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COMPLEX INTERPOLATION FOR PREDUAL SPACES OF MORREY-TYPE SPACES

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Abstract. In this paper, the author introduces the inhomogeneous Hausdorff type Besov and Triebel-Lizorkin spaces, which are the predual spaces of Besov-Morrey and Triebel-Lizorkin-Morrey spaces. By calculating the Calderón product of the sequence spaces related to Hausdorff type Besov and Triebel-Lizorkin spaces, the author obtains the complex interpolation of these spaces. In particular, the complex interpolation for the predual spaces of Morrey spaces is also obtained.

1. Introduction

The study of Morrey spaces is traced to C. B. Morrey in 1938, nowadays has become a useful tool in the study of the existence and regularity of some elliptic equations. Recall that the *Morrey space* $L^p_\tau(\mathbb{R}^n)$ is defined to be the set of all p-locally integrable functions f such that

$$\|f\|_{L^p_\tau(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{\int_P |f(x)|^p\,dx\right\}^{1/p} < \infty,$$

where P runs over all dyadic cubes in $\mathbb{R}^n$. Obviously, $L^p_0(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. In 1986, using atoms, Zorko [43] introduced a class of functions whose dual space is the Morrey space. Another description of the predual space of Morrey spaces was later given by Kalita [10] in 1998. In 2004, using the Hausdorff capacities, Adams and Xiao [2] introduced the third kind of the predual of Morrey spaces, and proved that these three predual spaces coincide. These spaces were also used in [2] (see also [3]) to

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calculate the Morrey type capacities. Recently, Rosenthal and Triebel [17] proved the boundedness of Calderón-Zygmund operators on Morrey spaces by first considering their boundedness on the predual spaces of Morrey spaces.

In a recent paper [42], via Hausdorff capacities, the Hausdorff type Besov space $\mathcal{B}\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and the Hausdorff type Triebel-Lizorkin space $\mathcal{F}\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ were introduced and proved to be the predual spaces of the homogeneous Besov-Morrey and Triebel-Lizorkin-Morrey spaces (see [13, 15, 27, 20]), respectively. The predual spaces of Morrey spaces (see [43, 10, 2, 3]) were also proved in [42] to be special cases of these spaces $\mathcal{B}\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{F}\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$. Moreover, these scales of Hausdorff type spaces also cover some Hardy-Hausdorff spaces, which are known to the predual spaces of $Q_\alpha(\mathbb{R}^n)$ spaces (see [8, 7, 33, 34]). Inspired by [17], these Hausdorff type spaces may serve as a useful tool in the study of boundedness of operators on Besov-Morrey and Triebel-Lizorkin-Morrey spaces. Indeed, because of the lack of the density of test functions, we can not prove the boundedness of operators on Besov-Morrey and Triebel-Lizorkin-Morrey spaces by first studying the mapping property of operators on test functions and then taking approximation. However, this can be done for their predual spaces since Hausdorff type Besov and Triebel-Lizorkin spaces support the density of test functions. Once we obtain the boundedness on predual spaces, a dual argument then gives the desired boundedness of operators on Besov-Morrey and Triebel-Lizorkin-Morrey spaces.

The main purpose of this paper is to consider the complex interpolation properties for the Hausdorff type Besov space and the Hausdorff type Triebel-Lizorkin space. The interpolation theory is known to be a very useful and important tool in various branches of mathematics such as harmonic analysis and the theory of operators. For Besov and Triebel-Lizorkin spaces, the study of their complex interpolation has been an attractive topic for long time; see, for example, Schechter [22, 23], Peetre [16], Bergh-Löfström [4], Triebel [28, 29, 30], Frazier-Jawerth [9], Kalton-Mayboroda-Mitrea [11], Bownik [5] and [26]. Of special importance for us is the paper [9] by Frazier and Jawerth, who transferred the interpolation problem from function spaces to the related sequences, and the latter are usually more easy to handle. Another important tool we need is an abstract interpolation theory for quasi-Banach function spaces developed by Kalton and Mitrea in [12] (see also [11]), which establishes the coincidence of Calderón products and the complex interpolations of quasi-Banach lattices under certain conditions.

We focus on the inhomogeneous version of Hausdorff type Besov and Hausdorff type Triebel-Lizorkin spaces in this paper. However, all results are also true for homogeneous cases. We first calculate the Calderón product of the sequence spaces related to the Hausdorff type Besov space $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and the Hausdorff type Triebel-Lizorkin space $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$, which are denoted by $bH^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $fH^{s,\tau}_{p,q}(\mathbb{R}^n)$. Then, applying Kalton and Mitrea's abstract interpolation approach, we obtain the complex interpolation property for $fH^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $bH^{s,\tau}_{p,q}(\mathbb{R}^n)$. Using the characterization of $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$

and $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ via wavelet decompositions, we further obtain the corresponding complex interpolation for $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ (see Theorem 2.8 below). As a special case, we obtain the complex interpolation for predual spaces of Morrey spaces (see Corollary 2.10 below).

One of the main obstacles in applying Kalton and Mitrea's approach (see [12]) to these Hausdorff type spaces is that it is unclear whether the spaces $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$, and also $fH^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $bH^{s,\tau}_{p,q}(\mathbb{R}^n)$ are analytically convex. Such property of analytically convex is needed in [12] when define the complex interpolation space. To overcome this, we introduce the tent spaces associated with $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and establish their atomic decomposition. With these atomic decomposition, the spaces $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$, and $fH^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $bH^{s,\tau}_{p,q}(\mathbb{R}^n)$, can be re-normed as Banach spaces, which are analytically convex.

The structure of this article is organized as follows. In Section 2, we recall the definition and some basic properties of the Hausdorff type spaces $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$, and also some notions on complex interpolations. The main result of this paper is also presented at the end of this section. In Section 3, we introduce the tent spaces related to $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and establish their atomic decomposition. Using this, we prove that the spaces $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and their related sequence spaces can be re-normed as Banach spaces. Finally, in Section 4, we present the proof of the complex interpolation property of the spaces $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$.

2. Preliminaries and the Main Result

In this section, we present the definition and some properties on Hausdorff type Besov and Triebel-Lizorkin spaces, and also complex interpolations. The main result of this paper is listed in the end of this section.

We begin with some notation. Denote by $\mathbb{N}_0$ the *natural numbers including* 0. Let $\mathcal{S}(\mathbb{R}^n)$ be the *space of all Schwartz functions* on $\mathbb{R}^n$ endowed with the classical topology and $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual space*, namely, the set of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ endowed with the weak-$*$ topology. We use $\widehat{f}$ to denote the *Fourier transform* of $f\in\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$. In what follows, for all $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $j\in\mathbb{N}$, we let $\varphi_j(\cdot):=2^{jn}\varphi(2^j\cdot)$. Let $\mathcal{Q}:=\{Q_{j,k}:=2^{-j}([0,1)^n+k):\ j\in\mathbb{Z},\ k\in\mathbb{Z}^n\}$ be the collection of all *dyadic cubes* in $\mathbb{R}^n$. We also let $\mathcal{Q}^*:=\{Q_{j,k}:\ j\in\mathbb{N}_0\}$.

For $E\subset\mathbb{R}^n$ and $d\in(0,n]$, the *d-dimensional Hausdorff capacity* of E is defined by

$$H^d(E):=\inf\left\{\sum_j r_j^d : E\subset\bigcup_j B(x_j,r_j)\right\}, \tag{2.1}$$

where the infimum is taken over all countable open ball coverings $\{B(x_j,r_j)\}_j$ of E; see, for example, [1, 35]. For any function $f:\mathbb{R}^n\to[0,\infty]$, the *Choquet integral* of f with respect to H^d is then defined by

$$\int_{\mathbb{R}^n} f(x)\,dH^d(x) := \int_0^\infty H^d(\{x\in\mathbb{R}^n : f(x)>\lambda\})\,d\lambda. \tag{2.2}$$

It is known that there exists a positive constant C_0 such that, for all nonnegative measurable functions $\{f_i\}_i$,

$$\int_{\mathbb{R}^n} \sum_i f_i(x)\,dH^d(x) \le C_0 \sum_i \int_{\mathbb{R}^n} f_i(x)\,dH^d(x). \tag{2.3}$$

For any measurable functions ν on $\mathbb{R}^{n+1}_{\mathbb{N}_0} := \mathbb{R}^n \times \{2^{-k} :\ k\in\mathbb{N}_0\}$ and $x\in\mathbb{R}^n$, its *nontangential maximal function* $N\nu$ is defined by

$$N\nu(x) := \sup\{|\nu(y,2^{-k})| :\ (y,2^{-k})\in\mathbb{R}^{n+1}_{\mathbb{N}_0},\ |y-x|<2^{-k}\}, \qquad x\in\mathbb{R}^n.$$

Definition 2.1. Let $s\in\mathbb{R}$, $p\in(1,\infty)$ and $\tau\in(0,\frac{1}{p'}]$. Assume that $\varphi_0,\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfy that

$$\operatorname{supp}\widehat{\varphi_0}\subset\{\xi\in\mathbb{R}^n :\ |\xi|\le 2\} \quad\text{and}\quad |\widehat{\varphi_0}(\xi)|\ge C \quad\text{if}\quad |\xi|\le 5/3 \tag{2.4}$$

and

$$\operatorname{supp}\widehat{\varphi}\subset\{\xi\in\mathbb{R}^n :\ 1/2\le|\xi|\le 2\}\ \text{and}\ |\widehat{\varphi}(\xi)|\ge C \quad\text{if}\quad 3/5\le|\xi|\le 5/3, \tag{2.5}$$

where C is a positive constant. The *Hausdorff type Besov space* $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $q\in[1,\infty)$ and the *Hausdorff type Triebel-Lizorkin space* $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $q\in(1,\infty)$ are defined, respectively, to be the sets of $f\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)} := \left\{\sum_{k\in\mathbb{N}_0} 2^{ksq}\inf_\nu\left[\int_{\mathbb{R}^n}|\varphi_k * f(x)|^p[\nu(x,2^{-k})]^{-p}\,dx\right]^{q/p}\right\}^{1/q} < \infty$$

and

$$\|f\|_{\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)} := \inf_\nu\left\{\int_{\mathbb{R}^n}\left[\sum_{k\in\mathbb{N}_0}2^{ksq}|\varphi_k * f(x)|^q[\nu(x,2^{-k})]^{-q}\right]^{p/q}dx\right\}^{1/p} < \infty,$$

where the infimums are taken over all nonnegative Borel measurable functions ν on $\mathbb{R}^{n+1}_{\mathbb{N}_0}$ satisfying that

$$\int_{\mathbb{R}^n}[N\nu(x)]^{p'}\,dH^{n\tau p'}(x)\le 1. \tag{2.6}$$

In what follows, we use $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ to denote either $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ or $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$. When $\mathcal{A}=\mathcal{F}$, then $q\in(1,\infty)$. Similar to the arguments for the homogeneous version of $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ in [42], we see that the space $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ is independent of the choice of φ_0 and φ; moreover, the space $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space, indeed, for all $f_1,\ f_2\in\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$,

$$\|f_1 + f_2\|_{\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)} \le (2C_0)^{p'} \left[\|f_1\|_{\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)} + \|f_2\|_{\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)}\right],$$

where C_0 is as in (2.3). Also, by an argument similar to that used for [41, Lemma 7.9], we know that the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Remark 2.2. It was proved in [42] that if $p \in [q, \infty)$, then the homogeneous version of $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ coincides with the homogeneous Triebel-Lizorkin-Hausdorff space $\dot{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ introduced in [36]. The same argument is also feasible for inhomogeneous versions, that is, $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n) = FH^{s,\tau}_{p,q}(\mathbb{R}^n)$ if $p \in [q, \infty)$, where $FH^{s,\tau}_{p,q}(\mathbb{R}^n)$ is introduced in [41].

We now recall the Besov-Morrey and Triebel-Lizorkin-Morrey spaces introduced and studied in [13, 15, 27]. Notice that the notation here is slightly different from those used in these papers.

Definition 2.3. Let $p \in (0, \infty)$, $s \in \mathbb{R}$, $q \in (0, \infty]$, $\tau \in [0, 1/p)$ and φ_0 and φ be as in Definition 2.1.

(i) The *Besov-Morrey space* $\mathcal{B}M^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{B}M^{s,\tau}_{p,q}(\mathbb{R}^n)} := \left\{\sum_{j\in\mathbb{N}_0} 2^{jsq} \|\varphi_j * f\|^q_{L^p_\tau(\mathbb{R}^n)}\right\}^{1/q} < \infty.$$

(ii) The *Triebel-Lizorkin-Morrey space* $\mathcal{F}M^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{F}M^{s,\tau}_{p,q}(\mathbb{R}^n)} := \left\|\left[\sum_{j\in\mathbb{N}_0} 2^{jsq} |\varphi_j * f|^q\right]^{1/q}\right\|_{L^p_\tau(\mathbb{R}^n)} < \infty.$$

Remark 2.4. (i) Obviously, when $\tau = 0$, then $\mathcal{B}M^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{F}M^{s,\tau}_{p,q}(\mathbb{R}^n)$ go back to the classical Besov and Triebel-Lizorkin spaces. Moreover, it holds true that $\mathcal{F}M^{0,\tau}_{p,2}(\mathbb{R}^n) = L^p_\tau(\mathbb{R}^n)$; see [20].

(ii) Let $s \in \mathbb{R}$, $p, q \in (0, \infty]$ ($p \in (0, \infty)$ for $F^{s,\tau}_{p,q}(\mathbb{R}^n)$), $\tau \in [0, \infty)$ and φ_0 and φ be as in Definition 2.1. Recall that the *Besov-type space* $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and the *Triebel-Lizorkin-type space* $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ are defined, respectively, to be be the sets of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{\sum_{j=\max\{-\log_2 \ell(P),0\}}^{\infty} 2^{jsq} \left[\int_P |\varphi_j * f(x)|^p \, dx\right]^{q/p}\right\}^{1/q} < \infty$$

and

$$\|f\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=\max\{-\log_2 \ell(P),0\}}^{\infty} 2^{jsq} |\varphi_j * f(x)|^q \right]^{p/q} dx \right\}^{1/p} < \infty,$$

where the supremums are taken over all dyadic cubes P.

It was proved in [41, Corollary 3.3] (see also [21, Theorem 1.1]) that if $\tau \in [0, 1/p)$, then $\mathcal{F}M^{s,\tau}_{p,q}(\mathbb{R}^n) = F^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\mathcal{B}M^{s,\tau}_{p,\infty}(\mathbb{R}^n) = B^{s,\tau}_{p,\infty}(\mathbb{R}^n)$ with equivalent quasi-norms and, when $q \in (0,\infty)$ and $\tau \in (0, 1/p)$, $\mathcal{B}M^{s,\tau}_{p,q}(\mathbb{R}^n)$ is a proper subspace of $B^{s,\tau}_{p,q}(\mathbb{R}^n)$. For more information on these spaces, we refer to [20, 18, 19, 36, 37, 40, 41, 24, 25].

Remark 2.5. Let $p \in (1,\infty)$ and $\tau \in (0, 1/p)$. The space $H^p_\tau(\mathbb{R}^n)$ is defined to be the set of all p-locally integrable functions f such that

$$\|f\|_{H^p_\tau(\mathbb{R}^n)} := \inf_w \left\{ \int_{\mathbb{R}^n} |f(x)|^p [w(x)]^{1-p}\, dx \right\}^{\frac{1}{p}} < \infty,$$

where the infimum is taken over all nonnegative measurable functions w on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} w(x)\, dH^{n\tau p'}(x) \le 1.$$

Recall that the space $H^p_\tau(\mathbb{R}^n)$ was introduced in [2] and proved therein to be the predual space of the Morrey space $L^{p'}_\tau(\mathbb{R}^n)$. Similar to the proof for [42, Theorem 1.11], we know that

$$H^p_\tau(\mathbb{R}^n) = \mathcal{F}H^{0,\tau}_{p,2}(\mathbb{R}^n)$$

with equivalent quasi-norms.

As the inhomogeneous version of [42, Theorem 1.10], one can prove that the dual space of $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ is just $\mathcal{A}M^{-s,\tau}_{p',q'}(\mathbb{R}^n)$. The proof is similar and we omit the details.

Theorem 2.6. *Let $s \in \mathbb{R}$, $p \in (1,\infty)$, $\tau \in (0, 1/p')$ and $q \in [1,\infty)$ ($q \in (1,\infty)$ for $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$). Then the dual space of $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ is $\mathcal{A}M^{-s,\tau}_{p',q'}(\mathbb{R}^n)$ in the following sense: for any $g \in \mathcal{A}M^{-s,\tau}_{p',q'}(\mathbb{R}^n)$, the linear functional*

$$L(f) = \int_{\mathbb{R}^n} f(x) g(x)\, dx, \tag{2.7}$$

defined initially for all $f \in \mathcal{S}(\mathbb{R}^n)$, has a bounded extension to $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$;

Conversely, if L is a bounded linear functional on $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$, then there exists $g \in \mathcal{A}M^{-s,\tau}_{p',q'}(\mathbb{R}^n)$ such that $\|g\|_{\mathcal{A}M^{-s,\tau}_{p',q'}(\mathbb{R}^n)}$ is not more than a positive constant multiple of $\|L\|$, and L can be represented in the form (2.7) *for all $f \in \mathcal{S}(\mathbb{R}^n)$.*

Let φ_0 and φ be as in Definition 2.1. For $s \in \mathbb{R}$, $q \in (0, \infty]$, $p \in (0, \infty)$ and ω being a nonnegative measurable function on $\mathbb{R}^n$, the *weighted Triebel-Lizorkin space* $F^s_{p,q}(\omega)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^s_{p,q}(\omega)} := \left\| \left\{ \sum_{j\in\mathbb{N}_0} 2^{jsq} |\varphi_j * f|^q \right\}^{1/q} \right\|_{L^p(\omega)} < \infty,$$

where, for any $\omega(x)\,dx$-measurable function g,

$$\|g\|_{L^p(\omega)} := \left[\int_{\mathbb{R}^n} |g(x)|^p \omega(x)\,dx \right]^{1/p}.$$

As the inhomogeneous counterpart of [42, Theorem 1.6], we have the following equivalent characterizations of $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Proposition 2.7. *Let* $s \in \mathbb{R}$, $p \in (1, \infty)$, $\tau \in (0, 1/p']$ *and* $q \in [1, \infty)$ *(*$q \neq 1$ *if* $\mathcal{A} = \mathcal{F}$*).*

(i) *There exists a positive constant* C *such that for all* $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$C^{-1}\|f\|_{\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)} \le \left\{ \sum_{j\in\mathbb{N}_0} \inf_{\omega} \|2^{js}\varphi_j * f\|^q_{L^p(\omega)} \right\}^{1/q} \le C\|f\|_{\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)}$$

and

$$C^{-1}\|f\|_{\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)} \le \inf_{\omega} \|f\|_{F^s_{p,q}(\omega)} \le C\|f\|_{\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)},$$

where the infimums are taken over all nonnegative Lebesgue measurable functions ω *on* $\mathbb{R}^n$ *satisfying that*

$$\int_{\mathbb{R}^n} [\omega(x)]^{-p'/p}\,dH^{n\tau p'}(x) \le 1. \tag{2.8}$$

(ii) *If* $\tau \in (0, 1/p')$, *then the infimums of* ω *in (i) can be further limited to all* $\omega \in A_p(\mathbb{R}^n)$ *satisfying* (2.8)*, here* $A_p(\mathbb{R}^n)$ *denotes the well-known Muckenhoupt weight class.*

Now we recall some basic notions about the classical complex interpolation of quasi-Banach spaces; see, for example, [6, 12, 11]. Consider a *couple of quasi-Banach spaces* X_0, X_1, which are continuously embedding into a large topological vector space Y. The *space* $X_0 + X_1$ is defined by

$$X_0 + X_1 := \{h \in Y : \exists\ h_i \in X_i,\ i \in \{0, 1\},\ \text{such that}\ h = h_0 + h_1\},$$

and its *norm* is defined by

$$\|h\|_{X_0+X_1} := \inf\{\|h_0\|_{X_0} + \|h_1\|_{X_1} :\ h = h_0 + h_1,\ h_0 \in X_1\ \text{and}\ h_1 \in X_1\}.$$

Let X be a quasi-Banach space, $U := \{z \in \mathbb{C} : \ 0 < \mathrm{Re} z < 1\}$ and $\overline{U}$ be its closure, here and in what follows, for any $z \in \mathbb{C}$, $\mathrm{Re} z$ denotes its *real part*. A map f: $U \to X$ is said to be *analytic* if, for any given $z_0 \in U$, there exists $\eta \in (0, \infty)$ such that $f(z) = \sum_{j=0}^{\infty} h_n (z - z_0)^n, \ h_n \in X$, is uniformly convergent for $|z - z_0| < \eta$.

A quasi-Banach space X is said to be *analytically convex* if there exists a positive constant C such that, for any analytic function $f : \ U \to X$ which is continuous on the closed strip $\overline{U}$,

$$\max_{z \in \overline{U}} \|f(z)\|_X \le C \max_{\mathrm{Re} z \in \{0,1\}} \|f(z)\|_X .$$

It is well known that all Banach spaces are analytically convex.

Suppose that $X_0 + X_1$ is analytically convex. The *set* $\mathcal{F} := \mathcal{F}(X_0, X_1)$ is defined to be the set of all functions f: $U \to X_0 + X_1$ satisfying that

(i) f is analytic and *bounded* in $X_0 + X_1$, which means that $f(U) := \{f(z) : \ z \in U\}$ is a bounded set of $X_0 + X_1$.

(ii) f is extended continuously to the closure $\overline{U}$ of the strip U such that the traces $t \mapsto f(j + it)$ are bounded continuous functions into X_j, $j \in \{0, 1\}$, $t \in \mathbb{R}$.

We endow $\mathcal{F}$ with the *quasi-norm*

$$\|f\|_{\mathcal{F}} := \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{X_0}, \quad \sup_{t \in \mathbb{R}} \|f(1 + it)\|_{X_1} \right\}.$$

Let $X_0, \ X_1$ be two quasi-Banach spaces such that $X_0 + X_1$ is analytically convex. Then the *complex interpolation space* $[X_0, X_1]_\theta$ with $\theta \in (0, 1)$ is defined by

$$[X_0, X_1]_\theta := \{g \in X_0 + X_1 : \ \exists \ f \in \mathcal{F} \text{ such that } f(\theta) = g\}$$

and its *norm* given by $\|g\|_{[X_0, X_1]_\theta} := \inf_{f \in \mathcal{F}} \{\|f\|_{\mathcal{F}} : f(\theta) = g\}$.

The main purpose of this paper reads as follows.

Theorem 2.8. *Let* $\theta \in (0, 1)$, $s_0, \ s_1 \in \mathbb{R}$, $p_0, \, p_1 \in (1, \infty)$, $q_0, \, q_1 \in [1, \infty)$ *(*$q_i \ge p_i$, $i \in \{0, 1\}$, *for Besov cases) and* $\tau_i \in (0, 1/p_i']$, $i \in \{0, 1\}$. *Let* $s = s_0(1 - \theta) + s_1 \theta$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ *and* $\tau = \tau_0(1 - \theta) + \tau_1 \theta$ *such that*

$$\tau p' = \tau_0 p_0' = \tau_1 p_1'. \tag{2.9}$$

Then

$$\left[\mathcal{A}H^{s_0, \tau_0}_{p_0, q_0}(\mathbb{R}^n), \mathcal{A}H^{s_1, \tau_1}_{p_1, q_1}(\mathbb{R}^n)\right]_\theta = \mathcal{A}H^{s, \tau}_{p, q}(\mathbb{R}^n).$$

Remark 2.9. Notice that for the Hausdorff Besov-type space we do not obtain the interpolation for full parameters. The reason we need the restriction $q_i \ge p_i$ is that without this condition, we can not prove the Hausdorff Besov-type space can be re-normed as Banach space, which is needed when we use the interpolation approach by Kalton and Mitrea. It is still unclear whether this restriction can be removed.

As an immediate consequence of Theorem 2.8 and Remark 2.5, we have the following interpolation for predual spaces of Morrey spaces.

Corollary 2.10. *Let* $\theta \in (0,1)$, $p_i \in (1,\infty)$, $\tau_i \in (0,1/p_i']$, $i \in \{0,1\}$, *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\tau = \tau_0(1-\theta) + \tau_1\theta$. *If* (2.9) *holds true, then*

$$\left[H^{p_0}_{\tau_0}(\mathbb{R}^n), H^{p_1}_{\tau_1}(\mathbb{R}^n)\right]_\theta = H^p_\tau(\mathbb{R}^n).$$

From Theorem 2.8, we also deduce the following interpolation property of linear operators on Hausdorff type Besov and Triebel-Lizorkin spaces.

Proposition 2.11. *Let all parameters be as in Theorem 2.8 and* (X_0, X_1) *be a couple of analytically convex quasi-Banach spaces.*

(i) *If the linear operator* T *is bounded from* X_i *to* $\mathcal{A}H^{s_i,\tau_i}_{p_i,q_i}(\mathbb{R}^n)$, $i \in \{0,1\}$, *then* T *is also bounded from* $[X_0, X_1]_\theta$ *to* $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$.

(ii) *If the linear operator* T *is bounded from* $\mathcal{A}H^{s_i,\tau_i}_{p_i,q_i}(\mathbb{R}^n)$ *to* X_i, $i \in \{0,1\}$, *then* T *is also bounded from* $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ *to* $[X_0, X_1]_\theta$.

As a special case, we obtain the corresponding interpolation property of operators on $H^p_\tau(\mathbb{R}^n)$. Recall that Adams and Xiao [3, Section 5.2] already obtained some interpolation property of operators between the predual spaces of Morrey spaces.

Finally, we make some conventions on notation. Throughout the paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbols* $A \lesssim B$ means $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its *characteristic function*.

3. Hausdorff Type Tent Spaces

In this section we study the tent spaces related to the space $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$, and establish their atomic decomposition, which are further used to show that $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ can be re-normed as Banach spaces for some parameters.

Definition 3.1. Let $s \in \mathbb{R}$, $p \in (1,\infty)$ and $\tau \in (0, \frac{1}{p'}]$. The *tent spaces* $\mathcal{B}T^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $q \in [1,\infty)$ and $\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $q \in (1,\infty)$ are defined, respectively, to be the sets of all functions F on $\mathbb{R}^{n+1}_{\mathbb{N}_0}$ such that

$$\|F\|_{\mathcal{B}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})} := \left\{\sum_{k\in\mathbb{N}_0} 2^{ksq} \inf_\nu \|F(\cdot, 2^{-k})[\nu(\cdot, 2^{-k})]^{-1}\|^q_{L^p(\mathbb{R}^n)}\right\}^{\frac{1}{q}} < \infty$$

and

$$\|F\|_{\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})} := \inf_\nu \left\| \left\{\sum_{k\in\mathbb{N}_0} 2^{ksq} |F(\cdot, 2^{-k})|^q [\nu(\cdot, 2^{-k})]^{-q}\right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where the infimums are taken over all nonnegative Lebesgue measurable functions ν on $\mathbb{R}^{n+1}_{\mathbb{N}_0}$ satisfying (2.6).

In what follows, we use $\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$ to denote either $\mathcal{B}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$ or $\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$. When $\mathcal{A}=\mathcal{F}$, then $q\in(1,\infty)$. Similar to the proof of [42, Theorem 1.6], one can prove the following conclusions.

Proposition 3.2. *Let $s\in\mathbb{R}$, $p\in(1,\infty)$, $\tau\in(0,1/p']$ and $q\in[1,\infty)$ ($q\neq 1$ if $\mathcal{A}=\mathcal{F}$). Then for all functions F on $\mathbb{R}^{n+1}_{\mathbb{N}_0}$, $\|F\|_{\mathcal{B}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})}$ and $\|F\|_{\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})}$ are equivalent to*

$$\left\{\sum_{k\in\mathbb{N}_0}2^{ksq}\inf_{\omega}\|F(\cdot,2^{-k})\|^q_{L^p(\omega)}\right\}^{\frac{1}{q}},$$

and

$$\inf_{\omega}\left\|\left\{\sum_{k\in\mathbb{N}_0}2^{ksq}\|F(\cdot,2^{-k})\|^q\right\}^{\frac{1}{q}}\right\|_{L^p(\omega)},$$

respectively, where the infimums are taken over all nonnegative Lebesgue measurable functions ω on $\mathbb{R}^n$ satisfying (2.8).

We now introduce atoms related to these tent spaces.

Definition 3.3. Let $s\in\mathbb{R}$, $p\in(1,\infty)$, $q\in[1,\infty)$ and $\tau\in(0,1/p']$. A function a on $\mathbb{R}^{n+1}_{\mathbb{N}_0}$ is called an $\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$-atom associated a ball B, if $\operatorname{supp} a\subset T(B):=\{(x,t)\in\mathbb{R}^{n+1}_{\mathbb{N}_0}:\ B(x,t)\subset B\}$ and satisfies that

$$\sum_{k\in\mathbb{N}_0}2^{ksq}\left[\int_{\mathbb{R}^n}|a(x,2^{-k})|^p\chi_{T(B)}(x,2^{-k})\,dx\right]^{q/p}\le|B|^{-\tau q}\quad\text{if}\quad\mathcal{A}=\mathcal{B}$$

or

$$\int_{\mathbb{R}^n}\left[\sum_{k\in\mathbb{N}_0}2^{ksq}|a(x,2^{-k})|^q\chi_{T(B)}(x,2^{-k})\right]^{p/q}dx\le|B|^{-\tau p}\quad\text{if}\quad\mathcal{A}=\mathcal{F}.$$

Similar to [41, Lemma 7.1], we can prove that all $\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$-atoms belong to $\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$ with uniform bound. To prove $\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$ can be characterized by atoms, we need the following lemma.

Lemma 3.4. *Let $s\in\mathbb{R}$, $p,q\in(1,\infty)$ and $\tau\in(0,1/p']$.*

(i) *If $\{G_j\}_j\subset\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$ and $\sum_j\|G_j\|_{\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})}<\infty$, then $G:=\sum_j G_j\in\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$ and there exists a positive constant C, independent of $\{G_j\}_j$, such that*

$$\|G\|_{\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})}\le C\sum_j\|G_j\|_{\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})}.$$

(ii) *The corresponding result for $\mathcal{B}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$ also holds true when $q\ge p$.*

Proof. (i) Without lost of generality, we may assume that $\lambda_j := \|G_j\|_{\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})} > 0$ for all j. Let $F_j := \lambda_j^{-1} G_j$. Then $\|F_j\|_{\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})} = 1$ and $G = \sum_j \lambda_j F_j$. For any $\varepsilon > 0$, by Proposition 3.2, we can take $\omega_j \geq 0$ such that

$$\int_{\mathbb{R}^n} [\omega_j(x)]^{-p'/p} \, dH^{n\tau p'}(x) \leq 1$$

and

$$\left\| \left\{ \sum_{k=0}^{\infty} 2^{ksq} |F_j(\cdot, 2^{-k})|^q \right\}^{1/q} \right\|_{L^p(\omega_j)} \leq 1 + \varepsilon.$$

Define

$$\omega := \left(\sum_j \lambda_j \right)^{p/p'} \left(\sum_j \lambda_j \omega_j^{-p'/p} \right)^{-p/p'}.$$

Then

$$\int_{\mathbb{R}^n} [\omega(x)]^{-p'/p} \, dH^{n\tau p'}(x) \lesssim \left(\sum_j \lambda_j \right)^{-1} \sum_j \lambda_j \int_{\mathbb{R}^n} \omega_j^{-p'/p} \, dH^{n\tau p'}(x) \lesssim 1.$$

Moreover, by the Minkowski inequality, we see that

$$\begin{aligned} \|G\|_{\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})} &\leq \left\{ \int_{\mathbb{R}^n} \left[\sum_{k=0}^{\infty} 2^{ksq} |G(\cdot, 2^{-k})|^q \right]^{p/q} \omega(x) \, dx \right\}^{1/q} \\ &\leq \left\{ \int_{\mathbb{R}^n} \left(\sum_j \lambda_j \left[\sum_{k=0}^{\infty} 2^{ksq} |F_j(\cdot, 2^{-k})|^q \right]^{1/q} \right)^p \omega(x) \, dx \right\}^{1/p}. \end{aligned}$$

Notice that, by the Hölder inequality,

$$\begin{aligned} &\left(\sum_j \lambda_j \left[\sum_{k=0}^{\infty} 2^{ksq} |F_j(\cdot, 2^{-k})|^q \right]^{1/q} \right)^p \\ &\leq \left(\sum_j \lambda_j \left[\sum_{k=0}^{\infty} 2^{ksq} |F_j(\cdot, 2^{-k})|^q \right]^{p/q} \omega_j \right) \left(\sum_j \lambda_j \omega_j^{-p/p'} \right)^{p/p'}. \end{aligned}$$

Hence

$$\begin{aligned}
&\|G\|_{\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})}\\
&\le \left(\sum_j \lambda_j\right)^{1/p'} \left\{\int_{\mathbb{R}^n} \sum_j \lambda_j \left[\sum_{k=0}^{\infty} 2^{ksq}|F_j(\cdot,2^{-k})|^q\right]^{p/q} \omega_j\,dx\right\}^{1/p}\\
&\lesssim (1+\varepsilon)\sum_j \lambda_j,
\end{aligned}$$

which proves (i).

(ii) We now consider the $\mathcal{B}$-space case. Let F_j be as above. Then for any $\varepsilon > 0$, we can take $\omega_{k,j} \ge 0$ such that

$$\int_{\mathbb{R}^n} [\omega_{k,j}(x)]^{-p'/p}\,dH^{n\tau p'}(x) \le 1$$

and

$$\left\|\left\{\sum_{k=0}^{\infty} 2^{ksq}|F_j(\cdot,2^{-k})|^q\right\}^{1/q}\right\|_{L^p(\omega_{k,j})} \le 1+\varepsilon.$$

Define

$$\omega_k := \left(\sum_j \lambda_j\right)^{p/p'} \left(\sum_j \lambda_j \omega_{k,j}^{-p'/p}\right)^{-p/p'}.$$

This time, by the Hölder inequality, we have

$$|G|^p = \left|\sum_j \lambda_j F_j\right|^p \le \left(\sum_j \lambda_j |F_j|^p \omega_{k,j}\right)\left(\sum_j \lambda_j \omega_{k,j}^{-p'/p}\right)^{p/p'}.$$

Since $q \ge p$, by the Minkowski inequality, we see that

$$\begin{aligned}
&\|G\|_{\mathcal{B}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})}\\
&\le \left\{\sum_{k=0}^{\infty} 2^{ksq}\left[\int_{\mathbb{R}^n} |G(x,2^{-k})|^p \omega_k(x)\,dx\right]^{q/p}\right\}^{1/q}\\
&\le \left\{\sum_{k=0}^{\infty} 2^{ksq}\left[\int_{\mathbb{R}^n} \sum_j \lambda_j |F_j(x,2^{-k})|^p \omega_{k,j}(x)\,dx\right]^{q/p}\right\}^{1/q} \left(\sum_j \lambda_j\right)^{1/p'}\\
&\le \left\{\sum_j \lambda_j \left(\sum_{k=0}^{\infty} 2^{ksq}\left[\int_{\mathbb{R}^n} |F_j(x,2^{-k})|^p \omega_{k,j}(x)\,dx\right]^{q/p}\right)^{p/q}\right\}^{1/p} \left(\sum_j \lambda_j\right)^{1/p'}\\
&\lesssim (1+\varepsilon)\sum_j \lambda_j,
\end{aligned}$$

which proves (ii), and then completes the proof. ∎

Using Lemma 3.4 instead of [41, Lemma 7.2], and similar to the proof of [41, Proposition 7.1], we obtain the following atomic decomposition of $\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$, the details being omitted.

Theorem 3.5. *Let $s \in \mathbb{R}$, $p, q \in (1, \infty)$, and $\tau \in (0, \frac{1}{p'}]$.*

(i) *A function $F \in \mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$ if and only if there exists a sequence $\{a_m\}_m$ of $\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$-atoms and an ℓ^1-sequence $\{\lambda_m\}_m \subset \mathbb{C}$ such that $F = \sum_m \lambda_m a_m$ pointwise and in $\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$. Moreover, $\|F\|_{\mathcal{F}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})}$ is equivalent to $\inf \sum_m |\lambda_m|$, where the infimum is taken over all admissible decompositions of F.*

(ii) *When $q \geq p$, the corresponding conclusion for $\mathcal{B}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$ also holds true.*

We denote the infimum $\inf \sum_m |\lambda_m|$ in Theorem 3.5 by $|||F|||_{\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})}$, which is an equivalent norm of $\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$, and then $(\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0}), |||\cdot|||_{\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})})$ become a Banach space ($q \geq p$ for $\mathcal{B}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})$).

Let φ_0 and φ be as in Definition 2.1 such that $\sum_{j\in\mathbb{N}_0} |\widehat{\varphi_j}|^2 \equiv 1$. We define an operator ρ_φ by setting $\rho_\varphi(f)(x, 2^{-j}) := \varphi_j * f(x)$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$. Conversely, for all functions F on $\mathbb{R}^{n+1}_{\mathbb{N}_0}$ and $x \in \mathbb{R}^n$, we define a map π_φ by

$$\pi_\varphi(F)(x) := \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} F(y,\, 2^{-k})\varphi_k(x-y)\, dy.$$

By the Calderón reproducing formula, we know that $\pi_\varphi \circ \rho_\varphi(f) = f$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$. Moreover, it is easy to see that $\|\rho_\varphi(f)\|_{\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})} = \|f\|_{\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)}$. In this sense, by the previous argument, we can endow $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ with an equivalent norm $|||\cdot|||_{\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)} := |||\rho_\varphi(f)|||_{\mathcal{A}T^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{N}_0})}$, under which $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ becomes a Banach space ($q \geq p$ for $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$), and hence are analytically convex.

Recall that the property that $\mathcal{F}H^{s,\tau}_{p,p}(\mathbb{R}^n)$ can be re-normed as Banach space was used in [38] to prove that the dual space of $\mathring{\mathcal{F}}M^{s,\tau}_{p,p}(\mathbb{R}^n) = \mathring{F}^{s,\tau}_{p,p}(\mathbb{R}^n)$ is $\mathcal{F}H^{-s,\tau}_{p',p'}(\mathbb{R}^n)$, where $s \in \mathbb{R}$, $p \in (1,\infty)$ and $\tau \in (0, 1/p)$, $\mathring{A}M^{s,\tau}_{p,q}(\mathbb{R}^n)$ denotes the closure of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{A}M^{s,\tau}_{p,q}(\mathbb{R}^n)$. Since this time we prove that $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ ($q \geq p$ for $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$) can be re-normed as Banach spaces, repeating the procedure used in [38], we can prove the following conclusion, the details being omitted.

Theorem 3.6. *Let $s \in \mathbb{R}$, $p, q \in (1,\infty)$ ($q \leq p$ for $\mathcal{B}$ cases) and $\tau \in (0, 1/p')$. Then the dual space of $\mathring{A}M^{s,\tau}_{p,q}(\mathbb{R}^n)$ is $\mathcal{A}H^{-s,\tau}_{p',q'}(\mathbb{R}^n)$ in the following sense: for any $g \in \mathcal{A}H^{-s,\tau}_{p',q'}(\mathbb{R}^n)$, the linear functional*

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx, \tag{3.1}$$

defined initially for all $f \in \mathcal{S}(\mathbb{R}^n)$*, has a bounded extension to* $\mathring{\mathcal{A}}M_{p,q}^{s,\tau}(\mathbb{R}^n)$*;*

Conversely, if L *is a bounded linear functional on* $\mathring{\mathcal{A}}M_{p,q}^{s,\tau}(\mathbb{R}^n)$*, then there exists* $g \in \mathcal{A}H_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ *such that* $\|g\|_{\mathcal{A}H_{p',q'}^{-s,\tau}(\mathbb{R}^n)}$ *is not more than a positive constant multiple of* $\|L\|$*, and* L *can be represented in the form* (3.1) *for all* $f \in \mathcal{S}(\mathbb{R}^n)$*.*

4. Proof of Theorem 2.8

In this section, we give the proof of Theorem 2.8. One of the main tool we used is the wavelet decomposition of the space $\mathcal{A}H_{p,q}^{s,\tau}(\mathbb{R}^n)$, which transfers the problem to the corresponding sequence spaces.

Let $\widetilde{\phi}$ be a scaling function on $\mathbb{R}$ with compact support and of sufficiently high regularity, and $\widetilde{\psi}$ the *corresponding orthonormal wavelet*. Then the tensor product ansatz yields a scaling function ϕ and associated wavelets $\psi_1, \ldots, \psi_{2^n-1}$, all defined on $\mathbb{R}^n$; see, e. g., [32, Proposition 5.2]. We suppose that $\phi \in C^{N_1}(\mathbb{R}^n)$ and $\operatorname{supp}\phi \subset [-N_2, N_2]^n$ for some natural numbers N_1 and N_2, which means that, for all $i \in \{1, \ldots, 2^n-1\}$, $\psi_i \in C^{N_1}(\mathbb{R}^n)$ and $\operatorname{supp}\psi_i \subset [-N_3, N_3]^n$ for some $N_3 \in \mathbb{N}$.

For $k \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$ and $i \in \{1, \ldots, 2^n-1\}$, define

$$\phi_{j,k}(x) := 2^{jn/2}\phi(2^jx-k) \qquad \text{and} \qquad \psi_{i,j,k}(x) := 2^{jn/2}\psi_i(2^jx-k), \quad x \in \mathbb{R}^n.$$

It is well known that

$$\int\limits_{\mathbb{R}^n} \psi_{i,j,k}(x)\,x^\gamma\,dx = 0 \qquad \text{if} \qquad |\gamma| \le N_1$$

(see [32, Proposition 3.1]), and

$$\{\phi_{0,k}:\ k \in \mathbb{Z}^n\} \cup \{\psi_{i,j,k}:\ k \in \mathbb{Z}^n,\ j \in \mathbb{N}_0,\ i \in \{1, \ldots, 2^n-1\}\}$$

forms an *orthonormal basis* of $L^2(\mathbb{R}^n)$ (see, for example, [31]). Hence

$$f = \sum_{k\in\mathbb{Z}^n} \lambda_k\,\phi_{0,k} + \sum_{i=1}^{2^n-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \lambda_{i,j,k}\,\psi_{i,j,k} \tag{4.1}$$

in $L^2(\mathbb{R}^n)$, where $\lambda_k := \langle f, \phi_{0,k}\rangle$ and $\lambda_{i,j,k} := \langle f, \psi_{i,j,k}\rangle$. In what follows, for convenience, we also write $\phi_{i,0,k} := \phi_{0,k}$ and $\lambda_{i,0,k} := \lambda_k$ for all $i \in \{1, \ldots, 2^n-1\}$, and $\lambda(f) := \{\lambda_{i,j,k}:\ i \in \{1, \ldots, 2^n-1\},\ j \in \mathbb{N}_0,\ k \in \mathbb{Z}^n\}$.

Next we recall the related sequence spaces.

Definition 4.1. Let $s \in \mathbb{R}$, $p, q \in (1, \infty)$ and $\tau \in (0, 1/p']$. The *sequence space* $aH^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the space of all sequences $t := \{t_{i,j,k} : \ i \in \{1, \ldots, 2^n - 1\}, \ j \in \mathbb{N}_0, \ k \in \mathbb{Z}^n\} \subset \mathbb{C}$ such that $\|t\|_{aH^{s,\tau}_{p,q}(\mathbb{R}^n)} < \infty$, where when $a = b$,

$$\|t\|_{bH^{s,\tau}_{p,q}(\mathbb{R}^n)} := \left\{ \sum_{j=0}^{\infty} \inf_{\nu} \left(\int_{\mathbb{R}^n} \sum_{i=1}^{2^n-1} \sum_{k\in\mathbb{Z}^n} 2^{jn(\frac{s}{n}+\frac{1}{2})p} |t_{i,j,k}|^p \chi_{Q_{j,k}}(x) [\nu(x, 2^{-j})]^{-p} \, dx \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}},$$

and when $a = f$,

$$\|t\|_{fH^{s,\tau}_{p,q}(\mathbb{R}^n)} := \inf_{\nu} \left\{ \int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} \sum_{i=1}^{2^n-1} \sum_{k\in\mathbb{Z}^n} 2^{jn(\frac{s}{n}+\frac{1}{2})q} |t_{i,j,k}|^q \chi_{Q_{j,k}}(x) [\nu(x, 2^{-j})]^{-q} \right)^{\frac{p}{q}} dx \right\}^{\frac{1}{p}},$$

and the infimum is taken over all nonnegative Borel measurable functions ν on $\mathbb{R}^{n+1}_{\mathbb{N}_0}$ satisfying (2.6).

Similar to the proof of [14, Theorem 6.4], we have the following wavelet characterization of $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Proposition 4.2. *Let $s \in \mathbb{R}$, $\tau \in (0, 1/p']$ and $p, q \in (1, \infty)$. Assume that*

$$\begin{aligned} N_1 + 1 \ > \max \big\{ & s + n\tau + n/(\max(p,q))' - n/p + np/(1+p\tau), \\ & -s + n\tau + 1/(\max(p,q))' + 1/p - 1 + 2np/(1+p\tau) \big\} \ . \end{aligned}$$

Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in \mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ if, and only if, f can be represented as (4.1) *in $\mathcal{S}'(\mathbb{R}^n)$ and $\|\lambda(f)\|_{aH^{s,\tau}_{p,q}(\mathbb{R}^n)} < \infty$. Moreover, $\|f\|_{\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)}$ is equivalent to $\|\lambda(f)\|_{aH^{s,\tau}_{p,q}(\mathbb{R}^n)}$.*

By this wavelet decomposition, we know that there exists a homeomorphism between $\mathcal{A}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $aH^{s,\tau}_{p,q}(\mathbb{R}^n)$. Therefore, to show Theorem 2.8, we only need to prove the corresponding interpolation for the sequence spaces $aH^{s,\tau}_{p,q}(\mathbb{R}^n)$. To this end, we first need to calculate the Calderón product of these spaces. For simplicity, in what follows, we redefine

$$\|t\|_{bH^{s,\tau}_{p,q}(\mathbb{R}^n)} := \left\{ \sum_{j=0}^{\infty} \inf_{\nu} \left(\int_{\mathbb{R}^n} \sum_{\ell(Q)=2^{-j}} 2^{jsp} |t_Q|^p \chi_Q(x) [\nu(x, 2^{-j})]^{-p} \, dx \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}},$$

and when $a = f$,

$$\|t\|_{fH^{s,\tau}_{p,q}(\mathbb{R}^n)} := \inf_{\nu} \left\{ \int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} \sum_{\ell(Q)=2^{-j}} 2^{jsq} |t_Q|^q \chi_Q(x) [\nu(x, 2^{-j})]^{-q} \right)^{\frac{p}{q}} dx \right\}^{\frac{1}{p}}.$$

Recall that a quasi-Banach space $(X, \|\cdot\|)$ of complex-valued measurable functions is called a *quasi-Banach lattice* if, for any $f \in X$ and a complex-valued measurable function g satisfying $|g| \le |f|$, then $g \in X$ and $\|g\|_X \le \|f\|_X$. Given two quasi-Banach lattices $(X_i, \|\cdot\|_i)$, $i \in \{0,1\}$, and $\theta \in (0,1)$, the *Calderón product* $X_0^{1-\theta}X_1^\theta$ is defined by

$$X_0^{1-\theta}X_1^\theta := \Big\{f \text{ is a complex-valued measurable function}: \ \exists\, f^0 \in X_0, \text{ and } f^1 \in X_1 \text{ such that } |f| \le |f^0|^{1-\theta}|f^1|^\theta\Big\},$$

and its *norm* is defined by $\|f\|_{X_0^{1-\theta}X_1^\theta} := \inf\left\{\|f^0\|_{X_0}^{1-\theta}\|f^1\|_{X_1}^\theta\right\}$, where the infimum is taken over all $f^i \in X_i$, $i \in \{0,1\}$ such that $|f| \le |f^0|^{1-\theta}|f^1|^\theta$. It is easy to see that the sequence spaces $aH_{p,q}^{s,\tau}(\mathbb{R}^n)$ are quasi-Banach lattices.

Theorem 4.3. *Let $\theta \in (0,1)$, $s_i \in \mathbb{R}$, $p_i, q_i \in (1,\infty)$ and $\tau_i \in (0, 1/p_i']$, $i \in \{0,1\}$. Let $s = s_0(1-\theta)+s_1\theta$, $\frac{1}{p} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0}+\frac{\theta}{q_1}$ and $\tau = \tau_0(1-\theta)+\tau_1\theta$. If* (2.9) *holds, then*

$$\left[aH_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n)\right]^{1-\theta}\left[aH_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)\right]^\theta = aH_{p,q}^{s,\tau}(\mathbb{R}^n).$$

To prove this result, we need the following conclusion, which when $\tau = 0$ goes back to [9, Proposition 2.7].

Lemma 4.4. *Let $s \in \mathbb{R}$, p, $q \in (1,\infty)$, $\tau \in (0, \frac{1}{p'}]$ and $\delta \in (0,1]$. Suppose that, for each dyadic cube Q with $Q \in \mathcal{Q}^*$, $E_Q \subset Q$ is a measurable set with $|E_Q| \ge \delta|Q|$. Then $t = \{t_Q\}_{Q\in\mathcal{Q}^*} \in fH_{p,q}^{s,\tau}(\mathbb{R}^n)$ if and only if $\|t\|_{\widetilde{fH_{p,q}^{s,\tau}}(\mathbb{R}^n)} < \infty$, where $\|t\|_{\widetilde{fH_{p,q}^{s,\tau}}(\mathbb{R}^n)}$ is defined the same as $\|t\|_{fH_{p,q}^{s,\tau}(\mathbb{R}^n)}$ with χ_Q replaced by χ_{E_Q}.*

Proof. We only prove $\|t\|_{fH_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|t\|_{\widetilde{fH_{p,q}^{s,\tau}}(\mathbb{R}^n)}$, since the inverse inequality is trivial. To this end, let ν satisfy (2.6) such that

$$\left\{\int_{\mathbb{R}^n}\left(\sum_{j=0}^\infty\sum_{\ell(Q)=2^{-j}} 2^{jsq}|\lambda_Q|^q\chi_{E_Q}(x)[\nu(x,2^{-j})]^{-q}\right)^{\frac{p}{q}}dx\right\}^{\frac{1}{p}} \lesssim \|\lambda\|_{\widetilde{fH_{p,q}^{s,\tau}}(\mathbb{R}^n)}.$$

For all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, define $\widetilde{\nu}(x,t) := \sup\{\nu(y,t): \ |y-x| < \sqrt{n}t\}$. Then by [41, Lemma 7.16], we know that a constant multiplication of $\widetilde{\nu}$ also satisfies (2.6). Moreover, for all $x \in Q$ with $\ell(Q) = 2^{-j}$,

$$\begin{aligned}\chi_Q(x)[\widetilde{\nu}(x,2^{-j})]^{-1} &\le \frac{1}{\delta}\frac{\int_Q \chi_{E_Q}(y)\,dy}{|Q|}[\widetilde{\nu}(x,2^{-j})]^{-1} \le \frac{1}{\delta}\frac{\int_Q \chi_{E_Q}(y)[\nu(y,2^{-j})]^{-1}\,dy}{|Q|}\\ &\lesssim \frac{1}{\delta}M\left(\chi_{E_Q}[\nu(\cdot,2^{-j})]^{-1}\right)(x),\end{aligned}$$

where M denotes the Hardy-Littlewood maximal operator. Then, applying the Fefferman-Stein vector-valued inequality, we obtain the desired conclusion. ■

Now we turn to the proof of Theorem 4.3.

Proof of Theorem 4.3. By similarity, we only consider the fH-case.

We first prove that $[fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)]^{1-\theta}[fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)]^{\theta} \subset fH^{s,\tau}_{p,q}(\mathbb{R}^n)$. To this end, let $\lambda := \{\lambda_Q\}_{Q\in\mathcal{Q}^*} \in [fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)]^{1-\theta}[fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)]^{\theta}$. Then there exist $\lambda_0 := \{\lambda^0_Q\}_{Q\in\mathcal{Q}^*} \in fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)$ and $\lambda_1 := \{\lambda^1_Q\}_{Q\in\mathcal{Q}^*} \in fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)$ such that $|\lambda_Q| \le |\lambda^0_Q|^{1-\theta}|\lambda^1_Q|^{\theta}$ for all $Q \in \mathcal{Q}^*$ and

$$\|\lambda_0\|^{1-\theta}_{fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)}\|\lambda_1\|^{\theta}_{fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)} \lesssim \|\lambda\|_{[fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)]^{1-\theta}[fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)]^{\theta}}.$$

Let ν_i satisfying

$$\int_{\mathbb{R}^n}[N\nu_i(x)]^{p_i'}\,dH^{n\tau_i p_i'}(x) \le 1, \quad i \in \{0,1\}.$$

such that

$$\left\{\int_{\mathbb{R}^n}\left(\sum_{j=0}^{\infty}\sum_{\ell(Q)=2^{-j}}2^{js_iq_i}|(\lambda_i)_Q|^{q_i}\chi_Q(x)[\omega_i(x,2^{-j})]^{-q_i}\right)^{\frac{p_i}{q_i}}dx\right\}^{\frac{1}{p_i}} \lesssim \|\lambda_i\|_{fH^{s_i,\tau}_{p_i,q_i}(\mathbb{R}^n)}$$

for $i \in \{0,1\}$. Define $\nu := \nu_0^{1-\theta}\nu_1^{\theta}$. It is easy to see that $N\nu \le (N\nu_0)^{1-\theta}(N\nu_1)^{\theta}$, which together with the Young inequality further implies that

$$(N\nu)^{p'} \le \frac{(1-\theta)p'}{p_0'}(N\nu_0)^{p_0'} + \frac{\theta p'}{p_1'}(N\nu_1)^{p_1'}.$$

Therefore ν satisfies (2.6), due to (2.9). Moreover, applying the Hölder inequality, we see that

$$\begin{aligned}\|\lambda\|_{fH^{s,\tau}_{p,q}(\mathbb{R}^n)} &\lesssim \left\|\left(\sum_{j=0}^{\infty}\sum_{\ell(Q)=2^{-j}}2^{jsq}|\lambda_Q|^{q}\chi_Q(x)[\nu(x,2^{-j})]^{-q}\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}\\ &\lesssim \left\|\left(\sum_{j=0}^{\infty}\sum_{\ell(Q)=2^{-j}}2^{js_0q_0}|(\lambda_0)_Q|^{q_0}\chi_Q(x)[\nu_0(x,2^{-j})]^{-q_0}\right)^{\frac{1}{q_0}}\right\|^{1-\theta}_{L^{p_0}(\mathbb{R}^n)}\\ &\quad\times\left\|\left(\sum_{j=0}^{\infty}\sum_{\ell(Q)=2^{-j}}2^{js_1q_1}|(\lambda_1)_Q|^{q_1}\chi_Q(x)[\nu_1(x,2^{-j})]^{-q_1}\right)^{\frac{1}{q_1}}\right\|^{\theta}_{L^{p_1}(\mathbb{R}^n)}\\ &\lesssim \|\lambda_0\|^{1-\theta}_{fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)}\|\lambda_1\|^{\theta}_{fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)} \lesssim \|\lambda\|_{[fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)]^{1-\theta}[fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)]^{\theta}}.\end{aligned}$$

Therefore, we have $[fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)]^{1-\theta}\,[fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)]^{\theta} \subset fH^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Next we show $[fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)]^{1-\theta}\,[fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)]^{\theta} \supset fH^{s,\tau}_{p,q}(\mathbb{R}^n)$. To this end, let $\lambda \in fH^{s,\tau}_{p,q}(\mathbb{R}^n)$ and ν satisfying (2.6) such that

$$\left\{\int_{\mathbb{R}^n}\left(\sum_{j=0}^{\infty}\sum_{\ell(Q)=2^{-j}} 2^{jsq}|\lambda_Q|^q\chi_Q(x)[\nu(x,2^{-j})]^{-q}\right)^{\frac{p}{q}}dx\right\}^{\frac{1}{p}} \lesssim \|\lambda\|_{fH^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

For all $x\in\mathbb{R}^n$ and $j\in\mathbb{N}_0$, we define

$$\widetilde{\nu}(x,2^{-j}) := \sup\left\{\nu(y,2^{-j}) :\ y\in Q_{jk},\ Q_{jk}\ni x\right\}.$$

Then, it it easy to check that a positive constant multiplication of $\widetilde{\nu}$ also satisfies (2.6). Moreover, $\nu\le\widetilde{\nu}$, and hence

$$\left\{\int_{\mathbb{R}^n}\left(\sum_{j=0}^{\infty}\sum_{\ell(Q)=2^{-j}} 2^{jsq}|\lambda_Q|^q\chi_Q(x)[\widetilde{\nu}(x,2^{-j})]^{-q}\right)^{\frac{p}{q}}dx\right\}^{\frac{1}{p}} \lesssim \|\lambda\|_{fH^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

For all $Q\in\mathcal{Q}^*$, let $E_Q\subset Q$ be a measurable set such that $|E_Q|=|Q|$. For all $k\in\mathbb{Z}$, define

$$A_k := \left\{x\in\mathbb{R}^n :\ \left(\sum_{j=0}^{\infty}\sum_{\ell(Q)=2^{-j}} 2^{jsq}|\lambda_Q|^q\chi_{E_Q}(x)[\widetilde{\nu}(x,2^{-j})]^{-q}\right)^{\frac{1}{q}} > 2^k\right\}$$

and

$$C_k := \left\{Q\in\mathcal{Q} :\ Q\in\mathcal{Q}^*,\ |E_Q\cap A_k|\ge\frac{1}{2}|E_Q| \text{ and } |E_Q\cap A_{k+1}|<\frac{1}{2}|E_Q|\right\}.$$

It is easy to see that if $Q\notin\cup_{k\in\mathbb{Z}}C_k$, then $\lambda_Q=0$. For all $Q\in\mathcal{Q}^*$, when $Q\in C_k$, define

$$\lambda^0_Q := \left(\frac{|\lambda_Q|}{A_Q}\right)^{q/q_0}\left(\sup_{y\in Q}\nu(y,\ell(Q))\right)^{p'/p'_0-q/q_0}$$

with $A_Q := 2^{k(1-\frac{pq_0}{qp_0})}|Q|^u$ and $u := \frac{s}{n}-\frac{q_0}{q}\frac{s_0}{n}$, and

$$\lambda^1_Q := \left(\frac{|\lambda_Q|}{B_Q}\right)^{q/q_1}\left(\sup_{y\in Q}\nu(y,\ell(Q))\right)^{p'/p'_1-q/q_1}$$

with $B_Q := 2^{k(1-\frac{pq_1}{qp_1})}|Q|^v$ and $v := \frac{s}{n}-\frac{q_1}{q}\frac{s_1}{n}$, and, when $Q\notin\cup_{k\in\mathbb{Z}}C_k$, $\lambda^0_Q=\lambda^1_Q:=0$. Since $\{C_k\}_{k\in\mathbb{Z}}$ are disjoint each other, we know that λ_0 and λ_1 are well defined. Moreover, it is easy to check that $|\lambda_Q|=|\lambda^0_Q|^{1-\theta}|\lambda^1_Q|^{\theta}$ for all $Q\in\mathcal{Q}^*$. Therefore, to complete the proof, it suffices to show that

(4.2) $$\|\lambda^0\|_{fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)} \lesssim \|\lambda\|^{p/p_0}_{fH^{s,\tau}_{p,q}(\mathbb{R}^n)} \quad \text{and} \quad \|\lambda^1\|_{fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)} \lesssim \|\lambda\|^{p/p_1}_{fH^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

We may assume that $\frac{p_0}{q_0} \le \frac{p_1}{q_1}$ by symmetry.

Define $\nu_0 := \widetilde{\nu}^{p'/p_0'}$ and $\widetilde{\nu}_1 := \nu^{p'/p_1'}$. By (2.9) we know that ν_i satisfies (2.6) with p' replaced by p_i', $i \in \{0,1\}$. By Lemma 4.4, we see that

$$\begin{aligned}
&\|\lambda^0\|_{fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)}\\
&\lesssim \left\{\int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}}\sum_{Q\in C_k}|Q|^{-\frac{s_0}{n}q_0}|\lambda_Q^0|^{q_0}\chi_{E_Q\cap A_k}(x)[\widetilde{\nu}_0(x,\ell(Q))]^{-q_0}\right)^{\frac{p_0}{q_0}}dx\right\}^{\frac{1}{p_0}}\\
&\sim \left\{\int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}}\chi_{A_k}(x)\sum_{Q\in C_k}2^{-k(1-\frac{pq_0}{qp_0})q}|Q|^{-\frac{s}{n}q}\right.\right.\\
&\qquad\left.\left.\times|\lambda_Q|^q\chi_{E_Q}(x)[\widetilde{\nu}(x,\ell(Q))]^{-q}\right)^{\frac{p_0}{q_0}}dx\right\}^{\frac{1}{p_0}}.
\end{aligned}$$

By the definition of A_k, and the fact that $\frac{p_0}{q_0} \le \frac{p_1}{q_1}$ implies $1-\frac{pq_0}{qp_0} \le 0$, we further see that

$$\|\lambda^0\|_{fH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n)} \lesssim \|\lambda\|^{p/p_0}_{fH^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

Similar to the above argument, with $\chi_{E_Q\cap A_k}$ replaced by $\chi_{E_Q\cap A^C_{k+1}}$, we obtain

$$\|\lambda^1\|_{fH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)} \lesssim \|\lambda\|^{p/p_1}_{fH^{s,\tau}_{p,q}(\mathbb{R}^n)},$$

which completes the proof of Theorem 4.3. ■

In 1998, Kalton and Mitrea [12, Theorem 3.4] proved the following conclusion, which has become a powerful tool to study the complex interpolation of quasi-Banach function spaces; see, for example, [11, 39, 26].

Proposition 4.5. *Let X_0, X_1 be a pair of quasi-Banach sequence lattices. If both X_0 and X_1 are analytically convex and at least one is separable, then $X_0 + X_1$ is also analytically convex and*

$$[X_0, X_1]_\Theta = X_0^{1-\Theta}\, X_1^{\Theta}, \qquad \Theta \in (0,1).$$

Since finite sequences are dense in $aH^{s,\tau}_{p,q}(\mathbb{R}^n)$, we know that these spaces are separable. Moreover, although $(aH^{s,\tau}_{p,q}(\mathbb{R}^n), \|\cdot\|_{aH^{s,\tau}_{p,q}(\mathbb{R}^n)})$ is only quasi-Banach space, by Proposition 4.2 and the argument in the end of Section 3, we know that $aH^{s,\tau}_{p,q}(\mathbb{R}^n)$ can be re-normed as Banach spaces ($q \ge p$ if $a = b$), and hence are analytically convex. Combining these observations with Theorem 4.3 and Proposition 4.5, we obtain the following interpolation for sequence spaces.

Theorem 4.6. *Let* $\theta \in (0,1)$*,* $s_i \in \mathbb{R}$*,* $p_i \in (1,\infty)$*,* $q_i \in [1,\infty)$ *and* $\tau_i \in (0, 1/p_i']$ *(*$q_i \ge p_i$ *for* bH*-spaces),* $i \in \{0,1\}$*. Let* $s = s_0(1-\theta) + s_1\theta$*,* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$*,* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ *and* $\tau = \tau_0(1-\theta) + \tau_1\theta$*. If* (2.9) *holds true, then*

$$\left[aH^{s_0,\tau_0}_{p_0,q_0}(\mathbb{R}^n), aH^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n)\right]_\theta = aH^{s,\tau}_{p,q}(\mathbb{R}^n).$$

Theorem 2.8 is then an immediate consequence of Theorem 4.3 and Proposition 4.2.

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