

APPROXIMATE SOLUTIONS FOR CONTINUOUS-TIME QUADRATIC FRACTIONAL PROGRAMMING PROBLEMS

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Abstract. In this article, a hybrid of the parametric method and discretization approach is proposed for a class of continuous-time quadratic fractional programming problems (CQFP). This approach leads to an approximation algorithm that solves the problem (CQFP) to any required accuracy. The analysis also shows that we can predetermine the size of discretization such that the accuracy of the corresponding approximate solution can be controlled within the predefined error tolerance. Hence, the trade-off between the quality of the results and the simplification of the problem can be controlled by the decision maker. Moreover, we prove the convergence of the searched sequence of approximate solutions.

1. INTRODUCTION

In this article, we shall pay our attention to a class of nonlinear optimal control problems with linear state constraints. Such a problem is called the *continuous-time quadratic fractional programming problem* (in short, the problem (CQFP)). The problem (CQFP), which will be defined in Section 2, is a generalization of the so-called *continuous-time linear programming problem* (in short, the problem (CLP)). The theory of the problem (CLP), which was originated from the “bottleneck problem” proposed by Bellman [3], has received considerable attention for a long time. Tyndall [30, 31], Levison [13] and Grinold [8] established strong duality results with varying algebraic restrictions on the problem. Meidan and Perold [14], Papageorgiou [17] and Schechter [26] have also obtained some interesting results of the problem (CLP). Anderson et al. [1, 2], Fleischer and Sethuraman [6], Pullan [18, 19] and Wang et al. [32] investigated a

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subclass of continuous-time linear programming problem, which is called the separated continuous-time linear programming problem and can be used to model the job-shop scheduling problems. In addition, Weiss [33] proposed a simplex-like algorithm to solve the separated continuous-time linear programming problem. Recently, Wen et al. [38] developed a numerical method to solve the non-separated continuous-time linear programming problem. On the other hand, the nonlinear type of continuous-time optimization problems was also studied by Farr and Hanson [4, 5], Grinold [8, 9], Hanson and Mond [12], Reiland [20, 21], Reiland and Hanson [22], Singh [27], Rojas-Medar et al. [23], Singh and Farr [28] and Nobakhtian and Pouryayevali [15, 16].

The optimization problem in which the objective function appears as a ratio of two real-valued function is known as a fractional programming problem. Due to its significance appearing in the information theory, stochastic programming and decomposition algorithms for large linear systems, the various theoretical and computational issues have received particular attention in the last decades. For more details on this topic, we may refer to Stancu-Minasian [29] and Schaible [24, 25]. In the literature, a number of optimality principles and duality models for fractional programming problems have been extended to some continuous-time fractional programming problems, one can consult Zalmai [42, 43, 44, 45]. However, in these works, the computational issues were not addressed. Recently, Wen and Wu [40, 41], Wen et al. [37] and Wen [34, 35, 36] have developed computational procedures by combining the parametric method and discrete approximation method to solve some classes of continuous-time fractional programming problems. To the limited knowledge of authors, the numerical methods for solving the problem (CQFP) are not studied so far. In this paper, by extending the methodology of [34], a hybrid of the parametric method and discretization approach is proposed for the problem (CQFP). This approach leads to an approximation algorithm that solves the problem (CQFP) to any required accuracy. The analysis also shows that we can predetermine the size of discretization such that the accuracy of the corresponding approximate solution can be controlled within the predefined error tolerance. Hence, the trade-off between the quality of the results and the simplification of the problem can be controlled by the decision maker. Moreover, we prove the convergence of the searched sequence of approximate solutions to the problem (CQFP).

The rest of this paper is organized as follows. In Section 2, we propose the auxiliary parametric quadratic problems, and establish many useful relations between the parametric problems and the problem (CQFP), which will be a cornerstone for designing a practical computational procedure. In Section 3, we propose a discrete approximation method for solving the auxiliary parametric quadratic problems. In Section 4, by using the different step sizes of discretization problems, we construct a sequence of continuous and strictly decreasing upper and lower bound functions with the unique zeros, respectively. Then, in Section 5, we use the zeros to determine a sequence of intervals which will shrink to the optimal value of the problem (CQFP) as

the size of discretization getting larger. Besides, we establish upper bounds of lengths of these intervals. Especially, we can predetermine the size of discretization such that the accuracy of the corresponding approximate solution to the problem (CQFP) can be controlled within the predefined error tolerance. Thereby, a practical approximation algorithm is proposed. Moreover, we prove the convergence of the searched sequence of approximate solutions to the problem (CQFP) in Section 6. The paper ends with conclusions in Section 7.

2. PARAMETRIC CONTINUOUS-TIME QUADRATIC PROGRAMMING PROBLEMS

Given $p, q \in \mathbb{N}$. Let $L^\infty([0, T], \mathbb{R}^p)$ be the space of all measurable and essentially bounded functions from a time space $[0, T]$ into the p -dimensional Euclidean space \mathbb{R}^p and let $C([0, T], \mathbb{R}^p)$ be the space of all continuous functions from $[0, T]$ into the \mathbb{R}^p . The problem (CQFP) is formulated as follows:

$$\begin{aligned}
 \text{(CQFP) maximize} \quad & \frac{\mu + \int_0^T \left\{ 1/2 \mathbf{x}(t)^\top D(t) \mathbf{x}(t) + \mathbf{f}(t)^\top \mathbf{x}(t) \right\} dt}{\xi + \int_0^T \left\{ 1/2 \mathbf{x}(t)^\top E(t) \mathbf{x}(t) + \mathbf{h}(t)^\top \mathbf{x}(t) \right\} dt} \\
 \text{subject to} \quad & B\mathbf{x}(t) \leq \mathbf{g}(t) + \int_0^t K\mathbf{x}(s)ds \text{ for all } t \in [0, T] \\
 & \mathbf{x}(t) \in L^\infty([0, T], \mathbb{R}_+^q),
 \end{aligned}$$

where

- $\mathbf{x}(t)$ is the decision variable, $T > 0$ is a given time horizon, and the superscript “ \top ” denotes the transpose operation of matrices.
- B and K are $p \times q$ matrices, $\mathbf{g} \in C([0, T], \mathbb{R}_+^p)$ and $\mathbb{R}_+^p = \{(x_1, \dots, x_p)^\top : x_i \geq 0 \text{ for } i = 1, \dots, p\}$.
- $D(t) = [d_{ij}(t)]_{q \times q}$ is a symmetric negative semi-definite matrix with $d_{ij}(t) \in C([0, T], \mathbb{R})$, $\mathbf{f} \in C([0, T], \mathbb{R}^q)$ and $\mu \in \mathbb{R}_+$; $E(t) = [e_{ij}(t)]_{q \times q}$ is a symmetric positive semi-definite matrix with $e_{ij}(t) \in C([0, T], \mathbb{R})$, $\mathbf{h} \in C([0, T], \mathbb{R}_+^q)$ and $\xi > 0$.

We also assume that $B = [B_{ij}]_{p \times q}$ and $K = [K_{ij}]_{p \times q}$ are $p \times q$ constant matrices satisfying

- $K_{ij} \geq 0$ for all $i = 1, \dots, p$ and $j = 1, \dots, q$;
- $B_{ij} \geq 0$ and $\sum_{i=1}^p B_{ij} > 0$ for all $i = 1, \dots, p$ and $j = 1, \dots, q$.

Let us write

$$(1) \quad \lambda = \frac{\mu + \int_0^T \left\{ 1/2 \mathbf{x}(t)^\top D(t) \mathbf{x}(t) + \mathbf{f}(t)^\top \mathbf{x}(t) \right\} dt}{\xi + \int_0^T \left\{ 1/2 \mathbf{x}(t)^\top E(t) \mathbf{x}(t) + \mathbf{h}(t)^\top \mathbf{x}(t) \right\} dt},$$

$$(2) \quad \Theta^{(\lambda)}(t) = [\theta_{ij}^{(\lambda)}(t)]_{q \times q} = D(t) - \lambda E(t)$$

and

$$(3) \quad \mathbf{a}^{(\lambda)}(t) = \mathbf{f}(t) - \lambda \mathbf{h}(t).$$

It is not difficult to see that the problem (CQFP) is equivalent to the following continuous-time optimization problem:

$$(CP) \quad \max \quad \lambda$$

subject to

$$\mu - \lambda \xi + \int_0^T \left\{ 1/2 \mathbf{x}(t)^\top \Theta^{(\lambda)}(t) \mathbf{x}(t) + \mathbf{a}^{(\lambda)}(t)^\top \mathbf{x}(t) \right\} dt = 0$$

$$B\mathbf{x}(t) \leq \mathbf{g}(t) + \int_0^t K\mathbf{x}(s) ds \text{ for all } t \in [0, T]$$

$$\mathbf{x}(t) \in L^\infty([0, T], \mathbb{R}_+^q) \text{ and } \lambda \in \mathbb{R}.$$

That is, if $\mathbf{x}(t)$ is feasible for the problem (CQFP) then $(\mathbf{x}(t), \lambda)$ is feasible for the problem (CP), where λ is defined as in (1). Conversely, if $(\mathbf{x}(t), \lambda)$ is feasible for the problem (CP) then $\mathbf{x}(t)$ is feasible for the problem (CQFP) with the objective value λ .

Remark 2.1. When we say that $(\mathbf{x}^*, \lambda^*)$ is an optimal solution of (CP), it means that the optimal objective value of (CP) is λ^* . However, when we say that the optimal objective value of (CP) is λ^* , it does not necessary say that the problem (CP) has an optimal solution $(\mathbf{x}^*, \lambda^*)$, and it just means that the optimal objective value λ^* is obtained by taking the supremum.

For convenience, given any optimization problem (P), we denote by $V(P)$ the optimal objective value of the problem (P); that is, $V(P)$ will be obtained by taking the supremum or infimum. In the sequel, we propose an auxiliary problem associated with the problem (CP) which will be proposed and formulated as the parametric continuous-time quadratic programming problem.

Given $\lambda \geq 0$, we consider the following continuous-time quadratic programming problem (in short, the problem (CQP $_\lambda$)):

$$\begin{aligned}
 (\text{CQP}_\lambda) \quad & \text{maximize} \quad \mu - \lambda\xi + \int_0^T \left\{ 1/2 \mathbf{x}(t)^\top \Theta^{(\lambda)}(t) \mathbf{x}(t) + \mathbf{a}^{(\lambda)}(t)^\top \mathbf{x}(t) \right\} dt \\
 & \text{subject to} \quad B\mathbf{x}(t) \leq \mathbf{g}(t) + \int_0^t K\mathbf{x}(s)ds \text{ for all } t \in [0, T] \\
 & \quad \mathbf{x}(t) \in L^\infty([0, T], \mathbb{R}_+^q).
 \end{aligned}$$

In the literature, the duality theorems of this kind of problems have already been established by Hanson [11] and Gogia and Gupta [10]. Based on these works, Wen et al. [39] provided an extended duality theorem and constructed a numerical solutions method. The numerical solutions method will be utilized to solve the problem (CQP_λ).

According to Wen et al. [39], the dual problem (DCQP_λ) of (CQP_λ) can be defined as follows:

$$\begin{aligned}
 (\text{DCQP}_\lambda) \quad & \text{minimize} \quad \mu - \lambda\xi + \int_0^T \left\{ -1/2 \mathbf{u}(t)^\top \Theta^{(\lambda)}(t) \mathbf{u}(t) + \mathbf{g}(t)^\top \mathbf{w}(t) \right\} dt \\
 & \text{subject to} \quad B^\top \mathbf{w}(t) - \int_t^T K^\top \mathbf{w}(s)ds \geq \Theta^{(\lambda)}(t) \mathbf{u}(t) + \mathbf{a}^{(\lambda)}(t) \text{ for } t \in [0, T], \\
 & \quad \mathbf{w}(\cdot) \in L^\infty([0, T], \mathbb{R}_+^p) \text{ and } \mathbf{u}(\cdot) \in L^\infty([0, T], \mathbb{R}^q).
 \end{aligned}$$

Since $\Theta^{(\lambda)}(t)$ is symmetric negative semi-definite for all $\lambda \geq 0$, by the same arguments given in Wen et al. [39], the weak and strong duality properties can be realized below.

Theorem 2.1. (Weak Duality between (CQP_λ) and (DCQP_λ)). *Let $\lambda \geq 0$. Considering the primal-dual pair problems (CQP_λ) and (DCQP_λ), for any feasible solutions $\mathbf{x}^{(0)}(t)$ and $(\mathbf{u}^{(0)}(t), \mathbf{w}^{(0)}(t))$ of problems (CQP_λ) and (DCQP_λ), respectively, we have*

$$\begin{aligned}
 & \mu - \lambda\xi + \int_0^T \left\{ 1/2 \mathbf{x}^{(0)}(t)^\top \Theta^{(\lambda)}(t) \mathbf{x}^{(0)}(t) + \mathbf{a}^{(\lambda)}(t)^\top \mathbf{x}^{(0)}(t) \right\} dt \\
 & \leq \mu - \lambda\xi + \int_0^T \left\{ -1/2 \mathbf{u}^{(0)}(t)^\top \Theta^{(\lambda)}(t) \mathbf{u}^{(0)}(t) + \mathbf{g}(t)^\top \mathbf{w}^{(0)}(t) \right\} dt;
 \end{aligned}$$

that is, $V(\text{CQP}_\lambda) \leq V(\text{DCQP}_\lambda)$.

Theorem 2.2. (Strong Duality between (CQP_λ) and (DCQP_λ)). *Let $\lambda \geq 0$. There exist optimal solutions $\bar{\mathbf{x}}^{(\lambda)}(t)$ and $(\bar{\mathbf{u}}^{(\lambda)}(t), \bar{\mathbf{w}}^{(\lambda)}(t))$ of the primal-dual pair problems (CQP_λ) and (DCQP_λ), respectively, such that $\bar{\mathbf{x}}^{(\lambda)}(t) = \bar{\mathbf{u}}^{(\lambda)}(t)$ and*

$$\begin{aligned}
 & \mu - \lambda\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda)}(t)^\top \Theta^{(\lambda)}(t) \bar{\mathbf{x}}^{(\lambda)}(t) + \mathbf{a}^{(\lambda)}(t)^\top \bar{\mathbf{x}}^{(\lambda)}(t) \right\} dt \\
 & = \mu - \lambda\xi + \int_0^T \left\{ -1/2 \bar{\mathbf{u}}^{(\lambda)}(t)^\top \Theta^{(\lambda)}(t) \bar{\mathbf{u}}^{(\lambda)}(t) + \mathbf{g}(t)^\top \bar{\mathbf{w}}^{(\lambda)}(t) \right\} dt;
 \end{aligned}$$

that is, $V(\text{CQP}_\lambda) = V(\text{DCQP}_\lambda)$.

In order to realize the relations between the problem (CP) and the problem (CQP_λ) , we define a function $\mathcal{F} : \mathbb{R}_+ \rightarrow \mathbb{R}$ by $\mathcal{F}(\lambda) = V(CQP_\lambda)$ for all $\lambda \geq 0$. Using the solvability of the problem (CQP_λ) and by a similar argument with [29, Theorem 4.5.2], we can obtain the following results.

Proposition 2.1. *The following statements hold true.*

- (i) *The real-valued function $\mathcal{F}(\lambda)$ is convex, hence is continuous.*
- (ii) *If $\lambda_1 < \lambda_2$, then $\mathcal{F}(\lambda_1) > \mathcal{F}(\lambda_2)$; that is, the real-valued function $\mathcal{F}(\cdot)$ is strictly decreasing.*

Many useful relations between (CQP_λ) and (CP) are given below. We omit the proof.

Proposition 2.2. *The following statements hold true.*

- (i) *Given any $\lambda \geq 0$, then $\mathcal{F}(\lambda) > 0$ if and only if $\lambda < V(\text{CP})$. Equivalently, $\mathcal{F}(\lambda) \leq 0$ if and only if $\lambda \geq V(\text{CP})$.*
- (ii) *Suppose that $(\bar{\mathbf{x}}(t), \lambda^*)$ is an optimal solution of (CP) with $V(\text{CP}) = \lambda^*$. Then $\bar{\mathbf{x}}(t)$ is an optimal solution of (CQP_{λ^*}) with $V(CQP_{\lambda^*}) = 0$; that is $\mathcal{F}(\lambda^*) = 0$.*
- (iii) *If there exists a $\lambda^* \geq 0$ such that $\mathcal{F}(\lambda^*) = 0$, then the optimal solution of the problem (CQP_{λ^*}) is also an optimal solution of (CQFP) and $V(\text{CQFP}) = \lambda^*$.*

By the above propositions, it can be shown that the problem (CQFP) is solvable. Let $\mathbf{1} = (1, 1, \dots, 1)^\top \in \mathbb{R}^p$ and

$$\hat{\rho} := \max_{j=1, \dots, q} \left\{ \frac{\sum_{i=1}^p K_{ij}}{\sum_{i=1}^p B_{ij}}, \frac{\max_{t \in [0, T]} f_j(t)}{\sum_{i=1}^p B_{ij}} \right\} \geq 0.$$

We define $\mathbf{w}^*(t) = \hat{\rho} e^{\hat{\rho}(T-t)} \mathbf{1}$ for all $t \in [0, T]$ and

$$(4) \quad \eta^* = \frac{1}{\xi} \left\{ \mu + \int_0^T \mathbf{g}(t)^\top \mathbf{w}^*(t) dt \right\} \geq 0.$$

Corollary 2.1. *There exists a unique λ^* in the closed interval $[\mu/\xi, \eta^*]$ such that $\mathcal{F}(\lambda^*) = 0$. That is,*

- $\frac{\mu}{\xi} \leq V(\text{CQFP}) \leq \eta^*$, and
- if $\bar{\mathbf{x}}^{(\lambda^*)}(t)$ is an optimal solution of the problem (CQP_{λ^*}) , then it is also an optimal solution of the problem (CQFP).

Proof. It is obvious that for all $\lambda \in \mathbb{R}_+$ the problem (CQP_λ) is feasible with the trivial feasible solution $\mathbf{0}(t) = 0$ for all $t \in [0, T]$. Hence,

$$\mathcal{F}\left(\frac{\mu}{\xi}\right) = V(\text{CQP}_{\frac{\mu}{\xi}}) \geq \mu - \frac{\mu}{\xi} \cdot \xi + \int_0^T \left\{ 1/2 \cdot \mathbf{0}(t)^\top \Theta^{(\lambda)}(t) \mathbf{0}(t) + \mathbf{a}^{(\lambda)}(t)^\top \mathbf{0}(t) \right\} dt = 0.$$

On the other hand, we claim that $(\mathbf{0}(t), \mathbf{w}^*(t))$ is feasible for (DCQP_λ) for all $\lambda \geq 0$. To see this, it is obvious that $\mathbf{w}^*(t) \geq 0$. By the definition of $\hat{\rho}$, $\hat{\rho}B^\top \mathbf{1} \geq K^\top \mathbf{1}$ and for $\lambda \geq 0$ we have

$$\hat{\rho}B^\top \mathbf{1} \geq \Theta^{(\lambda)}(t)\mathbf{0}(t) + \mathbf{a}^{(\lambda)}(t) \text{ for } t \in [0, T],$$

it follows that

$$\begin{aligned} & B^\top \mathbf{w}^*(t) - \int_t^T K^\top \mathbf{w}^*(s) ds \\ &= \hat{\rho}e^{\hat{\rho}(T-t)} B^\top \mathbf{1} - \int_t^T \hat{\rho}e^{\hat{\rho}(T-s)} ds K^\top \mathbf{1} \\ &= \hat{\rho}e^{\hat{\rho}(T-t)} B^\top \mathbf{1} + K^\top \mathbf{1} - e^{\hat{\rho}(T-t)} K^\top \mathbf{1} \\ &= e^{\hat{\rho}(T-t)} (\hat{\rho}B - K)^\top \mathbf{1} + K^\top \mathbf{1} \\ &\geq (\hat{\rho}B - K)^\top \mathbf{1} + K^\top \mathbf{1} = \hat{\rho}B^\top \mathbf{1} \\ &\geq \Theta^{(\lambda)}(t)\mathbf{0}(t) + \mathbf{a}^{(\lambda)}(t) \text{ for } t \in [0, T] \end{aligned}$$

and our claim is valid. Thus, by the definition of η^* we have

$$\mathcal{F}(\eta^*) = V(\text{DCQP}_{\eta^*}) \leq \mu - \eta^* \xi + \int_0^T \mathbf{g}(t)^\top \mathbf{w}^*(t) dt = 0.$$

Therefore, $\mathcal{F}(\eta^*) \leq 0 \leq \mathcal{F}(\frac{\mu}{\xi})$ and the corollary follows by Propositions 2.1 and 2.2. ■

From the above discussions, it follows that solving the problem (CQFP) is equivalent to determine the unique root of the nonlinear equation $\mathcal{F}(\lambda) = 0$. However, it is notoriously difficult to find the exact solution of every (CQP_λ) . In the next section, given a λ in the closed interval $[\mu/\xi, \eta^*]$, we shall utilize the discrete approximation procedure developed by Wen et al. [39] to find the approximate value of $\mathcal{F}(\lambda)$ and to estimate its error bound.

3. APPROXIMATE SOLUTIONS TO THE PROBLEM (CQP_λ)

Given $\lambda \in [\mu/\xi, \eta^*]$. For each $n \in \mathbb{N}$, we take

$$\mathcal{P}_n = \left\{ 0, \frac{T}{n}, \frac{2T}{n}, \dots, \frac{(n-1)T}{n}, T \right\}$$

as a partition of $[0, T]$, which divides $[0, T]$ into n subintervals with equal length T/n . For $l = 1, \dots, n$, we define

$$(5) \quad \Theta^{(\lambda, n, l)} = \left[\theta_{ij}^{(\lambda, n, l)} \right]_{q \times q},$$

$$(6) \quad \mathbf{a}_l^{(\lambda, n)} = \left(a_{1l}^{(\lambda, n)}, a_{2l}^{(\lambda, n)}, \dots, a_{ql}^{(\lambda, n)} \right)^\top \in \mathbb{R}^q$$

and

$$(7) \quad \mathbf{b}_l^{(n)} = \left(b_{1l}^{(n)}, b_{2l}^{(n)}, \dots, b_{pl}^{(n)} \right)^\top \in \mathbb{R}_+^p,$$

where

$$(8) \quad \begin{aligned} \theta_{ij}^{(\lambda, n, l)} &= \min \left\{ \theta_{ij}^{(\lambda)}(t) : t \in \left[\frac{l-1}{n}T, \frac{l}{n}T \right] \right\} \\ &= \min \left\{ d_{ij}(t) - \lambda \cdot e_{ij}(t) : t \in \left[\frac{l-1}{n}T, \frac{l}{n}T \right] \right\}, \end{aligned}$$

$$(9) \quad \begin{aligned} a_{jl}^{(\lambda, n)} &= \min \left\{ f_j(t) - \lambda h_j(t) : t \in \left[\frac{(l-1)T}{n}, \frac{lT}{n} \right] \right\} \\ &\text{for } j = 1, \dots, q \text{ and } l = 1, \dots, n \end{aligned}$$

and

$$(10) \quad b_{il}^{(n)} = \min \left\{ g_i(t) : t \in \left[\frac{(l-1)T}{n}, \frac{lT}{n} \right] \right\} \text{ for } i = 1, \dots, p \text{ and } l = 1, \dots, n.$$

We note that, since the parameter λ is nonnegative, the constant matrix $\Theta^{(\lambda, n, l)}$ is symmetric negative semi-definite for all n and l .

From Wen et al. [39], the discrete version of the problem (CQP_λ) can be defined as the following finite-dimensional quadratic programming problem

$$\begin{aligned} (\text{Q}_n^{(\lambda)}) \quad & \text{maximize} \quad \mu - \lambda \xi + \frac{T}{n} \sum_{l=1}^n \left\{ 1/2 \mathbf{x}_l^\top \Theta^{(\lambda, n, l)} \mathbf{x}_l + (\mathbf{a}_l^{(\lambda, n)})^\top \mathbf{x}_l \right\} \\ & \text{subject to} \quad B \mathbf{x}_l - \frac{T}{n} K \sum_{r=1}^{l-1} \mathbf{x}_r \leq \mathbf{b}_l^{(n)} \text{ for } l = 1, \dots, n \\ & \quad \mathbf{x}_l \in \mathbb{R}_+^q \text{ for } l = 1, \dots, n, \end{aligned}$$

where, the “empty sum” $\sum_{l=1}^0 \mathbf{x}_l$ is defined to be the zero vector. The dual problem $(DQ_n^{(\lambda)})$ of $(Q_n^{(\lambda)})$ is defined by

$$\begin{aligned}
 (DQ_n^{(\lambda)}) \quad & \text{minimize} \quad \mu - \lambda\xi + \frac{T}{n} \sum_{l=1}^n \left\{ -1/2 \mathbf{u}_l^\top \Theta^{(\lambda,n,l)} \mathbf{u}_l + (\mathbf{b}_l^{(n)})^\top \mathbf{w}_l \right\} \\
 & \text{subject to} \quad B^\top \mathbf{w}_l - \frac{T}{n} K^\top \sum_{r=l+1}^n \mathbf{w}_r \geq \Theta^{(\lambda,n,l)} \mathbf{u}_l + \mathbf{a}_l^{(\lambda,n)} \\
 (11) \quad & \text{for } l = 1, 2, \dots, n \\
 & \mathbf{w}_l \in \mathbb{R}_+^p \text{ for } l = 1, \dots, n \text{ and} \\
 & \mathbf{u}_l \in \mathbb{R}^q \text{ for } l = 1, \dots, n,
 \end{aligned}$$

where the “empty sum” $\sum_{l=n+1}^n \mathbf{y}_l$ is defined to be the zero vector.

The duality properties between $(Q_n^{(\lambda)})$ and $(DQ_n^{(\lambda)})$ can be established, one can refer to [39].

Proposition 3.1. *Let $\mathbf{x}^{(n)} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ and $(\mathbf{u}^{(n)}, \mathbf{w}^{(n)})$ with $\mathbf{u}^{(n)} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ and $\mathbf{w}^{(n)} = (\mathbf{w}_1, \dots, \mathbf{w}_n)$ be feasible solutions of $(Q_n^{(\lambda)})$ and $(DQ_n^{(\lambda)})$, respectively. Then*

$$\begin{aligned}
 & \mu - \lambda\xi + \frac{T}{n} \sum_{l=1}^n \{ 1/2 \mathbf{x}_l^\top \Theta^{(\lambda,n,l)} \mathbf{x}_l + (\mathbf{a}_l^{(\lambda,n)})^\top \mathbf{x}_l \} \\
 & \leq \mu - \lambda\xi + \frac{T}{n} \sum_{l=1}^n \{ -1/2 \mathbf{u}_l^\top \Theta^{(\lambda,n,l)} \mathbf{u}_l + (\mathbf{b}_l^{(n)})^\top \mathbf{w}_l \}.
 \end{aligned}$$

That is, $V(Q_n^{(\lambda)}) \leq V(DQ_n^{(\lambda)})$.

Proposition 3.2. *There exist a feasible solution $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_n)$ of primal problem $(Q_n^{(\lambda)})$ and a feasible solution $(\bar{\mathbf{u}}, \bar{\mathbf{w}})$ of dual problem $(DQ_n^{(\lambda)})$ with $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n)$ and $\bar{\mathbf{w}} = (\bar{\mathbf{w}}_1, \dots, \bar{\mathbf{w}}_n)$ such that $\bar{\mathbf{x}} = \bar{\mathbf{u}}$ and*

$$\begin{aligned}
 & \mu - \lambda\xi + \frac{T}{n} \sum_{l=1}^n \{ 1/2 \bar{\mathbf{x}}_l^\top \Theta^{(\lambda,n,l)} \bar{\mathbf{x}}_l + (\mathbf{a}_l^{(\lambda,n)})^\top \bar{\mathbf{x}}_l \} \\
 & = \mu - \lambda\xi + \frac{T}{n} \sum_{l=1}^n \{ -1/2 \bar{\mathbf{u}}_l^\top \Theta^{(\lambda,n,l)} \bar{\mathbf{u}}_l + (\mathbf{b}_l^{(n)})^\top \bar{\mathbf{w}}_l \}.
 \end{aligned}$$

That is, $\bar{\mathbf{x}}$ and $(\bar{\mathbf{u}}, \bar{\mathbf{w}})$ are optimal solutions of problems $(Q_n^{(\lambda)})$ and $(DQ_n^{(\lambda)})$, respectively.

By straightforward modifications of Lemma 3.1 and Lemma 3.2 in [39], we can see the boundedness of optimal solutions to the problems $(Q_n^{(\lambda)})$ and $(DQ_n^{(\lambda)})$. To see this, let

$$(12) \quad \sigma = \min \{ B_{ij} : B_{ij} > 0 \},$$

$$(13) \quad \nu = \max_{j=1, \dots, q} \left\{ \sum_{i=1}^p K_{ij} \right\},$$

$$(14) \quad \zeta = \max \{g_i(t) : i = 1, \dots, p \text{ and } t \in [0, T]\},$$

$$(15) \quad \tau(\lambda) = \max_{j=1, \dots, q} \max_{t \in [0, T]} \{f_j(t) - \lambda h_j(t), 0\},$$

$$(16) \quad \pi(\lambda) = \max_{i=1, \dots, q} \max_{j=1, \dots, q} \max_{t \in [0, T]} \{d_{ij}(t) - \lambda e_{ij}(t), 0\},$$

$$(17) \quad M_1 = \frac{q\zeta}{\sigma} \cdot \exp\left(\frac{q\nu T}{\sigma}\right)$$

and

$$(18) \quad M_2(\lambda) = \frac{1}{\sigma} \{M_1 \pi(\lambda) + \tau(\lambda)\} \exp\left(\frac{\nu T}{\sigma}\right).$$

Lemma 3.1. *Given any $n \in \mathbb{N}$, if $(\mathbf{x}_1^{(\lambda, n)}, \mathbf{x}_2^{(\lambda, n)}, \dots, \mathbf{x}_n^{(\lambda, n)})$ is a feasible solution of the primal problem $(Q_n^{(\lambda)})$, where $\mathbf{x}_l^{(\lambda, n)} = (x_{1l}^{(\lambda, n)}, x_{2l}^{(\lambda, n)}, \dots, x_{ql}^{(\lambda, n)})^\top \in \mathbb{R}_+^q$, then*

$$(19) \quad 0 \leq x_{jl}^{(\lambda, n)} \leq \frac{M_1}{q} \text{ for all } j = 1, \dots, q \text{ and } l = 1, \dots, n$$

and

$$(20) \quad \mu - \lambda\xi \leq V(Q_n^{(\lambda)}) \leq \mu - \lambda\xi + M_1\tau(\lambda)T.$$

This says that the feasible sets of the problems $(Q_n^{(\lambda)})$ are uniformly bounded in the sense that the bounds of $x_{jl}^{(\lambda, n)}$ are independent of n and λ .

Lemma 3.2. *The dual problem $(DQ_n^{(\lambda)})$ has an optimal solution $(\tilde{\mathbf{u}}^{(\lambda, n)}, \hat{\mathbf{w}}^{(\lambda, n)})$ with $\hat{\mathbf{w}}^{(\lambda, n)} = (\hat{\mathbf{w}}_1^{(\lambda, n)}, \dots, \hat{\mathbf{w}}_n^{(\lambda, n)})$ such that $\tilde{\mathbf{u}}^{(\lambda, n)}$ is also an optimal solution of $(Q_n^{(\lambda)})$ and*

$$(21) \quad 0 \leq \hat{w}_{il}^{(\lambda, n)} \leq M_2(\lambda)$$

for all $i = 1, \dots, p$ and $l = 1, \dots, n$. Moreover, we have

$$(22) \quad \mu - \lambda\xi \leq V(DQ_n^{(\lambda)}) \leq \mu - \lambda\xi + \left(\frac{1}{2}\pi(\lambda) M_1^2 + p\zeta M_2(\lambda)\right) T.$$

Besides, the optimal solutions of the problems $(Q_n^{(\lambda)})$ and $(DQ_n^{(\lambda)})$ can be utilized to construct the feasible solutions of the problems (CQP_λ) and $(DCQP_\lambda)$, respectively. To see this, let $\bar{\mathbf{x}}^{(\lambda,n)} = (\bar{\mathbf{x}}_1^{(\lambda,n)}, \bar{\mathbf{x}}_2^{(\lambda,n)}, \dots, \bar{\mathbf{x}}_n^{(\lambda,n)})$, where $\bar{\mathbf{x}}_l^{(\lambda,n)} = (\bar{x}_{1l}^{(\lambda,n)}, \dots, \bar{x}_{pl}^{(\lambda,n)})^\top$ for $l = 1, \dots, n$, be an optimal solution of the problem $(Q_n^{(\lambda)})$; and let $(\bar{\mathbf{x}}^{(\lambda,n)}, \bar{\mathbf{w}}^{(\lambda,n)})$ be an optimal solution of dual problem $(DQ_n^{(\lambda)})$, where $\bar{\mathbf{w}}^{(\lambda,n)} = (\bar{\mathbf{w}}_1^{(\lambda,n)}, \dots, \bar{\mathbf{w}}_n^{(\lambda,n)})$ and $\bar{\mathbf{w}}_l^{(\lambda,n)} = (\bar{w}_{1l}^{(\lambda,n)}, \dots, \bar{w}_{pl}^{(\lambda,n)})^\top$ for $l = 1, \dots, n$, such that Lemma 3.2 holds true. For $j = 1, \dots, q$, we define the step functions $\bar{x}_j^{(\lambda,n)} : [0, T] \rightarrow \mathbb{R}$ as follows:

$$(23) \quad \bar{x}_j^{(\lambda,n)}(t) = \begin{cases} \bar{x}_{jl}^{(\lambda,n)}, & \text{if } \frac{(l-1)T}{n} \leq t < \frac{lT}{n} \\ \bar{x}_{jn}^{(\lambda,n)}, & \text{if } t = T, \end{cases}$$

where $l = 1, \dots, n$. Then we can form a vector-valued function $\bar{\mathbf{x}}^{(\lambda,n)} : [0, T] \rightarrow \mathbb{R}^q$ by

$$(24) \quad \bar{\mathbf{x}}^{(\lambda,n)}(t) = \left(\bar{x}_1^{(\lambda,n)}(t), \bar{x}_2^{(\lambda,n)}(t), \dots, \bar{x}_q^{(\lambda,n)}(t) \right)^\top.$$

In this case, we say that $\bar{\mathbf{x}}^{(\lambda,n)}(t)$ is a *natural solution* of (CQP_λ) constructed from $\bar{\mathbf{x}}^{(\lambda,n)}$.

In order to construct a feasible solution of the problem $(DCQP_\lambda)$ by virtue of $(\bar{\mathbf{x}}^{(\lambda,n)}, \bar{\mathbf{w}}^{(\lambda,n)})$, we need some notations. For $i = 1, \dots, p$ and $j = 1, \dots, q$, we define the step functions as follows

$$(25) \quad a_j^{(\lambda,n)}(t) = \begin{cases} a_{jl}^{(\lambda,n)}, & \text{if } \frac{(l-1)T}{n} \leq t < \frac{lT}{n} \\ a_{jn}^{(\lambda,n)}, & \text{if } t = T \end{cases}$$

and

$$(26) \quad g_i^{(n)}(t) = \begin{cases} b_{il}^{(n)}, & \text{if } \frac{(l-1)T}{n} \leq t < \frac{lT}{n} \\ b_{in}^{(n)}, & \text{if } t = T, \end{cases}$$

where $l = 1, \dots, n$, and $a_{jl}^{(\lambda,n)}$ and $b_{il}^{(n)}$ are defined in (9) and (10), respectively.

We also define the function $\Theta^{(\lambda,n)} : [0, T] \mapsto \mathbb{R}^{q \times q}$ by

$$\Theta^{(\lambda,n)}(t) = \left[\theta_{ij}^{(\lambda,n)}(t) \right]_{q \times q},$$

where

$$(27) \quad \theta_{ij}^{(\lambda,n)}(t) = \begin{cases} \theta_{ij}^{(\lambda,n,l)}, & \text{if } t \in \left[\frac{l-1}{n}T, \frac{l}{n}T \right) \text{ for some } 1 \leq l \leq n, \\ \theta_{ij}^{(\lambda,n,n)}, & \text{if } t = T \end{cases}$$

and $\theta_{ij}^{(\lambda,n,l)}$ is defined in (8).

Remark 3.1. Since each $f_j(t) - \lambda h_j(t)$ is continuous on the compact interval $[0, T]$ for $j = 1, \dots, q$, it follows that each $f_j(t) - \lambda h_j(t)$ is also uniformly continuous on the compact interval $[0, T]$ for all j . Therefore, the sequence of step functions $\{a_j^{(\lambda, n)}(t)\}_{n=1}^\infty$ converges to $f_j(t) - \lambda h_j(t)$ on $[0, T]$ for all j . Similarly, we can also conclude that the sequence of step functions $\{\theta_{ij}^{(\lambda, n)}(t)\}_{n=1}^\infty$ converges to $\theta_{ij}^{(\lambda)}(t)$ on $[0, T]$ for all i, j , and the sequence $\{g_i^{(n)}(t)\}_{n=1}^\infty$ converges to $g_i(t)$ on $[0, T]$ for all i .

For further discussion, we also adopt the following notations:

$$(28) \quad \epsilon_n(\lambda) = \max_{j=1, \dots, q} \sup_{t \in [0, T]} \left\{ f_j(t) - \lambda h_j(t) - a_j^{(n, \lambda)}(t) \right\} \\ + M_1 \max_{i=1, \dots, q} \max_{j=1, \dots, q} \sup_{t \in [0, T]} \left\{ \theta_{ij}^{(\lambda)}(t) - \theta_{ij}^{(\lambda, n)}(t) \right\},$$

$$(29) \quad \bar{\epsilon}_n = \max_{i=1, \dots, p} \sup_{t \in [0, T]} \left\{ g_i(t) - g_i^{(n)}(t) \right\}$$

$$(30) \quad \delta_n(\lambda) = \max_{i=1, \dots, p} \max_{l=1, \dots, n} \left\{ \frac{T}{n} \bar{w}_{il}^{(\lambda, n)} \right\}.$$

By Remark 3.1 and Lemma 3.2, we see that for all $\lambda \geq 0$,

$$\epsilon_n(\lambda) \rightarrow 0, \quad \bar{\epsilon}_n \rightarrow 0, \quad \text{and} \quad \delta_n(\lambda) \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty.$$

Now, we are going to construct a feasible solution of the problem $(DCQP_\lambda)$ by virtue of $(\bar{\mathbf{x}}^{(\lambda, n)}, \bar{\mathbf{w}}^{(\lambda, n)})$. We define a function $\hat{\mathbf{w}}^{(\lambda, n)}(t) : [0, T] \mapsto \mathbb{R}^p$ as follows:

$$(31) \quad \hat{\mathbf{w}}^{(\lambda, n)}(t) = \bar{\mathbf{w}}_l^{(\lambda, n)} + \delta_n(\lambda) \rho e^{\rho(T-t)} \mathbf{1} \quad \text{for} \quad t \in \left[\frac{l-1}{n}T, \frac{l}{n}T \right)$$

and

$$\hat{\mathbf{w}}^{(\lambda, n)}(T) = \bar{\mathbf{w}}_n^{(\lambda, n)} + \delta_n(\lambda) \rho \mathbf{1},$$

where $\mathbf{1} = (1, 1, \dots, 1)^\top \in \mathbb{R}^p$ and

$$(32) \quad \rho = \max_{j=1, \dots, q} \left\{ \frac{\sum_{i=1}^p K_{ij}}{\sum_{i=1}^p B_{ij}}, \frac{1}{\sum_{i=1}^p B_{ij}} \right\}.$$

Moreover, we define

$$(33) \quad \tilde{\mathbf{w}}^{(\lambda, n)}(t) = \hat{\mathbf{w}}^{(\lambda, n)}(t) + \epsilon_n(\lambda) \rho e^{\rho(T-t)} \mathbf{1}$$

for all $t \in [0, T]$, where $\epsilon_n(\lambda)$ is defined as in (28). If $\bar{x}^{(\lambda, n)}(t)$ is the natural solution of (CQP_λ) constructed from $\bar{x}^{(\lambda, n)}$ defined as in (24), then we also say that $(\bar{x}^{(\lambda, n)}(t), \tilde{w}^{(\lambda, n)}(t))$ is a *natural solution* of problem $(DCQP_\lambda)$ constructed from the optimal solution $(\bar{x}^{(\lambda, n)}, \bar{w}^{(\lambda, n)})$ of problem $(DQ_n^{(\lambda)})$.

After some algebraic calculations, it is easy to show the feasibility of natural solutions of (CQP_λ) .

Lemma 3.3. *Let $\bar{x}^{(\lambda, n)}$ be an optimal solution of $(Q_n^{(\lambda)})$. Then the natural solution $\bar{x}^{(\lambda, n)}(t)$ of problem (CQP_λ) constructed from $\bar{x}^{(\lambda, n)}$ is a feasible solution of (CQP_λ) . Moreover, we have*

$$(34) \quad \mathcal{F}(\lambda) = V(CQP_\lambda) \geq V(Q_n^{(\lambda)})$$

for all $n \in \mathbb{N}$.

By a similar argument with the proof of [39, Lemma 4.2], we can establish the following results.

Lemma 3.4. *Let $\bar{x}^{(\lambda, n)}$ and $(\bar{x}^{(\lambda, n)}, \bar{w}^{(\lambda, n)})$ be optimal solutions of $(Q_n^{(\lambda)})$ and $(DQ_n^{(\lambda)})$, respectively. Let $\bar{x}^{(\lambda, n)}(t)$ and $\tilde{w}^{(\lambda, n)}(t)$ be defined as in (24) and (33), respectively. Then the following statements hold true.*

(i) *The natural solution $(\bar{x}^{(\lambda, n)}(t), \tilde{w}^{(\lambda, n)}(t))$ is a feasible solution of dual problem $(DCQP_\lambda)$.*

(ii) *We have*

$$(35) \quad 0 \leq \widehat{Obj}(\bar{x}^{(\lambda, n)}(t), \tilde{w}^{(\lambda, n)}(t)) - V(DQ_n^{(\lambda)}) \leq \delta_n(\lambda) \int_0^T \rho e^{\rho(T-t)} \mathbf{g}(t)^\top \mathbf{1} dt,$$

where $\widehat{Obj}(\bar{x}^{(\lambda, n)}(t), \tilde{w}^{(\lambda, n)}(t))$ is the objective value of $(DCQP_\lambda)$ at $(\bar{x}^{(\lambda, n)}(t), \tilde{w}^{(\lambda, n)}(t))$; that is,

$$\begin{aligned} & \widehat{Obj}(\bar{x}^{(\lambda, n)}(t), \tilde{w}^{(\lambda, n)}(t)) \\ &= \mu - \lambda \xi + \int_0^T \left\{ -1/2 \bar{x}^{(\lambda, n)}(t)^\top \Theta^{(\lambda)}(t) \bar{x}^{(\lambda, n)}(t) + \mathbf{g}(t)^\top \tilde{w}^{(\lambda, n)}(t) \right\} dt. \end{aligned}$$

Lemma 3.3 says that the natural solution $\bar{x}^{(\lambda, n)}(t)$ of problem (CQP_λ) constructed from an optimal solution of $(Q_n^{(\lambda)})$ is an approximate solution of (CQP_λ) . Based on Lemma 3.4, we can establish the estimation of its error bound by slightly modifying the arguments given in [39, Theorem 4.1].

Proposition 3.3. *The following statements hold true.*

(i) We have

$$(36) \quad 0 \leq \mathcal{F}(\lambda) - V(Q_n^{(\lambda)}) \leq \varepsilon_n(\lambda),$$

where

$$(37) \quad \begin{aligned} \varepsilon_n(\lambda) := & \bar{\varepsilon}_n p \delta_n(\lambda) (n + \exp(\rho T) - 1) \\ & + (\varepsilon_n(\lambda) + \delta_n(\lambda)) \int_0^T \rho \exp(\rho(T-t)) (\mathbf{g}(t))^\top \mathbf{1} dt. \end{aligned}$$

(ii) We have

$$\lim_{n \rightarrow \infty} V(DQ_n^{(\lambda)}) = \lim_{n \rightarrow \infty} V(Q_n^{(\lambda)}) = \mathcal{F}(\lambda).$$

(iii) Let $\bar{\mathbf{x}}^{(n,\lambda)}(t)$ be the natural solution of (CQP_λ) . Then the error between the optimal objective value of (CQP_λ) and the objective value at $\bar{\mathbf{x}}^{(n,\lambda)}(t)$ is less than or equal to $\varepsilon_n(\lambda)$.

4. LOWER AND UPPER BOUND FUNCTIONS FOR $\mathcal{F}(\lambda)$

Due to the difficulty of finding the exact value of $\mathcal{F}(\lambda)$, we shall construct lower and upper bound functions for $\mathcal{F}(\lambda)$. To see this, we define

$$(38) \quad \hat{\mathbf{a}}_l^{(\lambda,n)} = \left(\hat{a}_{1l}^{(\lambda,n)}, \hat{a}_{2l}^{(\lambda,n)}, \dots, \hat{a}_{ql}^{(\lambda,n)} \right)^\top \in \mathbb{R}^q$$

and

$$(39) \quad \hat{\Theta}^{(\lambda,n,l)} = \left[\hat{\theta}_{ij}^{(\lambda,n,l)} \right]_{q \times q},$$

where, for $i, j = 1, \dots, q$ and $l = 1, \dots, n$,

$$(40) \quad \hat{a}_{jl}^{(\lambda,n)} = \int_{\frac{l-1}{n}T}^{\frac{l}{n}T} (f_j(t) - \lambda h_j(t)) dt$$

and

$$(41) \quad \hat{\theta}_{ij}^{(\lambda,n,l)} = \int_{\frac{l-1}{n}T}^{\frac{l}{n}T} \{d_{ij}(t) - \lambda e_{ij}(t)\} dt.$$

We note that since $\lambda \geq 0$, the matrix $\hat{\Theta}^{(\lambda,n,l)}$ is also symmetric nonnegative semi-definite for all n and l .

Instead of the problems $(Q_n^{(\lambda)})$ and $(DQ_n^{(\lambda)})$, we consider the following relaxed problem:

$$\begin{aligned}
 (\Psi Q_n^{(\lambda)}) \quad & \text{maximize} \quad \mu - \lambda \xi + \sum_{l=1}^n \left\{ 1/2 \mathbf{x}_l^\top \widehat{\Theta}^{(\lambda,n,l)} \mathbf{x}_l + (\widehat{\mathbf{a}}_l^{(\lambda,n)})^\top \mathbf{x}_l \right\} \\
 & \text{subject to} \quad B \mathbf{x}_l - \frac{T}{n} K \sum_{r=1}^{l-1} \mathbf{x}_r \leq \mathbf{b}_l^{(n)} \text{ for } l = 1, \dots, n \\
 & \quad \mathbf{x}_l \in \mathbb{R}_+^q \text{ for } l = 1, \dots, n.
 \end{aligned}$$

Remark 4.1. We have the following observations.

- (i) Since the matrix $\widehat{\Theta}^{(\lambda,n,l)}$ is symmetric nonnegative semi-definite for all n and l , by the same arguments in [39], we see that $(\Psi Q_n^{(\lambda)})$ is also solvable.
- (ii) By the mean value theorem for definite integrals, for $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$, and $l = 1, 2, \dots, n$, there exist $t_{jl}^{(\lambda,n)}$ and $\bar{t}_{ijl}^{(\lambda,n)}$ in $[\frac{l-1}{n}T, \frac{l}{n}T]$ such that

$$\widehat{a}_{jl}^{(\lambda,n)} = \int_{\frac{l-1}{n}T}^{\frac{l}{n}T} \{f_j(t) - \lambda h_j(t)\} dt = \frac{T}{n} \{f_j(t_{jl}^{(\lambda,n)}) - \lambda h_j(t_{jl}^{(\lambda,n)})\}$$

and

$$\widehat{\theta}_{ij}^{(\lambda,n,l)} = \int_{\frac{l-1}{n}T}^{\frac{l}{n}T} \{d_{ij}(t) - \lambda e_{ij}(t)\} dt = \frac{T}{n} \{d_{ij}(\bar{t}_{ijl}^{(\lambda,n)}) - \lambda e_{ij}(\bar{t}_{ijl}^{(\lambda,n)})\}.$$

These imply that $\frac{T}{n} a_{jl}^{(\lambda,n)} \leq \widehat{a}_{jl}^{(\lambda,n)}$ and $\frac{T}{n} \theta_{ij}^{(\lambda,n,l)} \leq \widehat{\theta}_{ij}^{(\lambda,n,l)}$ for all i, j and l . Hence, $V(Q_n^{(\lambda)}) \leq V(\Psi Q_n^{(\lambda)})$ for all λ and n .

- (iii) Let $\bar{\mathbf{x}}^{(\lambda,n)} = (\bar{\mathbf{x}}_1^{(\lambda,n)}, \bar{\mathbf{x}}_2^{(\lambda,n)}, \dots, \bar{\mathbf{x}}_n^{(\lambda,n)})$ be an optimal solution of $(\Psi Q_n^{(\lambda)})$. Then the natural solution $\bar{\mathbf{x}}^{(n,\lambda)}(t)$, constructed from $\bar{\mathbf{x}}^{(\lambda,n)}$ and defined as in (24), is also a feasible solution of problem (CQP_λ) . Since the objective value of (CQP_λ) at $\bar{\mathbf{x}}^{(\lambda,n)}(t)$ is equal to $V(\Psi Q_n^{(\lambda)})$, it follows that

$$V(\Psi Q_n^{(\lambda)}) \leq V(CQP_\lambda) = \mathcal{F}(\lambda).$$

Moreover, since the problems $(\Psi Q_n^{(\lambda)})$ and $(Q_n^{(\lambda)})$ have the same feasible domain and by Lemma 3.1, we see that every component $\bar{x}_j^{(n,\lambda)}(t)$ of $\bar{\mathbf{x}}^{(n,\lambda)}(t)$ satisfying $0 \leq \bar{x}_j^{(n,\lambda)}(t) \leq \frac{M_1}{q}$.

In order to derive a lower bound function of $\mathcal{F}(\lambda)$, given any $n \in \mathbb{N}$, we define the function $L_n : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$(42) \quad L_n(\lambda) = V(\Psi Q_n^{(\lambda)}) \text{ for } \lambda \geq 0.$$

By the same arguments for proving Lemma 3.1, we can obtain

$$(43) \quad \mu - \lambda\xi \leq V(\Psi Q_n^{(\lambda)}) = L_n(\lambda) \leq \mu - \lambda\xi + M_1\tau(\lambda)T \text{ for all } \lambda \geq 0.$$

For further discussion, we define

$$(44) \quad c_1 = \max_{j=1, \dots, q} \max_{t \in [0, T]} \max \left\{ f_j(t) - \frac{\mu}{\xi} h_j(t), 0 \right\}$$

and

$$(45) \quad c_2 = c_1 M_1 T.$$

Since $h_{ij}(t) \geq 0$, we have

$$(46) \quad \tau(\lambda) \leq \max_{j=1, \dots, q} \max_{t \in [0, T]} \max \left\{ f_j(t) - \frac{\mu}{\xi} h_j(t), 0 \right\} = c_1 \text{ for all } \lambda \geq \mu/\xi.$$

Let

$$(47) \quad \alpha_d = \max_{i=1, \dots, q} \max_{j=1, \dots, q} \max_{t \in [0, T]} d_{ij}(t)$$

and

$$(48) \quad \alpha_e = \min_{i=1, \dots, q} \min_{j=1, \dots, q} \min_{t \in [0, T]} e_{ij}(t).$$

Then we have

$$d_{ij}(t) - \lambda e_{ij}(t) \leq \alpha_d - \lambda \alpha_e \text{ for all } \lambda \geq 0,$$

and this implies

$$(49) \quad \pi(\lambda) \leq \max\{\alpha_d - \lambda \alpha_e, 0\} \text{ for all } \lambda \geq 0.$$

Besides, by (43), (45) and (46), we have

$$(50) \quad L_n(\lambda) \leq \mu - \lambda\xi + c_2 \text{ for all } \lambda \geq \mu/\xi.$$

In the sequel, we shall provide some useful lemmas for further study.

Lemma 4.1. *Let λ_1 and λ_2 be two real numbers with $0 \leq \lambda_1 < \lambda_2$. Then*

$$(51) \quad L_n(\lambda_1) - L_n(\lambda_2) \geq (\lambda_2 - \lambda_1) \xi.$$

Proof. We note that $L_n(\lambda) = \max_{\mathbf{x} \in S(n)} \widehat{G}(\mathbf{x}, \lambda)$, where $S(n)$ is the feasible region of problem $(\Psi Q_n^{(\lambda)})$ and

$$\widehat{G}(\mathbf{x}, \lambda) = \mu - \lambda \xi + \sum_{l=1}^n \left\{ 1/2 \mathbf{x}_l^\top \widehat{\Theta}^{(\lambda, n, l)} \mathbf{x}_l + (\widehat{\mathbf{a}}_l^{(\lambda, n)})^\top \mathbf{x}_l \right\}.$$

Let $\mathbf{x}^{(n,1)}$ and $\mathbf{x}^{(n,2)}$ be feasible for $(\Psi Q_n^{(\lambda)})$ such that

$$(52) \quad L_n(\lambda_i) = \widehat{G}(\mathbf{x}^{(n,i)}, \lambda_i) \text{ for } i = 1, 2.$$

By the definition of $\mathbf{x}^{(n,1)}$, we have $\widehat{G}(\mathbf{x}^{(n,1)}, \lambda_1) - \widehat{G}(\mathbf{x}^{(n,2)}, \lambda_1) \geq 0$, and this implies

$$(53) \quad \begin{aligned} & L_n(\lambda_1) - L_n(\lambda_2) \\ &= \widehat{G}(\mathbf{x}^{(n,1)}, \lambda_1) - \widehat{G}(\mathbf{x}^{(n,2)}, \lambda_1) + \widehat{G}(\mathbf{x}^{(n,2)}, \lambda_1) - \widehat{G}(\mathbf{x}^{(n,2)}, \lambda_2) \\ &\geq \widehat{G}(\mathbf{x}^{(n,2)}, \lambda_1) - \widehat{G}(\mathbf{x}^{(n,2)}, \lambda_2). \end{aligned}$$

We define

$$(54) \quad \widehat{E}^{(n,l)} = \left[\widehat{e}_{ij}^{(n,l)} \right]_{q \times q}, \text{ where } \widehat{e}_{ij}^{(n,l)} = \int_{\frac{l-1}{n}T}^{\frac{l}{n}T} e_{ij}(t) dt$$

and

$$(55) \quad \widehat{\mathbf{h}}^{(n,l)} = (\widehat{h}_1^{(n,l)}, \dots, \widehat{h}_q^{(n,l)})^\top, \text{ where } \widehat{h}_j^{(n,l)} = \int_{\frac{l-1}{n}T}^{\frac{l}{n}T} h_j(t) dt \text{ for } j = 1, \dots, q.$$

We note that since $E(t)$ is positive semi-definite, so is $\widehat{E}^{(n,l)}$ for all n and l . Then

$$\begin{aligned} & \widehat{G}(\mathbf{x}^{(n,2)}, \lambda_1) - \widehat{G}(\mathbf{x}^{(n,2)}, \lambda_2) \\ &= (\lambda_2 - \lambda_1) \sum_{l=1}^n \left\{ 1/2 \mathbf{x}_l^{(n,2)\top} \widehat{E}^{(n,l)} \mathbf{x}_l^{(n,2)} + (\widehat{\mathbf{h}}^{(n,l)})^\top \mathbf{x}_l^{(n,2)} \right\} + (\lambda_2 - \lambda_1) \xi \\ &\geq (\lambda_2 - \lambda_1) \xi \text{ (since } \widehat{E}^{(n,l)} \text{ is positive semi-definite and } \widehat{\mathbf{h}}^{(n,l)} \geq \mathbf{0}.) \end{aligned}$$

Thus, by (53), we obtain (51). ■

Lemma 4.2. *The following statements hold true.*

- (i) For each $n \in \mathbb{N}$, $L_n(\lambda)$ is a continuous, convex and strictly decreasing function of λ .
- (ii) For each $n \in \mathbb{N}$ and $\lambda \geq 0$, we have $V(Q_n^{(\lambda)}) \leq L_n(\lambda) \leq \mathcal{F}(\lambda)$.

(iii) Let

$$(56) \quad \eta^L = \frac{\mu + c_2}{\xi}.$$

Then $\eta^L \geq \mu/\xi$ and there exists a unique $\lambda_n^L \in [\mu/\xi, \eta^L]$ such that $L_n(\lambda_n^L) = 0$ for each $n \in \mathbb{N}$.

Proof. It is easy to obtain (i), we omit the proof. The part (ii) follows by Remark 4.1 (ii) and (iii).

To prove part (iii), it is obvious that $\eta^L \geq \mu/\xi$. Since $\eta^L = \frac{\mu+c_2}{\xi}$ and by (50), we have

$$L_n(\eta^L) \leq \mu - \eta^L \xi + c_2 = 0.$$

On the other hand, by (43), we also have $L_n(\mu/\xi) \geq 0$. The continuity of $L_n(\lambda)$ says that there exists $\lambda_n^L \in [\mu/\xi, \eta^L]$ such that $L_n(\lambda_n^L) = 0$. Finally, by part (i), the strictly decreasing property of L_n shows the uniqueness of root λ_n^L . This completes the proof. ■

In order to derive the upper bound function of $\mathcal{F}(\lambda)$, let

$$(57) \quad \widehat{\delta}_n(\lambda) = \frac{T}{n} M_2(\lambda),$$

where $M_2(\lambda)$ is defined as in (18), and let

$$(58) \quad \begin{aligned} \widehat{\varepsilon}_n(\lambda) &= \bar{\varepsilon}_n \cdot p \cdot \widehat{\delta}_n(\lambda) \cdot (n + \exp(\rho T) - 1) \\ &+ \left(\varepsilon_n(\lambda) + \widehat{\delta}_n(\lambda) \right) \int_0^T \rho \cdot \exp(\rho(T-t)) (\mathbf{g}(t))^\top \mathbf{1} dt. \end{aligned}$$

Using (30) and (37), we immediately have

$$(59) \quad \delta_n(\lambda) \leq \widehat{\delta}_n(\lambda) \text{ and } \varepsilon_n(\lambda) \leq \widehat{\varepsilon}_n(\lambda).$$

Moreover, using (46) and (49), we obtain

$$(60) \quad \widehat{\delta}_n(\lambda) \leq \frac{T}{n\sigma} \cdot (M_1 \cdot \widetilde{\pi}(\lambda) + c_1) \cdot \exp\left(\frac{\nu T}{\sigma}\right) \text{ for all } \lambda \geq \frac{\mu}{\xi},$$

where $\widetilde{\pi} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a piecewise linear function defined as follows:

$$(61) \quad \widetilde{\pi}(\lambda) = \max\{\alpha_d - \lambda \alpha_e, 0\} \text{ for all } \lambda \geq 0.$$

We also note that $\widetilde{\pi}(\cdot)$ is increasing if $\alpha_e < 0$, and $\widetilde{\pi}(\cdot)$ is decreasing if $\alpha_e \geq 0$.

Lemma 4.3. *Suppose that the functions f_j, h_j, d_{ij} and e_{ij} are Lipschitz continuous for $1 \leq i \leq q$ and $1 \leq j \leq q$. Then, for all $n \in \mathbb{N}$ and $\lambda \geq \frac{\mu}{\xi}$, there exist $d \geq 0$, $r_n \geq 0$ and $s_n \geq 0$ such that*

$$(62) \quad 0 \leq \widehat{\epsilon}_n(\lambda) \leq \frac{d}{n} \cdot (1 + \lambda) + \widetilde{\pi}(\lambda) r_n + s_n$$

and $r_n \rightarrow 0, s_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. From (28), we have

$$(63) \quad \begin{aligned} \epsilon_n(\lambda) = & \max_{j=1, \dots, q} \max_{l=1, \dots, n} \left\{ \max_{t \in I_l^{(n)}} \{f_j(t) - \lambda h_j(t)\} - a_{jl}^{(\lambda, n)} \right\} \\ & + M_1 \max_{i=1, \dots, q} \max_{j=1, \dots, q} \max_{l=1, \dots, n} \left\{ \max_{t \in I_l^{(n)}} \theta_{ij}^{(\lambda)}(t) - \theta_{ij}^{(\lambda, n, l)} \right\}, \end{aligned}$$

where

$$I_l^{(n)} = \left[\frac{l-1}{n}T, \frac{l}{n}T \right].$$

Therefore, there exist $j_\alpha \in \{1, \dots, q\}$ and $t_1, t_2 \in I_l^{(n)}$ for some l such that

$$(64) \quad \begin{aligned} & \max_{j=1, \dots, q} \max_{l=1, \dots, n} \left\{ \max_{t \in I_l^{(n)}} \{f_j(t) - \lambda h_j(t)\} - a_{jl}^{(\lambda, n)} \right\} \\ & = f_{j_\alpha}(t_1) - \lambda h_{j_\alpha}(t_1) - [f_{j_\alpha}(t_2) - \lambda h_{j_\alpha}(t_2)]. \end{aligned}$$

Similarly, there exist $i_\beta, j_\gamma \in \{1, \dots, q\}$ and $t_3, t_4 \in I_l^{(n)}$ for some l such that

$$(65) \quad \begin{aligned} & \max_{i=1, \dots, q} \max_{j=1, \dots, q} \max_{l=1, \dots, n} \left\{ \max_{t \in I_l^{(n)}} \theta_{ij}^{(\lambda)}(t) - \theta_{ij}^{(\lambda, n, l)} \right\} \\ & = d_{i_\beta j_\gamma}(t_3) - \lambda e_{i_\beta j_\gamma}(t_3) - [d_{i_\beta j_\gamma}(t_4) - \lambda e_{i_\beta j_\gamma}(t_4)]. \end{aligned}$$

Let c_3 be a common Lipschitz constant of the functions $f_j(t), h_j(t), d_{ij}(t)$ and $e_{ij}(t)$ ($i = 1, \dots, q$ and $j = 1, \dots, q$.) Then, by (64) and (65), we have

$$(66) \quad \begin{aligned} \epsilon_n(\lambda) = & f_{j_\alpha}(t_1) - f_{j_\alpha}(t_2) - \lambda [h_{j_\alpha}(t_1) - h_{j_\alpha}(t_2)] \\ & + M_1 \{d_{i_\beta j_\gamma}(t_3) - d_{i_\beta j_\gamma}(t_4) - \lambda [e_{i_\beta j_\gamma}(t_3) - e_{i_\beta j_\gamma}(t_4)]\} \\ & \leq c_3 |t_1 - t_2| + \lambda c_3 |t_1 - t_2| + M_1 \{c_3 |t_3 - t_4| + \lambda c_3 |t_3 - t_4|\} \end{aligned}$$

$$(67) \quad \leq (1 + \lambda) (1 + M_1) \cdot c_3 \cdot \frac{T}{n}.$$

Now, we define

$$(68) \quad c_4 = \int_0^T \rho \cdot \exp(\rho(T-t)) (\mathbf{g}(t))^\top \mathbf{1} dt$$

$$(69) \quad r_n = \frac{T}{n\sigma} M_1 \exp\left(\frac{\nu T}{\sigma}\right) [p + \bar{\epsilon}_n(n + \exp(\rho T) - 1) + c_4]$$

$$(70) \quad s_n = \frac{T}{n\sigma} c_1 \exp\left(\frac{\nu T}{\sigma}\right) [p + \bar{\epsilon}_n(n + \exp(\rho T) - 1) + c_4]$$

$$(71) \quad d = c_3 c_4 (1 + M_1) T.$$

Then, for all $\lambda \geq \frac{\mu}{\xi}$, we have

$$\begin{aligned} \widehat{\epsilon}_n(\lambda) &= \bar{\epsilon}_n \cdot p \cdot \widehat{\delta}_n(\lambda) \cdot (n + \exp(\rho T) - 1) \\ &\quad + \left(\epsilon_n(\lambda) + \widehat{\delta}_n(\lambda) \right) \int_0^T \rho \cdot \exp(\rho(T-t)) (\mathbf{g}(t))^\top \mathbf{1} dt. \\ &= \widehat{\delta}_n(\lambda) \{p \cdot \bar{\epsilon}_n [n + \exp(\rho T) - 1] + c_4\} + c_4 \cdot \epsilon_n(\lambda) \\ &\leq \widetilde{\pi}(\lambda) r_n + s_n + c_4 \cdot \epsilon_n(\lambda) \text{ (by (60), (69) and (70))} \\ &\leq \widetilde{\pi}(\lambda) r_n + s_n + c_3 c_4 (1 + \lambda) (1 + M_1) \cdot \frac{T}{n} \text{ (by (67))} \\ &= \frac{d}{n} \cdot (1 + \lambda) + \widetilde{\pi}(\lambda) r_n + s_n. \end{aligned}$$

It is easy to see that $\widehat{\epsilon}_n(\lambda) \geq 0$ and $d \geq 0$. Finally, since $\bar{\epsilon}_n \rightarrow 0$ as $n \rightarrow \infty$, it says that $r_n \rightarrow 0$ and $s_n \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof. \blacksquare

Let η° be a positive number such that

$$(72) \quad \eta^\circ \geq \eta^* \text{ and } \xi \cdot \eta^\circ - \mu - c_2 > 0,$$

where η^* and c_2 are defined as in (4) and (45), respectively. Define

$$(73) \quad \widetilde{\pi}_e = \begin{cases} \widetilde{\pi}(\mu/\xi) = \max\{\alpha_d - \frac{\mu}{\xi} \cdot \alpha_e, 0\}, & \text{if } \alpha_e \geq 0, \\ \widetilde{\pi}(\eta^\circ) = \max\{\alpha_d - \eta^\circ \cdot \alpha_e, 0\}, & \text{if } \alpha_e < 0. \end{cases}$$

For $n \in \mathbb{N}$, we define the function $U_n(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ as follows:

$$(74) \quad U_n(\lambda) = L_n(\lambda) + \frac{d}{n}(1 + \lambda) + \widetilde{\pi}_e \cdot r_n + s_n,$$

where d , r_n and s_n are defined in (71), (69) and (70), respectively.

Lemma 4.4. *Suppose that the functions f_j , h_j , d_{ij} and e_{ij} are Lipschitz continuous for $1 \leq i \leq q$ and $1 \leq j \leq q$. The following statements hold true.*

- (i) For each $n \in \mathbb{N}$, $U_n(\lambda)$ is a continuous and convex function of λ . Moreover, if $n > d/\xi$ then $U_n(\lambda)$ is strictly decreasing.
- (ii) For each $n \in \mathbb{N}$, if $\alpha_e \geq 0$ then $L_n(\lambda) \leq \mathcal{F}(\lambda) \leq U_n(\lambda)$ for all $\lambda \geq \frac{\mu}{\xi}$; on the other hand, if $\alpha_e < 0$ then $L_n(\lambda) \leq \mathcal{F}(\lambda) \leq U_n(\lambda)$ for all $\frac{\mu}{\xi} \leq \lambda \leq \eta^\circ$.
- (iii) For each $n \in \mathbb{N}$ with $n > d/\xi$, we define

$$(75) \quad \eta_n^U = \frac{\tilde{\pi}_e \cdot r_n + \mu + c_2 + (d/n) + s_n}{\xi - (d/n)}.$$

Then $\frac{\mu}{\xi} \leq \eta_n^U$ and there exists a unique $\lambda_n^U \in [\mu/\xi, \eta_n^U]$ such that $U_n(\lambda_n^U) = 0$. Moreover, we have

$$(76) \quad \eta_n^U \rightarrow \frac{\mu + c_2}{\xi} \text{ as } n \rightarrow \infty.$$

Proof. By part (i) of Lemma 4.2, it is easy to see that $U_n(\lambda)$ is continuous and convex. Now, we are going to show that $U_n(\lambda)$ is strictly decreasing for n sufficiently large. To see this, let $\lambda_1 < \lambda_2$, then by Lemma 4.1 we have

$$\begin{aligned} U_n(\lambda_1) - U_n(\lambda_2) &= L_n(\lambda_1) - L_n(\lambda_2) + \frac{d}{n}(\lambda_1 - \lambda_2) \\ &\geq (\lambda_2 - \lambda_1)\xi + \frac{d}{n}(\lambda_1 - \lambda_2) \\ &= (\lambda_2 - \lambda_1) \left(\xi - \frac{d}{n} \right), \end{aligned}$$

and this implies that $U_n(\lambda)$ is strictly decreasing for $n > \frac{d}{\xi}$.

To prove part (ii), we have

$$\begin{aligned} (77) \quad \mathcal{F}(\lambda) &\leq V(Q_n^{(\lambda)}) + \varepsilon_n(\lambda) \text{ (by part (i) of Proposition 3.3)} \\ &\leq L_n(\lambda) + \hat{\varepsilon}_n(\lambda) \text{ (by (59) and part (ii) of Lemma 4.2)} \\ &\leq L_n(\lambda) + \frac{d}{n} \cdot (1 + \lambda) + \tilde{\pi}(\lambda) r_n + s_n \text{ for all } \lambda \geq \frac{\mu}{\xi} \text{ (by (62))} \end{aligned}$$

We note that if $\alpha_e < 0$ then

$$\tilde{\pi}(\lambda) \leq \tilde{\pi}(\eta^\circ) \text{ for all } \frac{\mu}{\xi} \leq \lambda \leq \eta^\circ.$$

Hence, by Lemma 4.2 (ii), (74) and (77), we see that $L_n(\lambda) \leq \mathcal{F}(\lambda) \leq U_n(\lambda)$ for all $\frac{\mu}{\xi} \leq \lambda \leq \eta^\circ$. Similarly, if $\alpha_e \geq 0$ then

$$\tilde{\pi}(\lambda) \leq \tilde{\pi}\left(\frac{\mu}{\xi}\right) \text{ for all } \lambda \geq \frac{\mu}{\xi},$$

and this implies that $L_n(\lambda) \leq \mathcal{F}(\lambda) \leq U_n(\lambda)$ for all $\lambda \geq \frac{\mu}{\xi}$.

Finally, we prove part (iii) by the following two cases:

- $\alpha_e \geq 0$. Since $n > d/\xi$, i.e., $\xi - (d/n) > 0$, it follows that

$$\eta_n^U \geq \frac{\tilde{\pi}(\mu/\xi) \cdot r_n + \mu + c_2 + (d/n) + s_n}{\xi} \geq \frac{\mu}{\xi}.$$

We claim that $U_n(\mu/\xi) \geq 0$ and $U_n(\eta_n^U) \leq 0$. To see this, we first obtain, from (43) and (62), that $L_n(\mu/\xi) \geq 0$ and $\frac{d}{n} \cdot (1 + \lambda) + \tilde{\pi}(\mu/\xi) \cdot r_n + s_n \geq 0$. Hence, by (74), we have $U_n(\mu/\xi) \geq 0$. On the other hand, by (50) and (74), we have

$$U_n(\lambda) \leq \mu - \lambda\xi + c_2 + \frac{d}{n} \cdot (1 + \lambda) + \tilde{\pi}(\mu/\xi) \cdot r_n + s_n \text{ for all } \lambda \geq \frac{\mu}{\xi}.$$

Hence, by (75), we have

$$U_n(\eta_n^U) \leq \mu - \eta_n^U \xi + c_2 + \frac{d}{n} \cdot (1 + \eta_n^U) + \tilde{\pi}(\mu/\xi) \cdot r_n + s_n = 0.$$

Therefore, by our claim and the continuity of $U_n(\lambda)$, there exists $\lambda_n^U \in [\mu/\xi, \eta_n^U]$ such that $U_n(\lambda_n^U) = 0$. Besides, by part (i), the strictly decreasing property of U_n shows that there exists a unique $\lambda_n^U \in [\mu/\xi, \eta_n^U]$ such that $U_n(\lambda_n^U) = 0$. Moreover, since $r_n \rightarrow 0$ and $s_n \rightarrow 0$ as $n \rightarrow \infty$ and by (75), we can obtain (76).

- $\alpha_e < 0$. By a similar argument with the case that $\alpha_e \geq 0$.

Moreover, from (75), it is easy to see $\eta_n^U \rightarrow \frac{\mu+c_2}{\xi}$ as $n \rightarrow \infty$. We complete this proof. ■

Remark 4.2. Let λ_n^L and λ_n^U be the roots of equations $L_n(\lambda) = 0$ and $U_n(\lambda) = 0$, respectively.

- (i) Since $\frac{\mu}{\xi} \leq \lambda_n^L \leq \eta^\circ$ and by Lemma 4.4 (ii), we have $L_n(\lambda_n^L) \leq \mathcal{F}(\lambda_n^L) \leq U_n(\lambda_n^L)$ for all $n \in \mathbb{N}$.
- (ii) By the following two cases, we see that $L_n(\lambda_n^U) \leq \mathcal{F}(\lambda_n^U) \leq U_n(\lambda_n^U)$ for all n large enough.

Given $n \in \mathbb{N}$ with $n > d/\xi$.

- For the case of $\alpha_e \geq 0$, since $\mu/\xi \leq \lambda_n^U$ and by Lemma 4.4 (ii), we have $L_n(\lambda_n^U) \leq \mathcal{F}(\lambda_n^U) \leq U_n(\lambda_n^U)$.
- For the case of $\alpha_e < 0$, since $\tilde{\pi}_e \cdot r_n + \frac{d}{n} \cdot (1 + \eta^\circ) + s_n \rightarrow 0$ as $n \rightarrow \infty$ and $\xi \cdot \eta^\circ - \mu - c_2 > 0$ by (72), it follows that, for all n large enough,

$$(78) \quad \tilde{\pi}_e \cdot r_n + \frac{d}{n} \cdot (1 + \eta^\circ) + s_n \leq \xi \cdot \eta^\circ - \mu - c_2.$$

Hence, if $n \in \mathbb{N}$ satisfies the inequalities $n > d/\xi$ and (78), then, by (75), we have

$$\frac{\mu}{\xi} \leq \eta_n^U \leq \eta^\circ,$$

and this implies $L_n(\lambda_n^U) \leq \mathcal{F}(\lambda_n^U) \leq U_n(\lambda_n^U)$ by Lemma 4.4 (ii).

For the remainder of this paper, we adopt the following notation:

$$(79) \quad N_e = \begin{cases} \min \left\{ n \in \mathbb{N} : n > \frac{d}{\xi} \right\}, & \text{if } \alpha_e \geq 0 \\ \min \left\{ n \in \mathbb{N} : n > \frac{d}{\xi} \text{ and } \tilde{\pi}_e \cdot r_n + \frac{d}{n} \cdot (1 + \eta^\circ) \right. \\ \left. + s_n \leq \xi \cdot \eta^\circ - \mu - c_2 \right\}, & \text{if } \alpha_e < 0. \end{cases}$$

Therefore, we see that $L_n(\lambda_n^U) \leq \mathcal{F}(\lambda_n^U) \leq U_n(\lambda_n^U)$ for all $n \in \mathbb{N}$ with $n \geq N_e$.

(iii) We have $\frac{\mu}{\xi} \leq \lambda_n^L \leq \lambda_n^U$ for all $n \in \mathbb{N}$ with $n \geq N_e$. To see this, since $L_n(\lambda_n^L) = U_n(\lambda_n^U) = 0$ and by part (ii), we have

$$L_n(\lambda_n^U) \leq U_n(\lambda_n^U) = L_n(\lambda_n^L).$$

This implies $\lambda_n^U \geq \lambda_n^L$, since $L_n(\lambda)$ is strictly decreasing.

5. APPROXIMATE SOLUTIONS TO (CQFP)

In this section, we are going to show that it is possible to generate an approximate solution of (CQFP) according to a pre-determined error bound.

Lemma 5.1. *Given $n \in \mathbb{N}$ with $n \geq N_e$. Let λ_n^L and λ_n^U be the roots of equations $L_n(\lambda) = 0$ and $U_n(\lambda) = 0$, respectively. Then the following statements hold true.*

(i) *The sequences $\{\lambda_n^L\}_{n=1}^\infty$ and $\{\lambda_n^U\}_{n=1}^\infty$ are bounded and*

$$(80) \quad \frac{d}{n} (1 + \eta_n^U) + \tilde{\pi}_e \cdot r_n + s_n \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where η_n^U and $\tilde{\pi}_e$ are given in (75) and (73), respectively.

(ii) *We have*

$$(81) \quad 0 \leq \lambda_n^U - \lambda_n^L \leq \frac{1}{\xi} \left[\frac{d}{n} (1 + \eta_n^U) + \tilde{\pi}_e \cdot r_n + s_n \right]$$

and $\lambda_n^U - \lambda_n^L \rightarrow 0$ as $n \rightarrow \infty$.

Proof. To prove part (i), since $\lambda_n^L \in [\mu/\xi, \eta^L]$ for all n and η^L is independent on n by Lemma 4.2, the sequence $\{\lambda_n^L\}_{n=1}^\infty$ is bounded. Similarly, since $\lambda_n^U \in [\mu/\xi, \eta_n^U]$ and the sequence $\{\eta_n^U\}$ is convergent by Lemma 4.4, the sequence $\{\lambda_n^U\}_{n=1}^\infty$ is also bounded. Since $r_n \rightarrow 0$ and $s_n \rightarrow 0$ as $n \rightarrow \infty$ by Lemma 4.3 and the sequence $\{\eta_n^U\}_{n=1}^\infty$ is bounded by (76), we obtain (80).

For proving part (ii), from Remark 4.2, we have $\lambda_n^U \geq \lambda_n^L$. Since

$$(82) \quad L_n(\lambda_n^L) = 0$$

and

$$(83) \quad U_n(\lambda_n^U) = L_n(\lambda_n^U) + \frac{d}{n}(1 + \lambda_n^U) + \tilde{\pi}_e \cdot r_n + s_n = 0,$$

by subtracting (83) from (82), we obtain

$$\begin{aligned} 0 &= L_n(\lambda_n^L) - L_n(\lambda_n^U) - \frac{d}{n}(1 + \lambda_n^U) - \tilde{\pi}_e \cdot r_n - s_n \\ &\geq (\lambda_n^U - \lambda_n^L)\xi - \frac{d}{n}(1 + \lambda_n^U) - \tilde{\pi}_e \cdot r_n - s_n \text{ (by Lemma 4.1)} \end{aligned}$$

and this implies

$$(84) \quad 0 \leq \lambda_n^U - \lambda_n^L \leq \frac{1}{\xi} \left[\frac{d}{n}(1 + \lambda_n^U) + \tilde{\pi}_e \cdot r_n + s_n \right].$$

Thus we obtain (81), since $\lambda_n^U \leq \eta_n^U$. Finally, from (80), we have $\lambda_n^U - \lambda_n^L \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof. ■

The following results are very useful for designing a practical algorithm.

Theorem 5.1. *Suppose that the functions f_j, h_j, d_{ij} and e_{ij} are Lipschitz continuous for $1 \leq i \leq q$ and $1 \leq j \leq q$. Let $\lambda^* = V(\text{CQFP})$. Given any $n \in \mathbb{N}$ with $n \geq N_e$, then the following statements hold true.*

(i) *We have*

$$(85) \quad -\frac{d}{n}(1 + \lambda_n^U) - \tilde{\pi}_e \cdot r_n - s_n \leq \mathcal{F}(\lambda_n^U) \leq 0 \leq \mathcal{F}(\lambda_n^L) \leq \frac{d}{n}(1 + \lambda_n^L) + \tilde{\pi}_e \cdot r_n + s_n.$$

This also implies $\lambda_n^L \leq \lambda^ \leq \lambda_n^U$.*

(ii) *For the given n , we take a number λ_n^* from the interval $[\lambda_n^L, \lambda_n^U]$. Let $\bar{\mathbf{x}}^{(\lambda_n^*)}(t)$ be the natural solution of problem $(\text{CQP}_{\lambda_n^*})$ constructed from the optimal solution of problem $(\Psi Q_n^{(\lambda_n^*)})$ and defined as in (23). Then $\bar{\mathbf{x}}^{(\lambda_n^*)}(t)$ is feasible for the problems $(\text{CQP}_{\lambda_n^*})$ and (CQFP) . Let*

$$(86) \quad \begin{aligned} & \hat{\varphi}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \\ &= \mu - \lambda_n^* \xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top \Theta^{(\lambda_n^*)}(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{a}^{(\lambda_n^*)}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt \end{aligned}$$

be the objective value of (CQP) $_{\lambda_n^*}$ at the feasible solution $\bar{\mathbf{x}}^{(\lambda_n^*)}(t)$, and let

$$\varphi(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) = \frac{\mu + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top D(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{f}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt}{\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt}$$

be the objective value of (CQFP) at the feasible solution $\bar{\mathbf{x}}^{(\lambda_n^*)}(t)$. Then

$$(87) \quad 0 \leq V(\text{CQFP}) - \varphi(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \leq \text{Er}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)),$$

where

$$(88) \quad \begin{aligned} & \text{Er}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \\ &= \lambda_n^U - \lambda_n^L + \frac{|\hat{\varphi}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t))|}{\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt}. \end{aligned}$$

Moreover, we have

$$(89) \quad 0 \leq \text{Er}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \leq \frac{2}{\xi} \left[\frac{d}{n} (1 + \eta_n^U) + \tilde{\pi}_e \cdot r_n + s_n \right]$$

and $\text{Er}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \rightarrow 0$ as $n \rightarrow \infty$. In other words, the natural solution $\bar{\mathbf{x}}^{(\lambda_n^*)}(t)$ is an approximate solution of (CQFP) with error bound $\text{Er}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t))$.

Proof. To prove part (i), since $L_n(\lambda_n^L) = 0$, by (36), (59), (62) and by remark 4.1-(iii), it follows that

$$(90) \quad \begin{aligned} 0 \leq \mathcal{F}(\lambda_n^L) &= \mathcal{F}(\lambda_n^L) - L_n(\lambda_n^L) \leq \mathcal{F}(\lambda_n^L) - V(\mathbf{Q}_n^{(\lambda_n^L)}) \\ &\leq \varepsilon_n(\lambda_n^L) \leq \hat{\varepsilon}_n(\lambda_n^L) \leq \frac{d}{n} (1 + \lambda_n^L) + \tilde{\pi}_e \cdot r_n + s_n. \end{aligned}$$

Similarly, we also have

$$0 \leq \mathcal{F}(\lambda_n^U) - L_n(\lambda_n^U) \leq \hat{\varepsilon}_n(\lambda_n^U) \leq \frac{d}{n} (1 + \lambda_n^U) + \tilde{\pi}_e \cdot r_n + s_n,$$

which implies

$$(91) \quad L_n(\lambda_n^U) \leq \mathcal{F}(\lambda_n^U) \leq L_n(\lambda_n^U) + \frac{d}{n}(1 + \lambda_n^U) + \tilde{\pi}_e \cdot r_n + s_n.$$

Since $L_n(\lambda_n^U) + \frac{d}{n}(1 + \lambda_n^U) + \tilde{\pi}_e \cdot r_n + s_n = U_n(\lambda_n^U) = 0$, from (91), we have

$$-\frac{d}{n}(1 + \lambda_n^U) - \tilde{\pi}_e \cdot r_n - s_n \leq \mathcal{F}(\lambda_n^U) \leq 0.$$

Therefore, from (90), we obtain the desired inequalities (85).

To prove part (ii), it is obvious that $\bar{\mathbf{x}}^{(\lambda_n^*)}(t)$ is a feasible solution of (CQFP). From (86), we obtain

$$\begin{aligned} & \mu + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top D(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{f}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt \\ &= \hat{\varphi} \left(\bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right) + \lambda_n^* \left(\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt \right), \end{aligned}$$

which implies

$$\begin{aligned} & \frac{\mu + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top D(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{f}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt}{\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt} \\ &= \lambda_n^* + \frac{\hat{\varphi} \left(\bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right)}{\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt}, \end{aligned}$$

i.e.,

$$\varphi \left(\bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right) = \lambda_n^* + \frac{\hat{\varphi} \left(\bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right)}{\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt}.$$

Since $\lambda^* \geq \varphi \left(\bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right)$ and $\lambda_n^L \leq \lambda^* \leq \lambda_n^U$, we obtain

$$\begin{aligned} 0 &\leq \lambda^* - \varphi \left(\bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right) \\ &= (\lambda^* - \lambda_n^*) - \frac{\hat{\varphi} \left(\bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right)}{\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt} \end{aligned}$$

$$\begin{aligned} &\leq |\lambda^* - \lambda_n^*| + \frac{|\hat{\varphi}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t))|}{\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt} \\ &\leq \lambda_n^U - \lambda_n^L + \frac{|\hat{\varphi}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t))|}{\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_n^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_n^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_n^*)}(t) \right\} dt} \\ &= Er(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)). \end{aligned}$$

Furthermore, since $L_n(\lambda_n^L) = 0$, $\lambda_n^* \geq \lambda_n^L$ and by remark 4.1-(iii), we have $L_n(\lambda_n^*) \leq L_n(\lambda_n^L) = 0$ and

$$(92) \quad \hat{\varphi}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) = V(\Psi \mathbf{Q}_n^{(\lambda_n^*)}) = L_n(\lambda_n^*) \leq 0.$$

Hence,

$$\begin{aligned} &Er(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \\ &\leq \lambda_n^U - \lambda_n^L + \frac{1}{\xi} \left[-\hat{\varphi}(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \right] \quad (\text{by (88) and (92)}) \\ &\leq \frac{1}{\xi} \left[\frac{d}{n} (1 + \lambda_n^U) + \tilde{\pi}_e \cdot r_n + s_n \right] + \frac{1}{\xi} [-L_n(\lambda_n^*)] \quad (\text{by (84) and (92)}) \\ &\leq \frac{1}{\xi} \left[\frac{d}{n} (1 + \lambda_n^U) + \tilde{\pi}_e \cdot r_n + s_n - L_n(\lambda_n^U) \right] \\ &\quad (\text{since } \lambda_n^* \leq \lambda_n^U \text{ and } L_n(\cdot) \text{ is strictly decreasing.}) \\ &= \frac{2}{\xi} \left[\frac{d}{n} (1 + \lambda_n^U) + \tilde{\pi}_e \cdot r_n + s_n \right] \quad (\text{since } L_n(\lambda_n^U) + \frac{d}{n} (1 + \lambda_n^U) + \tilde{\pi}_e \cdot r_n + s_n = 0) \\ &\leq \frac{2}{\xi} \left[\frac{d}{n} (1 + \eta_n^U) + \tilde{\pi}_e \cdot r_n + s_n \right] \quad (\text{since } \lambda_n^U \leq \eta_n^U). \end{aligned}$$

Finally, using (80) and (89), we have $Er(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof. ■

According to Theorem 5.1, we are in a position to provide a computational procedure to obtain the approximate solution of (CQFP). For $n \geq N_e$, we define

$$(93) \quad \omega_n = \frac{2}{\xi} \left[\frac{d}{n} (1 + \eta_n^U) + \tilde{\pi}_e \cdot r_n + s_n \right],$$

where $d, r_n, s_n, \tilde{\pi}_e$ and η_n^U are defined in (71), (69), (70), (73) and (75), respectively. By (89), we have

$$0 \leq Er(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \leq \omega_n.$$

Suppose that the error tolerance ϵ is pre-determined by the decision-makers. By calculating ω_n according to (93), we can determine the natural number $n \in \mathbb{N}$ such that

$$\omega_n \leq \epsilon \text{ and } n \geq N_e,$$

which also says that

$$0 \leq Er(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \leq \epsilon.$$

This also means that the corresponding approximate solution $\bar{\mathbf{x}}^{(\lambda_n^*)}(t)$ is acceptable, since the error tolerance ϵ is attained. Now, the computational procedure is given below.

Computational Procedure:

- **Step 1.1.** Set the error tolerance ϵ and the initial number n such that $n \geq N_e$, where N_e is defined in (79).
- **Step 1.2.** Evaluate ω_n as defined in (93).
- **Step 1.3.** If $\omega_n > \epsilon$ then set $n \leftarrow n + 1$ and go to Step 1.2; otherwise go to Step 1.4.
- **Step 1.4.** Find a number λ_n^* from the interval $[\lambda_n^L, \lambda_n^U]$, where λ_n^L and λ_n^U are the roots of equations $L_n(\lambda) = 0$ and $U_n(\lambda) = 0$, respectively.
- **Step 1.5.** Find the optimal solution of finite-dimensional quadratic programming problem $(\Psi Q_n^{(\lambda_n^*)})$ using well-known efficient algorithms. Use this optimal solution to construct the natural solution $\bar{\mathbf{x}}^{(\lambda_n^*)}(t)$ according to (24). Evaluate the error bound $Er(\bar{\mathbf{x}}^{(\lambda_n^*)}(t))$ defined in (88).
- **Step 1.6.** Return $\bar{\mathbf{x}}^{(\lambda_n^*)}(t)$ as an approximate optimal solution of the original problem (CQFP) with error bound $Er(\bar{\mathbf{x}}^{(\lambda_n^*)}(t)) \leq \epsilon$.

For Step 1.4, by using the convexity of $L_n(\lambda)$ and inequality (51), we can utilize the *regula falsi* method to find a number $\lambda_n^* \in [\lambda_n^L, \lambda_n^U]$. Note that $\lambda_n^* \in [\lambda_n^L, \lambda_n^U]$ is equivalent to that λ_n^* satisfies one of the following conditions:

- $L_n(\lambda_n^*) = 0$;
- $U_n(\lambda_n^*) = 0$;
- $L_n(\lambda_n^*) < 0$ and $U_n(\lambda_n^*) > 0$.

This method starts with two given numbers β^L and β^U , where

$$\beta^L = \frac{\mu}{\xi} \text{ and } \beta^U = \beta^L + \frac{1}{\xi} \cdot L_n(\beta^L).$$

It is obvious that $L_n(\beta^L) \geq 0$. By Lemma 4.1, we have

$$L_n(\beta^U) \leq L_n(\beta^L) - \xi \cdot (\beta^U - \beta^L) = L_n(\beta^L) - \xi \cdot \left\{ \beta^L + \frac{1}{\xi} \cdot L_n(\beta^L) - \beta^L \right\} = 0.$$

Calculate $U_n(\beta^U)$. If $U_n(\beta^U) \geq 0$ then $\beta^U \in [\lambda_n^L, \lambda_n^U]$, since, in this case, $L_n(\beta^U) \leq 0$ and $U_n(\beta^U) \geq 0$. Otherwise, a straight line is drawn between the two points $(\beta^L, L_n(\beta^L))$ and $(\beta^U, L_n(\beta^U))$. The intersection between this line and the λ -axis defines a new β^U , calculated according to the following expression

$$\beta^U \leftarrow \beta^L + \frac{\beta^L - \beta^U}{L_n(\beta^U) - L_n(\beta^L)} \cdot L_n(\beta^L).$$

Thus, we have the following subroutine for finding $\lambda_n^* \in [\lambda_n^L, \lambda_n^U]$.

Subroutine for finding $\lambda_n^* \in [\lambda_n^L, \lambda_n^U]$:

- **Step 2.1.** Let $\beta^L = \frac{\mu}{\xi}$ and $\beta^U = \beta^L + \frac{1}{\xi} \cdot L_n(\beta^L)$. Calculate $L_n(\beta^L)$.
- **Step 2.2.** Calculate $L_n(\beta^U)$ and $U_n(\beta^U)$.
- **Step 2.3.** If $U_n(\beta^U) \geq 0$ then STOP and return $\lambda_n^* = \beta^U$. Otherwise, set

$$\beta^U \leftarrow \beta^L + \frac{\beta^L - \beta^U}{L_n(\beta^U) - L_n(\beta^L)} \cdot L_n(\beta^L),$$

and go to Step 2.2.

We have to mention that the evaluations of Step 1.2 are independent of Step 1.4 and Step 1.5, i.e., we can estimate the rough error bound ω_n of the desired approximate solution $\bar{x}^{(\lambda_n^*)}(t)$ without using the results of Step 1.4 and Step 1.5. It also means that we can save the computational time, since the main successive iterations occur in Steps 1.1-1.3, where the workload does not need the heavy computation.

6. THE CONVERGENCE OF APPROXIMATE SOLUTIONS

Finally, we shall demonstrate the convergent property of the sequence $\{\bar{x}^{(\lambda_n^*)}(t)\}$ that are natural solutions of $(CQP_{\lambda_n^*})$ constructed from the optimal solutions of problems $(\Psi Q_n^{(\lambda_n^*)})$. We recall that the dual space of the separable Banach space $L^1[0, T]$ can be identified with $L^\infty[0, T]$. The following lemmas are very useful for further discussion.

Lemma 6.1. (Friedman [7]). *Let $\{f_k\}$ be a sequence in $L^\infty([0, T], \mathbb{R})$. If the sequence $\{f_k\}$ is uniformly bounded with respect to $\|\cdot\|_\infty$, then there exists a subsequence $\{f_{k_j}\}$ which weakly-star converges to $f_0 \in L^\infty([0, T], \mathbb{R})$. In other words, for any $g \in L^1([0, T], \mathbb{R})$, we have*

$$\lim_{k_j \rightarrow \infty} \int_0^T f_{k_j}(t)g(t)dt = \int_0^T f_0(t)g(t)dt.$$

Lemma 6.2. *If the sequence $\{f_k\}_{k=1}^\infty$ is uniformly bounded on $[0, T]$ with respect to $\|\cdot\|_\infty$, and weakly-star converges to $f_0 \in L^\infty([0, T], \mathbb{R})$, then*

$$f_0(t) \leq \limsup_{k \rightarrow \infty} f_k(t) \text{ a.e. in } [0, T]$$

and

$$f_0(t) \geq \liminf_{k \rightarrow \infty} f_k(t) \text{ a.e. in } [0, T].$$

Proof. The results follow from the similar arguments of Levinson [13, Lemma 2.1]. ■

Theorem 6.1. *We consider the sequence $\{\bar{\mathbf{x}}^{(\lambda_n^*)}(t)\}$ that is obtained according to part (ii) of Theorem 5.1. Then the sequence $\{\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)\}$ which weakly-star converges to an optimal solution $\bar{\mathbf{x}}^{(*)}(t)$ of (CQFP).*

Proof. According to Remark 4.1 (iii), we see that the sequence $\{\bar{\mathbf{x}}^{(\lambda_n^*)}(t)\}$ of vector-valued functions are uniformly bounded with respect to $\|\cdot\|_\infty$ in which the bounds are independent of n . Using Lemma 6.1, there exists a subsequence $\{\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)\}$ which weakly-star converges to $\mathbf{x}^{(*)}(t)$. Since $\bar{x}_j^{(\lambda_{n_\nu}^*)}(t) \geq 0$ for all $t \in [0, T]$ and $j = 1, \dots, q$, using Lemma 6.2, it follows that

$$x_j^{(*)}(t) \geq \liminf_{n_\nu \rightarrow \infty} \bar{x}_j^{(\lambda_{n_\nu}^*)}(t) \geq 0 \text{ a.e. in } [0, T],$$

i.e., $\mathbf{x}^{(*)}(t) \geq \mathbf{0}$ a.e. in $[0, T]$. Considering the feasibility of $\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)$, we have

$$(94) \quad B\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \leq \mathbf{g}(t) + \int_0^t K\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(s)ds \text{ for all } t \in [0, T]$$

From (94), since B is nonnegative, by taking the limit superior and applying Lemma 6.2, it follows that

$$(95) \quad B\mathbf{x}^{(*)}(t) \leq \limsup_{n_\nu \rightarrow \infty} B\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \leq \int_0^t K\mathbf{x}^{(*)}(s)ds + \mathbf{g}(t) \text{ a.e. in } [0, T]$$

Let \mathcal{N}_0 be the subset of $[0, T]$ such that the inequality of (95) is violated and let \mathcal{N}_1 be the subset of $[0, T]$ such that $\mathbf{x}^{(*)}(t) \not\geq \mathbf{0}$. Then, we define $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1$ and

$$\bar{\mathbf{x}}^{(*)}(t) = \begin{cases} \mathbf{x}^{(*)}(t) & \text{if } t \notin \mathcal{N} \\ \mathbf{0} & \text{if } t \in \mathcal{N}, \end{cases}$$

where the set \mathcal{N} has measure zero. We see that the subsequence $\{\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)\}$ is also weakly-star converges to $\bar{\mathbf{x}}^{(*)}(t)$. We remain to show that $\bar{\mathbf{x}}^{(*)}(t)$ is an optimal solution of (CQFP). It is obvious that $\bar{\mathbf{x}}^{(*)}(t) \geq \mathbf{0}$ for all $t \in [0, T]$ and $\bar{\mathbf{x}}^{(*)}(t) = \mathbf{x}^{(*)}(t)$ a.e. in $[0, T]$. We consider the following cases.

- For $t \notin \mathcal{N}$, from (95), we have

$$B\bar{\mathbf{x}}^{(*)}(t) = B\mathbf{x}^{(*)}(t) \leq \mathbf{g}(t) + \int_0^t K\mathbf{x}^{(*)}(s)ds = \mathbf{g}(t) + \int_0^t K\bar{\mathbf{x}}^{(*)}(s)ds.$$

- For $t \in \mathcal{N}$, since B is nonnegative, using (94) and weak-star convergence, we also have

$$\begin{aligned} B\bar{\mathbf{x}}^{(*)}(t) &= \mathbf{0} \leq \limsup_{n_\nu \rightarrow \infty} B\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \\ &\leq \mathbf{g}(t) + \int_0^t K\mathbf{x}^{(*)}(s)ds = \mathbf{g}(t) + \int_0^t K\bar{\mathbf{x}}^{(*)}(s)ds. \end{aligned}$$

Therefore, we obtain

$$B\bar{\mathbf{x}}^{(*)}(t) \leq \mathbf{g}(t) + \int_0^t K\bar{\mathbf{x}}^{(*)}(s)ds \text{ for all } t \in [0, T],$$

which says that $\bar{\mathbf{x}}^{(*)}(t)$ is a feasible solution of (CQFP).

Now we are going to show that $\bar{\mathbf{x}}^{(*)}(t)$ is an optimal solution of (CQFP). To see this, for given $t \in [0, T]$, we let the function $\phi_{D(t)}(\cdot) : \mathbb{R}^q \rightarrow \mathbb{R}$ be defined by $\phi_{D(t)}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top D(t)\mathbf{x} + \mathbf{f}(t)^\top \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^q$. Since $D(t)$ is a negative semi-definite matrix for all t , $\phi_{D(t)}(\cdot)$ is concave. Hence, we have

$$\begin{aligned} &\phi_{D(t)}\left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)\right) \\ &\leq \phi_{D(t)}\left(\bar{\mathbf{x}}^{(*)}(t)\right) + \nabla\phi_{D(t)}\left(\bar{\mathbf{x}}^{(*)}(t)\right)^\top \left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) - \bar{\mathbf{x}}^{(*)}(t)\right) \text{ a.e. in } [0, T], \end{aligned}$$

where $\nabla\phi_{D(t)}$ is the gradient of $\phi_{D(t)}$. This implies

$$\begin{aligned} (96) \quad &\int_0^T \phi_{D(t)}\left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)\right) dt \\ &\leq \int_0^T \phi_{D(t)}\left(\bar{\mathbf{x}}^{(*)}(t)\right) dt + \int_0^T \left[\nabla\phi_{D(t)}\left(\bar{\mathbf{x}}^{(*)}(t)\right)^\top \left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) - \bar{\mathbf{x}}^{(*)}(t)\right)\right] dt. \end{aligned}$$

Since $\{\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)\}$ weakly-star converges to $\bar{\mathbf{x}}^{(*)}(t)$ and $\nabla\phi_{D(t)}\left(\bar{\mathbf{x}}^{(*)}(t)\right) \in L^1([0, T], \mathbb{R}^q)$, we have

$$(97) \quad \lim_{n_\nu \rightarrow \infty} \int_0^T \left[\nabla\phi_{D(t)}\left(\bar{\mathbf{x}}^{(*)}(t)\right)^\top \left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) - \bar{\mathbf{x}}^{(*)}(t)\right)\right] dt = 0.$$

Therefore, by (96) we obtain

$$\begin{aligned} (98) \quad &\lim_{n_\nu \rightarrow \infty} \mu + \int_0^T \left\{1/2 \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)^\top D(t) \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) + \mathbf{f}(t)^\top \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)\right\} dt \\ &\leq \mu + \int_0^T \left\{1/2 \bar{\mathbf{x}}^{(*)}(t)^\top D(t) \bar{\mathbf{x}}^{(*)}(t) + \mathbf{f}(t)^\top \bar{\mathbf{x}}^{(*)}(t)\right\} dt. \end{aligned}$$

Similarly, we can also show that

$$(99) \quad \begin{aligned} & \lim_{n_\nu \rightarrow \infty} \xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \right\} dt \\ & \geq \mu + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(*)}(t)^\top E(t) \bar{\mathbf{x}}^{(*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(*)}(t) \right\} dt. \end{aligned}$$

Furthermore, by (87), we have

$$(100) \quad 0 \leq V(\text{CQFP}) - \varphi \left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \right) \leq Er \left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \right),$$

where

$$(101) \quad \begin{aligned} & \varphi \left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \right) \\ & = \frac{\mu + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)^\top D(t) \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) + \mathbf{f}(t)^\top \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \right\} dt}{\xi + \int_0^T \left\{ 1/2 \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t)^\top E(t) \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) + \mathbf{h}(t)^\top \bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \right\} dt}. \end{aligned}$$

Since $Er \left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \right) \rightarrow 0$ as $n_\nu \rightarrow \infty$, we obtain

$$V(\text{CQFP}) = \lim_{n_\nu \rightarrow \infty} \varphi \left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \right).$$

By considering the weak-star convergence on (101), we obtain that

$$\begin{aligned} V(\text{CQFP}) & = \lim_{n_\nu \rightarrow \infty} \varphi \left(\bar{\mathbf{x}}^{(\lambda_{n_\nu}^*)}(t) \right) \\ & \leq \varphi \left(\bar{\mathbf{x}}^{(*)}(t) \right) \quad (\text{by (98) and (99)}) \\ & \leq V(\text{CQFP}) \quad (\text{since } \bar{\mathbf{x}}^{(*)}(t) \text{ is a feasible solution of (CQFP)}), \end{aligned}$$

which also says that $\bar{\mathbf{x}}^{(*)}(t)$ is an optimal solution of (CQFP), and the proof is complete. \blacksquare

7. CONCLUSIONS

Based on the theoretical properties and computational method presented in [34, 39], an interval-type algorithm has been successfully proposed to solve a class of continuous-time quadratic fractional programming problems. The proposed computational procedure is a hybrid of the parametric method and discretization approach. Fortunately,

the estimate for the size of discretization and the error bound of approximate solutions have also been obtained. Thereby, we can predetermine the size of discretization such that the accuracy of the corresponding approximate solution can be controlled within the predefined error tolerance. Hence, the trade-off between the quality of the results and the simplification of the problem can be controlled by the decision-makers.

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