2(V_0)$, where $\Gamma_2(V) = \inf_{u_1(\cdot),\dots u_m(\cdot),\dots} \inf_{i\in I} \|x_i(\theta) - y(\theta)\|$, where $u_i(\cdot)$ are admissible controls of the pursuers P_i, $i \in I$, and $V$is an admissible strategy of the evader$E$.

If $\Gamma_1(U_{10},\, ...,\, U_{m0},\, ...) = \Gamma_2(V_0) = \gamma$, then we say that the game has the value γ [17].

Problem 5. *Find the optimal strategies U_{i0} and V_0 of the players P_i, $i \in I$, and E, respectively, and the value of the game.*

Instead of differential game described by (2.1) we can consider an equivalent differential game with the same payoff function and described by the following system (see, for example, [8]):

$$P_i : \dot{x}_i(t) = (\theta - t)u_i(t),\, x_i(0) = x_{i0} \doteq x_i^1\theta + x_i^0,\, i \in I,$$
$$E : \dot{y}(t) = (\theta - t)v(t),\, y(0) = y_0 \doteq y^1\theta + y^0.$$

and the attainability sets of the pursuer P_i and the evader E at the time θ are the closed balls $H\left(x_{i0},\; \rho_i\left(\theta^3/3\right)^{1/2}\right)$ and $H\left(y_0,\, \sigma\left(\theta^3/3\right)^{1/2}\right)$ respectively.

3. Main Result

We study the stated problem under the following assumption.

Assumption 6. There exists a nonzero vector $p_0$such that $(y_0 - x_{i0},\, p_0) \ge 0$ for all $i \in I$.

Note that in the case of finite set $I = \{1,\, ...,\, m\}$ Assumption 6 is true since all vectors $y_0 - x_{i0}$, $i \in I$, lie on one plane in l_2 and the normal vector of this plane can be taken as the vector p_0. Let

$$\gamma = \inf\left\{l \ge 0 \,|\quad H\left(y_0,\, \sigma(\theta^3/3)^{1/2}\right) \subset \bigcup_{i\in I} H\left(x_{i0},\, \rho_i(\theta^3/3)^{1/2} + l\right)\right\}. \tag{3.1}$$

Clearly, such number γ exists. We'll prove the following statement.

Theorem 7. *If Assumption 6 is true, then the number γ given by (3.1) is the value of the game.*

Proof of the theorem relies on Lemmas 8, 9, and 10 in the following subsections.

3.1. Auxiliary differential game

Here we study a differential game of two players, the Pursuer P and the Evader E, described by equations:

$$P : \dot{x} = (\theta - t)u, \quad x(0) = x_0,$$
$$E : \dot{y} = (\theta - t)v, \quad y(0) = y_0,$$

where $x,\, y,\, u,\, v,\, x_0,\, y_0 \in \mathbb{R}^n$, $x_0 \neq y_0$, u and v are control parameters of the pursuer and the evader, respectively, θ is a given positive number. Admissible control of the pursuer and the evader are defined as the functions $u(\cdot) \in B(\rho)$ and $v(\cdot) \in B(\sigma)$, respectively. We require that the state of the evader at the time θ must belong to the half-space X defined by

$$X = \left\{ z \in l_2 \mid 2(z, y_0 - x_0) \leq \frac{1}{3}\theta^3 \left(\rho^2 - \sigma^2\right) + \|y_0\|^2 - \|x_0\|^2 \right\}. \tag{3.2}$$

We assume that $x_0 \in X$, i.e., initial position of the pursuer belongs to the half-space X.

The Pursuer tries to realize the equality $x(\theta) = y(\theta)$ and the evader tries to avoid this. It should be noted that the number σ need not to be less than ρ. The problem is to construct a strategy for the pursuer such that $x(\theta) = y(\theta)$ for any control of the evader. We prove the following statement.

Lemma 8. *There exists a strategy of the pursuer such that if $y(\theta) \in X$, then $x(\theta) = y(\theta)$.*

Proof. 1°. *Construction of the strategy of the pursuer*. We define the strategy of the pursuer by formula

$$u(t,\ v) = 3\theta^{-3}(\theta - t)(y_0 - x_0) + v(t). \tag{3.3}$$

2°. *Admissibility of the constructed strategy*. Since $y(\theta) \in X$, then from (3.2) we obtain,

$$2(y_0 - x_0,\, y(\theta)) \leq \frac{1}{3}\theta^3(\rho^2 - \sigma^2) + \|y_0\|^2 - \|x_0\|^2,$$

hence

$$2\left(y_0 - x_0,\, y_0 + \int_0^\theta (\theta - t)v(t)dt\right) \leq \frac{1}{3}\theta^3(\rho^2 - \sigma^2) + \|y_0\|^2 - \|x_0\|^2.$$

It can be rewritten as follows

$$(3.4)\quad 2\left(y_0-x_0,\ \int_0^\theta(\theta-t)v(t)dt\right)\le\frac{1}{3}\theta^3(\rho^2-\sigma^2)-\left(\|y_0\|^2-2(x_0,\,y_0)+\|x_0\|^2\right).$$

According to (3.3) we have

$$\int_0^\theta ||u(t)||^2dt=\frac{3}{\theta^3}||y_0-x_0||^2+\frac{3}{\theta^3}\cdot 2\left(y_0-x_0,\ \int_0^\theta(\theta-t)v(t)dt\right)+\int_0^\theta ||v(t)||^2dt.$$

Combining this with (3.4) we obtain the following one

$$\begin{aligned}\int_0^\theta ||u(t)||^2dt&\le\frac{3}{\theta^3}||y_0-x_0||^2+\frac{3}{\theta^3}\cdot\left(\frac{\theta^3}{3}(\rho^2-\sigma^2)-||y_0-x_0||^2\right)+\sigma^2\\&=\rho^2-\sigma^2+\sigma^2=\rho^2.\end{aligned}$$

Thus, strategy of the pursuer is admissible.

3° Proof that $x(\theta)=y(\theta)$. Indeed, by (3.3)

$$\begin{aligned}x(\theta)&=x_0+\int_0^\theta(\theta-t)u(t)dt\\&=x_0+\int_0^\theta(\theta-t)\cdot\left(\frac{3}{\theta^3}(\theta-t)(y_0-x_0)+v(t)\right)dt\\&=y_0+\int_0^\theta(\theta-t)v(t)dt=y(\theta).\end{aligned}$$

Hence $x(\theta)=y(\theta)$. This completes the proof of Lemma 8.

It should be noted that in construction of the pursuer's strategy we have not required the inequality $\rho\ge\sigma$.

3.2. Some properties of balls and half-spaces in Hilbert space

Suppose that we have finitely or countably many closed balls $H(y_0,r)$ and $H(x_{i0},\,R_i)$, $i\in I$. Let

$$I_0=\{i\in I\mid S(y_0,\ r)\cap H(x_{i0},\,R_i)\ne\emptyset\},$$

$$X_i=\left\{z\in l_2\ \middle|\ 2(y_0-x_{i0},\,z)\le R_i^2-r^2+\|y_0\|^2-\|x_{i0}\|^2\right\}\quad if\ x_{i0}\ne y_0,\quad i\in I_0,$$

and

$$(3.5)\qquad X_i=\{z\in l_2\mid\ (z-y_0,\,p_0)\le R_i\}\ \ if\ x_{i0}=y_0,\ i\in I_0.$$

Note that the half-space (3.5) contains the ball $H(x_{i0},\,R_i)=H(y_0,\,R_i)$.

Lemma 9. *If Assumption 6 is valid and*

$$H(y_0, r) \subset \bigcup_{i \in I} H(x_{i0}, R_i), \tag{3.6}$$

then

$$H(y_0, r) \subset \bigcup_{i \in I_0} X_i. \tag{3.7}$$

Proof. Indeed, if $x_{i0} \neq y_0$, $i \in I_0$, then by the Corollary to Assertion 2 [7, p. 634] the inclusion

$$S(y_0, r) \cap H(x_{i0}, R_i) \subset X_i, \qquad i \in I_0, \tag{3.8}$$

is valid. We show that the inclusion (3.8) is also true for the case $x_{i0} = y_0$, $i \in I_0$. If this is the case, then either $r > R_i$ or $r \le R_i$. For $r > R_i$, the intersection $S(y_0, r) \cap H(y_0, R_i) = \emptyset$, and therefore by definition of I_0 we get $i \notin I_0$. We drop this case since we deal only with $i \in I_0$. In the latter case, i.e. if $r \le R_i$, we have

$$S(y_0, r) \cap H(x_{i0}, R_i) = S(x_{i0}, r) \cap H(x_{i0}, R_i) = H(x_{i0}, R_i) \subset X_i, i \in I_0.$$

Thus, for all $i \in I_0$ the inclusion (3.8) is true and therefore from it we get

$$\bigcup_{i \in I_0} (S(y_0, r) \cap H(x_{i0}, R_i)) \subset \bigcup_{i \in I_0} X_i.$$

Hence,

$$S(y_0, r) \cap \left(\bigcup_{i \in I_0} H(x_{i0}, R_i) \right) \subset \bigcup_{i \in I_0} X_i. \tag{3.9}$$

On the other hand, from (3.6) we obtain

$$H(y_0, r) \cap S(y_0, r) \subset \left(\bigcup_{i \in I} H(x_{i0}, R_i) \right) \cap S(y_0, r).$$

Consequently,

$$\begin{aligned} S(y_0, r) &\subset \bigcup_{i \in I} (H(x_{i0}, R_i) \cap S(y_0, r)) \\ &= \bigcup_{i \in I_0} (H(x_{i0}, R_i) \cap S(y_0, r)) \subset \bigcup_{i \in I_0} H(x_{i0}, R_i), \end{aligned} \tag{3.10}$$

since $H(x_{i0},\ R_i) \cap S(y_0,\ r) = \emptyset,\ i \in I \backslash I_0$. Then combining (3.9) and (3.10) yields

$$S(y_0,\ r) \subset \bigcup_{i \in I_0} X_i. \tag{3.11}$$

We proceed to show the inclusion (3.7). Assume the contrary. Then there exists a point $\bar{y} \in H(y_0, r)$ such that $\bar{y} \notin \bigcup_{i \in I_0} X_i$. This implies that $\bar{y} \notin X_i$ for all $i \in I_0$. Then it follows from the definition of X_i that

$$2(y_0 - x_{i0},\, \bar{y}) > R_i^2 - r^2 + \|y_0\|^2 - \|x_{i0}\|^2,\ if\ x_{i0} \neq y_0, \tag{3.12}$$

$$(\bar{y} - y_0,\, p_0) > R_i,\ if\ \ x_{i0} = y_0.$$

Since by Assumption 6 $(y_0 - x_{i0},\, p_0) \geq 0,\ i \in I_0$, then by (3.12) for all points of the ray $\xi(t) = \bar{y} + p_0 t,\ t \geq 0$, we have

$$\begin{aligned} 2(y_0 - x_{i0}, \xi(t)) &= 2(y_0 - x_{i0},\, \bar{y}) + 2(y_0 - x_{i0},\, p_0)t \\ &\geq 2(y_0 - x_{i0},\, \bar{y}) \\ &> R_i^2 - r^2 + \|y_0\|^2 - \|x_{i0}\|^2,\ i \in I_0, \end{aligned}$$

for $x_{i0} \neq y_0$, and

$$(\xi(t) - y_0,\, p_0) = (\bar{y} - y_0,\, p_0) + t \geq (\bar{y} - y_0,\, p_0) > R_i,\ i \in I_0,$$

for $x_{i0} = y_0$. Thus, $\xi(t) \notin X_i$ for all $i \in I_0$ and $t \geq 0$, and hence $\xi(t) \notin \bigcup_{i \in I_0} X_i$, for all $t \geq 0$.

On the other hand, the ray $\xi(t),\ t \geq 0$, intersect the sphere $S(y_0,\ r)$ at some point $\xi_0 \in S(y_0,\ r)$, which according to (3.11) must be in $\bigcup_{i \in I_0} X_i$. Contradiction. The proof of Lemma 9 is complete.

Lemma 10. (see Assertion 5 [7, p. 635]). *Let* $\inf_{i \in I} R_i = R_0 > 0$. *If Assumption 6 is true and for any* $0 < \varepsilon < R_0$ *the set* $\bigcup_{i \in I} H(x_{i0},\, R_i - \varepsilon)$ *does not contain the ball* $H(y_0,\, r)$, *then there exists a point* $\bar{y} \in S(y_0,\, r)$ *such that* $\|\bar{y} - x_{i0}\| \geq R_i$ *for all* $i \in I$.

3.3. Proof of the theorem

1. *Construction of the Pursuer's strategies.* We introduce counterfeit pursuers (CP) $z_i,\ i \in I$ whose motions are described by the equations

$$\dot{z}_i = (\theta - t)w_i,\ z_i(0) = x_{i0},$$

$$\left(\int_0^\infty \|w_i(s)\|^2\, ds \right)^{1/2} \leq \bar{\rho}_i = \rho_i + \gamma \left(\frac{3}{\theta^3} \right)^{1/2}.$$

Clearly, the attainability set of the CP z_i from the initial state x_{i0} at $t = 0$ to the time θ is the ball $H\left(x_{i0},\ \bar{\rho}_i(\theta^3/3)^{1/2}\right) = H\left(x_{i0},\ \rho_i(\theta^3/3)^{1/2} + \gamma\right)$. Define the strategies of CPs z_i, $i \in I$, on $0 \le t \le \theta$ as follows

$$w_i(t,\ v) = 3\theta^{-3}(\theta - t)(y_0 - x_{i0}) + v(t). \tag{3.13}$$

This strategy needs to some comments. The maximum energy of the CP z_i, $i \in I$, is $\bar{\rho}_i^2$. CP z_i might not be able to move according to (3.13). If for the strategy (3.13)

$$\int_0^{\tau} ||3\theta^{-3}(\theta - t)(y_0 - x_{i0}) + v(t)||^2 dt = \bar{\rho}_i^2$$

at the first time $t = \tau \in [0,\ \theta]$, then the energy of the CP z_i is exhausted at this time and the CP z_i cannot move any more. Then, automatically $w(t, v) = 0,\ \tau < t \le \theta$, and therefore $z_i(\tau) = z_i(t) = z_i(\theta)$, $\tau \le t \le \theta$. The strategies of the real pursuers x_i are defined as follows

$$u_i(t,\ v) = \frac{\rho_i}{\bar{\rho}_i} w_i(t,\ v),\ i \in I,\ 0 \le t \le \theta. \tag{3.14}$$

2. *Proof that the value γ is guaranteed for the pursuers.* Let us show that the strategies of the pursuers (3.14) ensure the inequality

$$\sup_{v(\cdot)} \inf_{i \in I} \|y(\theta) - x_i(\theta)\| \le \gamma.$$

Indeed, by definition of γ, we have

$$H\left(y_0,\ \sigma(\theta^3/3)^{1/2}\right) \subset \bigcup_{i \in I} H\left(x_{i0},\ \rho_i(\theta^3/3)^{1/2} + \gamma\right). \tag{3.15}$$

Then it follows from Lemma 9 where $R_i = \rho_i(\theta^3/3)^{1/2} + \gamma$, $r = \sigma(\theta^3/3)^{1/2}$ that

$$H\left(y_0,\ \sigma(\theta^3/3)^{1/2}\right) \subset \bigcup_{i \in I_0} X_i,$$

where

$$\begin{aligned} I_0 &= \left\{ i \in I \,\middle|\, S\left(y_0,\ \sigma(\theta^3/3)^{1/2}\right) \cap H\left(x_{i0},\ \rho_i(\theta^3/3)^{1/2} + \gamma\right) \ne \emptyset \right\}, \\ X_i &= \left\{ z \,\middle|\, 2(y_0 - x_{i0}, z) \le \left(\rho_i(\theta^3/3)^{1/2} + \gamma\right)^2 - \sigma^2(\theta^3/3) + \|y_0\|^2 - \|x_{i0}\|^2 \right\} \end{aligned} \tag{3.16}$$

if $x_{i0} \ne y_0$, and

$$X_i = \left\{ z \,\middle|\, (z - y_0,\ p_0) \le \rho_i(\theta^3/3)^{1/2} + \gamma \right\}, \tag{3.17}$$

if $x_{i0} = y_0$ and $\rho_i(\theta^3/3)^{1/2} + \gamma \geq \sigma(\theta^3/3)^{1/2}$. Consequently, the point $y(\theta) \in H\left(y_0, \sigma(\theta^3/3)^{1/2}\right)$ belongs to a half-space X_s, $s \in I_0$. The half-space X_s can be of the either form (3.16) or (3.17). In the former case $x_{s0} \neq y_0$, and so according to Lemma 8 for the strategy (3.13) of the CP z_s we obtain

$$\int_0^\theta ||3\theta^{-3}(\theta - t)(y_0 - x_{s0}) + v(t)||^2 dt \leq \bar{\rho}_s^2$$

and $z_s(\theta) = y(\theta)$.

In the latter case, $x_{s0} = y_0$, $\bar{\rho}_i = \rho_i + (3/\theta^3)^{1/2}\gamma \geq \sigma$, and the strategy (3.13) takes the form $w_s(t, v) = v$.

Then

$$\begin{aligned} z_s(\theta) &= x_{s0} + \int_0^\theta (\theta - t)w(t)dt \\ &= y_0 + \int_0^\theta (\theta - t)v(t)dt = y(\theta). \end{aligned}$$

What CP z_s can use strategy $w_s(t,\ v\ ) = v(t)$ on the time interval $0 \leq t \leq \theta$ follows from

$$\int_0^\theta \|v(s)\|^2\, ds \leq \sigma^2 \leq \bar{\rho}^2.$$

Thus, $z_s(\theta) = y(\theta)$ in both cases. We'll now show that $\|x_s(\theta) - y(\theta)\| \leq \gamma$. Indeed, by (3.14)

$$\begin{aligned} \|x_s(\theta) - y(\theta)\| &= \|x_s(\theta) - z(\theta)\| \\ &= \left\| \int_0^\theta (\theta - t)u_s(t)dt - \int_0^\theta (\theta - t)w_s(t)dt \right\| \\ &= \left\| \int_0^\theta (\theta - t)\left(\frac{\rho_s}{\bar{\rho}_s} - 1\right) w_s(t)dt \right\| \\ &\leq \left(\frac{3}{\theta^3}\right)^{1/2} \gamma \cdot \frac{1}{\bar{\rho}_s} \int_0^\theta (\theta - t)\, \|w_s(t)\|\, dt. \end{aligned} \tag{3.18}$$

Using the Cauchy-Schwartz inequality, yields

$$\int_0^\theta (\theta - t)\, \|w_s(t)\|\, dt \leq \left(\int_0^\theta (\theta - t)^2 dt\right)^{1/2} \left(\int_0^\theta \|w_s(t)\|^2\, dt\right)^{1/2} \leq \bar{\rho}_s \left(\frac{\theta^3}{3}\right)^{1/2}.$$

Combining this inequality with (3.18) we obtain

$$\|x_s(\theta) - y(\theta)\| \leq \gamma.$$

Thus, the value γ is guaranteed by the actual pursuers.

3. *Proof that the value γ is guaranteed for the evader.* Let us construct the evader's strategy ensuring that

$$\inf_{u_1(\cdot),\,...,\,u_m(\cdot),\,...}\ \inf_{i\in I} \|y(\theta)-x_i(\theta)\| \geq \gamma, \tag{3.19}$$

where $u_1(\cdot)$, ..., $u_m(\cdot)$, ... are arbitrary admissible controls of the pursuers. If $\gamma = 0$, then the inequality (3.19) is obviously valid for any admissible control of the evader. Let $\gamma > 0$. By definition of γ, for any $0 < \varepsilon < \gamma$ the set

$$\bigcup_{i\in 1} H\left(x_{i0},\, \rho_i\left(\theta^3/3\right)^{1/2} + \gamma - \varepsilon\right),$$

does not contain the ball $H\left(y_0,\, \sigma(\theta^3/3)^{1/2}\right)$. Then, by Lemma 9 there exists a point $\bar{y} \in S\left(y_0,\, \sigma(\theta^3/3)^{1/2}\right)$, that is, $\|\bar{y}-y_0\| = \sigma(\theta^3/3)^{1/2}$ such that $\|\bar{y}-x_{i0}\| \geq \rho_i(\theta^3/3)^{1/2} + \gamma$, $i \in I$. On the other hand,

$$\|x_i(\theta)-x_{i0}\| \leq \left(\frac{\theta^3}{3}\right)^{1/2}\left(\int_0^\theta \|u_i(t)\|^2\,dt\right)^{1/2} \leq \rho_i\left(\frac{\theta^3}{3}\right)^{1/2}, i \in I.$$

Consequently,

$$\|\bar{y}-x_i(\theta)\| \geq \|\bar{y}-x_{i0}\| - \|x_i(\theta)-x_{i0}\| \geq \rho_i(\theta^3/3)^{1/2} + \gamma - \rho_i(\theta^3/3)^{1/2} = \gamma,\ i \in I.$$

therefore, if the evader comes to the point $\bar{y}$ at the time θ, then the value γ will be guaranteed him. The control of the evader

$$v(t) = \sigma(\theta^3/3)^{1/2}(\theta-t)e,\ 0 \leq t \leq \theta,\ e = \frac{\bar{y}-y_0}{\|\bar{y}-y_0\|},$$

brings him to the point $\bar{y}$ at the time θ. Indeed, we have

$$\begin{aligned} y(\theta) &= y_0 + \int_0^\theta (\theta-s)v(s)ds \\ &= y_0 + e\int_0^\theta (\theta-s)^2\sigma(\theta^3/3)^{1/2}ds \\ &= y_0 + \sigma(\theta^3/3)^{1/2}e = \bar{y}. \end{aligned}$$

Thus, the value of the game is not less than γ, and inequality (3.19) holds. The proof of the theorem is complete.

4. Conclusion

A pursuit-evasion differential game of fixed duration with countably many pursuers has been studied. Control functions satisfy integral constraints. The value of the game has been found, and the optimal strategies of players have been constructed. The proof of the main result relies on the solution of an auxiliary differential game problem in half-space and on some properties of spheres.

It should be noted that the strategy (3.3) guarantees the equality $x(\theta) = y(\theta)$ whenever $y(\theta) \in X$ (see Lemma 8), even though $\sigma > \rho$, that is, the pursuer whose energy less than that of the evader, can "capture" the evader.

Note that Lemma 9 is a modification of Assertion 4 ([6, p.634]. We recall that the conclusion of the Assertion 4 was the inclusion $H(y_0,\ r) \subset \bigcup_{i\in I} X_i$. The important point to note for conclusion of Lemma 9 of present paper is that if we restrict the set I to I_0 by excluding all numbers $i \in I$ for which $S(y_0,\ r) \cap H(x_{i0},\ R_i) = \emptyset$, the inclusion $H(y_0,\ r) \subset \bigcup_{i\in I_0} X_i$ still holds. Applying this result to the differential game we obtained that all pursuers $P_i,\ i \in I\backslash I_0$, can be removed from the differential game and only the pursuers $P_i,\ i \in I_0$, can guarantee the desired result.

The advantage of using the strategy (3.14) lies in the fact that ρ_i need not to be greater than σ. In the work [8], ρ_i must be $\geq \sigma$ since otherwise constructed strategies of the pursuers in general are not defined. For example, if $\sigma > \rho$ in the formula (3.5) of [8, p.5], $u(t)$ is not defined at $v(t) = 0$.

We discuss now the strategy (3.3), $u(t, v) = 3\theta^{-3}(\theta - t)(y_0 - x_0) + v(t)$. In this formula, the first summand $3\theta^{-3}(\theta - t)(y_0 - x_0)$ is chosen so that if $v(t) = 0$, the pursuer reaches the point y_0 for the time θ. Therefore, the strategy (3.3) guarantees the equality $x(\theta) = y(\theta)$. Then due to the inclusion $y(\theta) \in X$ this strategy is admissible. Clearly, the strategy (3.3) is linear with respect to t and $v(t)$, and simpler than the strategy (3.5) in Ibragimov and Salimi [8], moreover, admits the case $\sigma > \rho$.

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