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DENSITY AND UPPER SEMICONTINUITY OF STRONG SOLUTIONS FOR PARAMETRIC GENERALIZED SYSTEM

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Abstract. In this paper, we obtain some stability results for a class of parametric generalized system. Under weaker conditions, a density theorem of positive proper efficient solutions in the set of strong solutions is obtained. Then, by using the density result, we establish the upper semicontinuity of strong solution mappings to parametric generalized system without monotonicity. Our results are different from the corresponding ones in the literature. Some examples are given to illustrate the results.

1. INTRODUCTION

Many results on the existence of solutions to various kinds of vector variational inequalities and vector equilibrium problems have been widely established, see [5, 6, 7, 8] and the references therein.

The stability analysis of solution maps to parametric vector variational inequalities (PVVI, in short) and parametric vector equilibrium problems (PVEP, in short) is another important topic in optimization theory and applications. There are some papers to discuss the upper and/or lower semicontinuity of solution maps. Cheng and Zhu [9] obtained a result on the lower semicontinuity of the solution set map to a parametric vector variational inequality in finite-dimensional spaces based on a scalarization method. Recently, by virtue of a density result and scalarization technique, Gong and Yao [11] have first discussed the lower semicontinuity of the efficient solution to PVEP, which are called generalized systems in their paper. By using the ideas of Cheng and

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Zhu [9], Gong [12] has discussed the continuity of the solution set mapping for a class of parametric weak vector equilibrium problems in topological vector spaces. Huang et al.[10] used local existence results of the models considered and additional assumptions to establish the lower semicontinuity of solution mappings for parametric implicit vector equilibrium problems. Kimura and Yao [13] discussed the semicontinuity of solution maps for parametric vector quasi-equilibrium problems. In 2009, Xu and Li [14] proved the lower semicontinuity for PVEP by using a new proof method which is different from the one used in Gong and Yao [11]. Chen et al. [15] studied the continuity of solution sets for parametric generalized systems without the uniform compactness assumption, which improves the corresponding results in [11, 12].

We observed that the semicontinuity of solution maps to PVEP has been discussed under assumption of C-strict/strong monotonicity for the mappings, which implies that the f-solution set of the PVEP is a singleton (see [9, 11, 12, 14, 15, 19]). However, it is well known that the f-solution set of (weak) PVEPs should be general, but not a singleton. Moreover, to the best of our knowledge, there are few upper semicontinuous results have been concerned for strong solution mappings to PVEPs in the literature. So, in this paper, by using a density result, we aim at discussing the upper semicontinuity of the strong solution map for a classes of parametric generalized system by removing the assumption of C-strict monotonicity, where the f-solution set may be general. Our results are new and different from the corresponding ones in the literature ([11, 12, 14, 15, 19]). Some examples are given for illustration of the results.

The rest of the paper is organized as follows. In Sect. 2, we introduce a class of parametric generalized system (PGS), and recall some concepts and their properties. In Sect. 3, we obtain the density theorem of strong solution sets to (PGS), which does not need whether the f-solution set is singleton. In Sect. 4, we discuss the upper semicontinuity of the strong solution mappings to (PGS), and compare our main results with the corresponding ones in the recent literature.

2. Preliminaries

Throughout this paper, unless specified otherwise, let X, Y and Z be real linear metric spaces. Let Y^* be the topological dual space of Y, and C be a closed convex pointed cone in Y with nonempty topological interior intC. Let $d(\cdot,\cdot)$ denotes the distance in metric space. The notation $B(\mu,\delta)$ denotes the open ball with center $\mu \in \Lambda$ and the radius $\delta > 0$ and the cl$C$ denotes the closure of C.

Let

$$C^* := \{f \in Y^* : f(y) \geq 0, \ \forall y \in C\}$$

be the dual cone of C. Denote the quasi-interior of C^* by $C^\sharp$, i.e.,

$$C^\sharp := \{f \in Y^* : f(y) > 0, \ \forall y \in C \setminus \{0\}\}.$$

A nonempty convex set $B \subset C$ is said to be a base of C if

(i) $0 \notin \text{cl}B$; (ii) $C = \text{cone}B$.
It is easy to see that $C^\sharp \neq \emptyset$ if and only if C has a base.

Let A be a nonempty subset of X and $F : A \times A \to Y$ be a vector-valued mapping. We consider the following generalized system (GS)

$$\text{Find } x \in A \text{ such that } F(x, y) \notin -K, \forall y \in A,$$

where $K \cup \{0\}$ is a convex cone in Y.

When the subset A and the mapping F are perturbed by a parameter $\mu \in \Lambda$, which Λ is a nonempty subset of Z, we consider the following parametric generalized system (PGS)

$$\text{Find } x \in A(\mu) \text{ such that } F(x, y, \mu) \notin -K, \forall y \in A(\mu),$$

where $A : \Lambda \to 2^X \setminus \{\emptyset\}$ is a set-valued mapping, $F : B \times B \times \Lambda \subset X \times X \times Z \to Y$ is a vector-valued mapping with $A(\Lambda) = \bigcup_{\mu \in \Lambda} A(\mu) \subset B$.

Definition 2.1. (i) A vector $x \in A$ is called a weak solution to the (GS), iff

$$F(x, y) \notin -\text{int}C, \ \forall y \in A.$$

The set of the weak solutions to the (GS) is denoted by V_w.

(ii) A vector $x \in A(\mu)$ is called a weak solution to the (PGS), iff

$$F(x, y, \mu) \notin -\text{intC}, \forall y \in A(\mu).$$

The set of the weak solutions to the (PGS) is denoted by $V_w(\mu)$.

Definition 2.2. (i) A vector $x \in A$ is called a strong solution to the (GS), iff

$$F(x, y) \notin -C \setminus \{0\}, \ \forall y \in A.$$

The set of the strong solutions to the (GS) is denoted by V_s.

(ii) A vector $x \in A(\mu)$ is called a strong solution to the (PGS), iff

$$F(x, y, \mu) \notin -C \setminus \{0\}, \ \forall y \in A(\mu).$$

The set of the strong solutions to the (PGS) is denoted by $V_s(\mu)$.

Definition 2.3. (i) Let $f \in C^* \setminus \{0\}$. A vector $x \in A$ is called an f-solution to the (GS), iff

$$f(F(x, y)) \geq 0, \ \forall y \in A.$$

The set of the f-solution to the (GS) is denoted by V_f.

(ii) Let $f \in C^* \setminus \{0\}$. A vector $x \in A(\mu)$ is called an f-solution to the (PGS), iff

$$f(F(x, y, \mu)) \geq 0, \ \forall y \in A(\mu).$$

The set of the f-solution to the (PGS) is denoted by $V_f(\mu)$.

(iii) A vector $x \in A(\mu)$ is called a positive proper efficient solution to the (PGS) if there exists $f \in C^\sharp$ such that

$$f(F(x, y, \mu)) \geq 0, \ \forall \ y \in A(\mu).$$

Let $\mu \in \Lambda, x \in A(\mu)$. Define

$$F(x, A(\mu), \mu) = \{F(x, y, \mu) : \ y \in A(\mu)\}.$$

Let $\mu \in \Lambda$ and $A(\mu) \subset X$ be a nonempty set, and let $\varphi : A(\mu) \times A(\mu) \times \Lambda \to Y$, $\psi : A(\mu) \times \Lambda \to Y$ be two vector-valued mappings. Throughout the rest of this note, for each $\mu \in \Lambda$, we always assume

$$F(x, y) = \psi(y) + \varphi(x, y) - \psi(x), \ x, y \in A.$$

and

$$F(x, y, \mu) = \psi(y, \mu) + \varphi(x, y, \mu) - \psi(x, \mu), \ x, y \in A(\mu).$$

Next, we recall other basic concepts and properties.

Definition 2.4. Let $F : X \times X \times \Lambda \to Y$ be a vector-valued mapping.

(i) $F(\cdot, \cdot, \cdot)$ is called C-monotone on $A(\mu) \times A(\mu) \times \Lambda$, iff for any given $\mu \in \Lambda$, for each $x, y \in A(\mu)$, $F(x, y, \mu) + F(y, x, \mu) \in -C$.

(ii) $F(\cdot, \cdot, \cdot)$ is called C-strictly monotone (i.e., C-strongly monotone in [19]) on $A(\mu) \times A(\mu) \times \Lambda$, iff F is a C-monotone on $A(\mu) \times A(\mu) \times \Lambda$, and for any given $\mu \in \Lambda$, for each $x, y \in A(\mu)$ with $x \neq y$, $F(x, y, \mu) + F(y, x, \mu) \in -intC$.

(iii) $F(x, \cdot, \mu)$ is called C-convex if, for each $y_1, y_2 \in A(\mu)$ and $t \in [0, 1]$, $tF(x, y_1, \mu) + (1 - t)F(x, y_2, \mu) \in F(x, ty_1 + (1 - t)y_2, \mu) + C$.

(iv) $F(x, \cdot, \mu)$ is called C-convexlike on $A(\mu)$, iff for any $y_1, y_2 \in A(\mu)$ and any $t \in [0, 1]$, there exists $y_3 \in A(\mu)$ such that $tF(x, y_1, \mu) + (1 - t)F(x, y_2, \mu) \in F(x, y_3, \mu) + C$.

(v) A set $D \subset Y$ is called a C-convex set, iff $D + C$ is a convex set in Y.

Definition 2.5. [3]. Let $F : \Lambda \to 2^X$ be a set-valued mapping, and given $\bar{\mu} \in \Lambda$.

(i) F is called lower semicontinuous(l.s.c, in short) at $\bar{\mu}$, iff for any open set V satisfying $V \bigcap F(\bar{\mu}) \neq \emptyset$, there exists $\delta > 0$, such that for every $\mu \in B(\bar{\mu}, \delta)$, $V \bigcap F(\mu) \neq \emptyset$.

(ii) F is called upper semicontinuous(u.s.c, in short) at $\bar{\mu}$, iff for any open set V satisfying $F(\bar{\mu}) \subset V$, there exists $\delta > 0$, such that for every $\mu \in B(\bar{\mu}, \delta)$, $F(\mu) \subset V$.

We say F is l.s.c(resp. u.s.c) on Λ, iff it is l.s.c(resp. u.s.c) at each $\mu \in \Lambda$. F is said to be continuous on Λ, iff it is both l.s.c and u.s.c on Λ.

Proposition 2.1. [3]. *Let $F : \Lambda \to 2^X$ be a set-valued mapping, and given $\bar{\mu} \in \Lambda$.*

(i) *F is l.s.c at $\bar{\mu}$ if and only if for any sequence $\{\mu_n\} \subset \Lambda$ with $\mu_n \to \bar{\mu}$ and any $\bar{x} \in F(\bar{\mu})$, there exists $x_n \in F(\mu_n)$, such that $x_n \to \bar{x}$.*

(ii) *If F has compact values(i.e., $F(\mu)$ is a compact set for each $\mu \in \Lambda$), then F is u.s.c at $\bar{\mu}$ if and only if for any sequences $\{\mu_n\} \subset \Lambda$ with $\mu_n \to \bar{\mu}$ and $\{x_n\}$ with $x_n \in F(\mu_n)$, there exist $\bar{x} \in F(\bar{\mu})$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \to \bar{x}$.*

The following Lemma gives the relationships between $V_w(\mu)$ and $V_f(\mu)$.

Lemma 2.1. [16]. *Let $F(x, A(\mu), \mu)$ be a C-convex set for each $\mu \in \Lambda$ and $x \in A(\mu)$. If intC $\neq \emptyset$, then $V_w(\mu) = \bigcup_{f \in C^* \setminus \{0\}} V_f(\mu)$.*

Lemma 2.2. [2, Property 3, p. 238] *For any neighborhood U of 0_X, there exists an balanced open neighborhood U_1 such that $U_1 + U_1 \subset U$.*

Throughout this paper, we always assume $V_f(\mu) \neq \emptyset$ and $V_s(\mu) \neq \emptyset$ for all $\mu \in \Lambda$. This paper aims at investigating the density of positive proper efficient solutions and upper semicontinuity of the strong solution mappings to (PGS).

3. Density in Strong Solution Sets to (PGS)

Define the set-valued mapping $H : C^* \setminus \{0\} \to 2^A$ by $H(f) = V_f, \forall f \in C^* \setminus \{0\}$, we can get the following lemma.

Lemma 3.1. *For each $f \in C^* \setminus \{0\}$. Suppose the following conditions are satisfied:*

(i) *A is a compact set;*

(ii) *$\varphi(\cdot, \cdot)$ is continuous on $A \times A, \psi(\cdot)$ is continuous on A;*

(iii) *$M = \{\varphi(x, y) : x, y \in A\}$ and $\psi(A)$ are bounded subsets of Y;*

(iv) *For each $x \in A \setminus V_f$, there exists $y \in V_f$ such that*

$$\varphi(x, y) + \varphi(y, x) + B(0, d^r(x, y)) \subset -C,$$

where r is a positive constant.

Then, we have $H(\cdot)$ is l.s.c on $C^* \setminus \{0\}$.

Proof. Suppose to the contrary that there exists $f_0 \in C^* \setminus \{0\}$ such that $H(\cdot)$ is not l.s.c at f_0. Then, there exist a sequence $\{f_n\}$ with $f_n \to f_0$ with respect to the strong topology $\beta(Y^*, Y)$ and $x_0 \in H(f_0) = V_{f_0}$ such that for any $x_n \in H(f_n) = V_{f_n}, x_n \nrightarrow x_0$.

Since A is compact, then there exists $\bar{x}_n \in A$ such that $\bar{x}_n \to x_0$. Obviously, $\bar{x}_n \in A \setminus V_{f_n}$. By (iv), there exists $y_n \in V_{f_n}$ such that

$$\varphi(\bar{x}_n, y_n) + \varphi(y_n, \bar{x}_n) + B(0, d^r(\bar{x}_n, y_n)) \subset -C.$$

Since $y_n \in V_{f_n}$ implies $y_n \in A$ and the compactness of A, then there exists $y_0 \in A$ such that $y_n \to y_0$(taking a subset of y_n if necessary). By (ii) and the closeness of C, taking the limit as $n \to \infty$, we have

$$\varphi(x_0, y_0) + \varphi(y_0, x_0) + B(0, d^r(x_0, y_0)) \subset -C. \tag{3.1}$$

Noting that $x_0 \in V_{f_0}$ and $y_0 \in A$, we have

$$f_0(\psi(y_0) + \varphi(x_0, y_0) - \psi(x_0)) \geq 0. \tag{3.2}$$

From $y_n \in V_{f_n}$ and $\bar{x}_n \in A$, we can get

$$f_n(\psi(\bar{x}_n) + \varphi(y_n, \bar{x}_n) - \psi(y_n)) \geq 0. \tag{3.3}$$

From assumption (ii) and the continuity of f_n, we have

$$\lim_{n\to\infty} f_n(\psi(\bar{x}_n) + \varphi(y_n, \bar{x}_n) - \psi(y_n)) = f_0(\psi(x_0) + \varphi(y_0, x_0) - \psi(y_0)). \tag{3.4}$$

So, we get

$$f_0(\psi(x_0) + \varphi(y_0, x_0) - \psi(y_0)) \geq 0.$$

By assumption (iii), $M = \{\varphi(x, y) : x, y \in A\}$ and $\psi(A)$ are bounded subsets of Y. Let $S = \{\psi(x) : x \in A\}$, define

$$P_{M+S-\psi(A)}(y^*) = \sup_{y \in M+S-\psi(A)} |y^*(y)|, \ y^* \in Y^*.$$

We know that $P_{M+S-\psi(A)}$ is a seminorm of Y^*. For arbitrary $\epsilon > 0$, let

$$U = \{y^* \in Y^* : P_{M+S-\psi(A)}(y^*) < \epsilon\}$$

is a neighborhood of 0 with respect to $\beta(Y^*, Y)$. Since $f_n - f_0 \to 0$, then there exists N such that $f_n - f_0 \in U$, for all $n > N$. Thus,

$$P_{M+S-\psi(A)}(f_n - f_0) = \sup_{y \in M+S-\psi(A)} |(f_n - f_0)(y)| < \epsilon, \text{whenever } n > N.$$

We also have

$$\begin{aligned} & |(f_n - f_0)(\psi(\bar{x}_n) + \varphi(y_n, \bar{x}_n) - \psi(y_n))| \\ = & |f_n(\psi(\bar{x}_n) + \varphi(y_n, \bar{x}_n) - \psi(y_n)) - f_0(\psi(\bar{x}_n) + \varphi(y_n, \bar{x}_n) - \psi(y_n))| < \epsilon. \end{aligned}$$

Hence,

$$\begin{aligned} & \lim_{n\to\infty} (f_n - f_0)(\psi(\bar{x}_n) + \varphi(y_n, \bar{x}_n) - \psi(y_n)) \\ = & \lim_{n\to\infty} (f_n(\psi(\bar{x}_n) + \varphi(y_n, \bar{x}_n) - \psi(y_n)) - f_0(\psi(\bar{x}_n) + \varphi(y_n, \bar{x}_n) - \psi(y_n))) = 0. \end{aligned} \tag{3.5}$$

From (3.3), we have

$$f_n(\psi(\bar{x}_n)+\varphi(y_n,\bar{x}_n)-\psi(y_n))-f_0(\psi(\bar{x}_n)+\varphi(y_n,\bar{x}_n)-\psi(y_n))$$
$$\geq -f_0(\psi(\bar{x}_n)+\varphi(y_n,\bar{x}_n)-\psi(y_n)).$$

So

$$\lim_{n\to\infty}[f_n(\psi(\bar{x}_n)+\varphi(y_n,\bar{x}_n)-\psi(y_n))-f_0(\psi(\bar{x}_n)+\varphi(y_n,\bar{x}_n)-\psi(y_n))]$$
$$\geq \lim_{n\to\infty}(-f_0(\psi(\bar{x}_n)+\varphi(y_n,\bar{x}_n)-\psi(y_n))).$$

Thus, combining with (3.4), (3.5), we can obtain

$$f_0(\psi(x_0)+\varphi(y_0,x_0)-\psi(y_0))\geq 0. \tag{3.6}$$

According to (3.2) and (3.6), one has

$$f_0(\varphi(x_0,y_0)+\varphi(y_0,x_0))\geq 0. \tag{3.7}$$

From (3.1), if $x_0\neq y_0$, we get $\varphi(x_0,y_0)+\varphi(y_0,x_0)\in -\text{int}C$. Thus, it follows from $f\in C^*\setminus\{0\}$ that

$$f_0(\varphi(x_0,y_0)+\varphi(y_0,x_0))<0,$$

which contradicts (3.7). Therefore, $x_0=y_0$. Since $y_n\in V_{f_n}, y_n\to y_0=x_0$, this contradicts that for any $x_n\in V_{f_n}, x_n\nrightarrow x_0$. Hence, the proof is complete. ■

Theorem 3.1. *For any given $f\in C^*\setminus\{0\}$. Suppose that C has a base and the following conditions are satisfied:*

(*i*) *A is a compact set;*

(*ii*) *For each $x\in A$, $\psi(\cdot)+\varphi(x,\cdot)$ is C-convexlike on A;*

(*iii*) *$\varphi(\cdot,\cdot)$ is continuous on $A\times A, \psi(\cdot)$ is continuous on A;*

(*iv*) *For each $x\in A\setminus V_f$, there exists $y\in V_f$ such that*

$$\varphi(x,y)+\varphi(y,x)+B(0,d^r(x,y))\subset -C,$$

where r is a positive constant;

(*v*) *$M=\{\varphi(x,y):x,y\in A\}$ and $\psi(A)$ are bounded subsets of Y.*

Then,

$$\bigcup_{f\in C^\sharp}V_f\subset V_s\subset \text{cl}(\bigcup_{f\in C^\sharp}V_f).$$

Proof. By virtue of Gong's method in Theroem 2.1 of [19], we only need to prove

$$\bigcup_{f\in C^*\setminus\{0\}}V_f\subset \text{cl}(\bigcup_{f\in C^\sharp}V_f). \tag{3.10}$$

One can define set-valued mapping $H : C^* \setminus \{0\} \to 2^A$ by

$$H(f) = V_f,\ f \in C^* \setminus \{0\}.$$

By Lemma 3.1, $H(\cdot)$ is lower semicontinuous on $C^* \setminus \{0\}$.

Let $x_0 \in \bigcup_{f \in C^* \setminus \{0\}} V_f$. Then, there exists $f_0 \in C^* \setminus \{0\}$ such that

$$x_0 \in V_{f_0} = H(f_0).$$

Since C has a base, then $C^\sharp \neq \emptyset$. Let $g \in C^\sharp$ and set

$$f_n = f_0 + (1/n)g.$$

One has, $f_n \in C^\sharp$. Then, using the same way in Theroem 2.1 of [19], we can get $\{f_n\}$ converges to f_0 with respect to the topology $\beta(Y^*, Y)$.

Since $H(f)$ is l.s.c at f_0, by Proposition 2.1 (i), for sequence $\{f_n\} \subset C^* \setminus \{0\}$, $f_n \to f_0$ and for any $x_0 \in H(f_0)$, there exists $x_n \in H(f_n) = V_{f_n} \subset \bigcup_{f \in C^\sharp} V_f$ such that $x_n \to x_0$. This means that

$$x_0 \in \mathrm{cl}(\bigcup_{f \in C^\sharp} V_f).$$

By the arbitrariness of $x_0 \in \bigcup_{f \in C^* \setminus \{0\}} V_f$, we have

$$\bigcup_{f \in C^* \setminus \{0\}} V_f \subset \mathrm{cl}(\bigcup_{f \in C^\sharp} V_f).$$

This completes the proof. ■

By virtue of Theorem 3.1, we can get the following result easily.

Theorem 3.2. *For any given $f \in C^* \setminus \{0\}$. Suppose that C has a base and the following conditions are satisfied:*

- (i) *$A(\cdot)$ is continuous with compact values on Λ;*
- (ii) *For each $\mu \in \Lambda$ and $x \in A(\mu)$, $\psi(\cdot, \mu) + \varphi(x, \cdot, \mu)$ is C-convexlike on $A(\mu)$;*
- (iii) *For each $\mu \in \Lambda$, $\varphi(\cdot, \cdot, \mu)$ is continuous on $B \times B$, $\psi(\cdot, \mu)$ is continuous on B;*
- (iv) *For each $\mu \in \Lambda, x \in A(\mu) \setminus V_f(\mu)$, there exists $y \in V_f(\mu)$ such that*

$$\varphi(x, y, \mu) + \varphi(y, x, \mu) + B(0, d^r(x, y)) \subset -C,$$

  where r is a positive constant;
- (v) *$M = \{\varphi(x, y, \mu) : x, y \in A(\mu)\}$ and $\psi(A(\mu), \mu)$ are bounded subsets of Y for each $\mu \in \Lambda$.*

Then,

$$\bigcup_{f\in C^\sharp} V_f(\mu) \subset V_s(\mu) \subset \mathrm{cl}(\bigcup_{f\in C^\sharp} V_f(\mu)).$$

Remark 3.1. Our result improves and extends Theorem 2.1 of [19](or Lemma 1.2 of [11]). In [19], under the condition of C-strong monotonicity, the f-solution set to (PGS) is confined to be a singleton (see [[19], Lemma 2.2] or [[17], Theorem 3.2]). In Theorem 3.1, we use condition (iv) to weaken this condition, which can get that f-solution set may be a general set, but not a singleton. Moreover, the condition (iv) of Theorem 3.2 is weaker than C-strong monotonicity. The following example is given to illustrate the case.

Example 3.1. Let $X = R, Y = R^2, C = R^2_+ := [0,+\infty) \times [0,+\infty), \Lambda = [1,2], A(\mu) = [-1,1]$. For each $\mu \in \Lambda, x \in A(\mu), y \in A(\mu)$, let $\varphi(x,y,\mu) = (-\mu^2 + 2\mu - 2, \mu x), \psi(x,\mu) = (\frac{1}{3}, \frac{3}{2}\mu)$. For any given $\mu \in \Lambda$, let $f((x,y)) = \frac{1}{\mu}y$. It follows from a direct computation that $V_f(\mu) = [0,1]$. Obviously, the f-solution set to the (PGS) is not a singleton, but a general set. It is clear that conditions (i) (ii) (iii) and (v) of Theorem 3.2 are satisfied. The condition (iv) in Theorem 3.2 can be checked as follows: For any $x \in A(\mu) \setminus V_f(\mu) = [-1,0)$, there exists $y = 0 \in V_f(\mu)$, such that

$$\begin{aligned}&\varphi(x,y,\mu) + \varphi(y,x,\mu) + B(0, d^r(x,y))\\ &= (-2\mu^2 + 4\mu - 4, \mu x) + B(0, d^r(x,0)) \subset -C.\end{aligned}$$

However, the condition of C-strong monotonicity in [19] does not hold. Indeed, for any $x \in A(\mu) \setminus V_f(\mu) = [-1,0)$, there exists $y = -x \in V_f(\mu) = [0,1]$ such that

$$\varphi(x,y,\mu) + \varphi(y,x,\mu) = (-2\mu^2 + 4\mu - 4, 0) \in -\partial C \setminus \{0\}$$

where ∂C is the boundary of C. Obviously, $\varphi(x,y,\mu) + \varphi(y,x,\mu) \notin -\mathrm{int}C$, which implies that $\varphi(\cdot,\cdot,\mu)$ is not C-strongly monotone on $A(\mu) \times A(\mu)$. Then, Lemma 1.2 of [11], Theorem 2.1 of [19] are not applicable.

4. Upper Semicontinuity of Strong Solution Sets to (PGS)

Theorem 4.1. *For each* $f \in C^* \setminus \{0\}$. *Suppose the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with compact values on* Λ;

(ii) *For each* $\mu \in \Lambda$, $\varphi(\cdot,\cdot,\mu)$ *is continuous on* $B \times B$, $\psi(\cdot,\mu)$ *is continuous on* B.

Then, $V_f(\cdot)$ is u.s.c on Λ.

Proof. Suppose to the contrary that there exists $\mu_0 \in \Lambda$ such that $V_f(\cdot)$ is not u.s.c at μ_0. Then, there exist an open set V satisfying $V_f(\mu_0) \subset V$ and a sequence $\mu_n \to \mu_0, x_n \in V_f(\mu_n)$, such that for any $n, x_n \notin V$.

Since $x_n \in V_f(\mu_n)$ implies $x_n \in A(\mu_n)$, from the upper semicontinuity and compactness of $A(\cdot)$ at μ_0, there exist $x_0 \in A(\mu_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \to x_0$.

Now we first prove $x_0 \in V_f(\mu_0)$. Assume that $x_0 \notin V_f(\mu_0)$, then there exists $y_0 \in A(\mu_0)$ such that

$$f(\psi(y_0, \mu_0) + \varphi(x_0, y_0, \mu_0) - \psi(x_0, \mu_0)) < 0. \tag{4.1}$$

Since $A(\cdot)$ is lower semicontinuous at μ_0 and $y_0 \in A(\mu_0)$, there exists $y_n \in A(\mu_n)$ such that $y_n \to y_0$.

For $x_n \in V_f(\mu_n)$ and $y_n \in A(\mu_n)$, we have $f(\psi(y_n, \mu_n) + \varphi(x_n, y_n, \mu_n) - \psi(x_n, \mu_n)) \geq 0$. From assumption (ii) and the continuity of f, taking the limit as $n \to \infty$, we obtain that

$$f(\psi(y_0, \mu_0) + \varphi(x_0, y_0, \mu_0) - \psi(x_0, \mu_0)) \geq 0.$$

which contradicts (4.1). Therefore, we have $x_0 \in V_f(\mu_0)$.

Since $V_f(\mu_0) \subset V$, thus $x_0 \in V$. It follows $x_n \to x_0$ that there exists enough large positive number N, such that

$$x_n \in V, \ \forall n \geq N,$$

which is a contradiction. Thus, our result holds and the proof is complete. ■

Proposition 4.2. *For any $f \in C^* \setminus \{0\}$. Suppose the following conditions are satisfied:*

(i) $A(\cdot)$ *is compact values on* Λ;

(ii) *For each* $\mu \in \Lambda, y \in A(\mu)$, $\varphi(\cdot, y, \mu)$ *and* $\psi(\cdot, \mu)$ *are continuous on* B.

Then, $V_f(\mu)$ is a closed set. Moreover, $V_f(\mu)$ is a compact set.

Proof. For any sequence $x_\alpha \in V_f(\mu)$ with $x_\alpha \to x_0$, we have $x_\alpha \in A(\mu)$ and

$$f(\psi(y, \mu) + \varphi(x_\alpha, y, \mu) - \psi(x_\alpha, \mu)) \geq 0.$$

By the compactness of $A(\mu)$, it follows that $x_0 \in A(\mu)$.

Since for any $y \in A(\mu)$, $\varphi(\cdot, y, \mu)$ and $\psi(\cdot, \mu)$ are continuous on $A(\mu)$, we can get

$$\psi(y, \mu) + \varphi(x_\alpha, y, \mu) - \psi(x_\alpha, \mu) \to \psi(y, \mu) + \varphi(x_0, y, \mu) - \psi(x_0, \mu).$$

It follows from the continuity of f that $f(\psi(y, \mu) + \varphi(x_0, y, \mu) - \psi(x_0, \mu)) \geq 0, \ \forall y \in A(\mu)$, i.e., $x_0 \in V_f(\mu)$. Therefore, $V_f(\mu)$ is a closed set.

Moreover, by virtue of $V_f(\mu) \subset A(\mu)$ and the compactness of $A(\mu)$, we can obtain $V_f(\mu)$ is a compact set. This completes the proof. ■

Theorem 4.2. *Let $f \in C^* \setminus \{0\}$. Suppose that C has a base and the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with compact values on* Λ;

(ii) *For each* $\mu \in \Lambda$ *and* $x \in A(\mu)$, $\psi(\cdot, \mu) + \varphi(x, \cdot, \mu)$ *is* C*-convexlike on* $A(\mu)$;

(iii) *For each* $\mu \in \Lambda$, $\varphi(\cdot, \cdot, \mu)$ *is continuous on* $B \times B$, $\psi(\cdot, \mu)$ *is continuous on* B;

(iv) *For each* $\mu \in \Lambda, x \in A(\mu) \setminus V_f(\mu)$, *there exist* $y \in V_f(\mu)$ *such that*

$$\varphi(x, y, \mu) + \varphi(y, x, \mu) + B(0, d^r(x, y)) \subset -C,$$

where r *is a positive constant;*

(v) $M = \{\varphi(x, y, \mu) : x, y \in A(\mu)\}$ *and* $\psi(A(\mu), \mu)$ *are bounded subsets of* Y *for each* $\mu \in \Lambda$.

Then, $V_s(\cdot)$ is u.s.c on Λ.

Proof. Suppose to the contrary that there exists $\mu_0 \in \Lambda$ such that $V_s(\cdot)$ is not u.s.c at μ_0. Then, there exist an open set V satisfying $V_s(\mu_0) \subset V$ and a sequence $\{\mu_n\}$ with $\mu_n \to \mu_0,\ x_n \in V_s(\mu_n)$ such that for any $n, x_n \notin V$.

By Theorem 3.2, for each $\mu \in \Lambda$, we have

$$\bigcup_{f \in C^\sharp} V_f(\mu) \subset V_s(\mu) \subset \text{cl}(\bigcup_{f \in C^\sharp} V_f(\mu)).$$

For any neighborhood of $x_n + U(0)$ of x_n, where $U(0)$ is a neighborhood of 0 in X, from Lemma 2.2, we can take a neighborhood $U_1(0)$ of 0 such that $U_1(0) + U_1(0) \subset U(0)$. Since

$$x_n \in V_s(\mu_n) \subset \text{cl}(\bigcup_{f \in C^\sharp} V_f(\mu_n)).$$

Thus, we have

$$(x_n + U(0)) \cap \bigcup_{f \in C^\sharp} V_f(\mu_n) \neq \emptyset.$$

Hence, there exists $z_n \in \bigcup_{f \in C^\sharp} V_f(\mu_n)$ such that

$$z_n - x_n \in U_1(0).$$

Thus, there exists $f' \in C^\sharp$ such that $z_n \in V_{f'}(\mu_n)$. By Theorem 4.1 and Proposition 4.1, $V_f(.)$ is upper semicontinuous with compact values at μ_0. Hence, there exist $x_0 \in V_{f'}(\mu_0)$ and a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to x_0$. Therefore, there exists N_0 such that

$$z_{n_k} - x_0 \in U_1(0),\ \forall n_k > N_0.$$

Thus, for subsequence $\{x_{n_k}\}$ of $\{x_n\}$, we have

$$x_{n_k} - x_0 = x_{n_k} - z_{n_k} + z_{n_k} - x_0 \in -U_1(0) + U_1(0) \subset U_1(0) + U_1(0) \subset U(0).$$

By the arbitrariness of $U(0)$, this means $x_{n_k} \to x_0$.

On the other hand, $x_0 \in \bigcup_{f\in C^\sharp} V_f(\mu_0) \subset V_s(\mu_0) \subset V$. Thus, for given number N sufficiently large, it follows that $x_N \in V$, which contradicts the assumption. Therefore, $V_s(\cdot)$ is u.s.c on Λ. This completes the proof. ∎

Now, we give an example to illustrate that the result of Theorem 4.2 is applicable.

Example 4.1. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+$, $\Lambda = [-1, 1]$ be a subset of Z. Let $\varphi : X \times X \times \Lambda \to Y$ and $\psi : X \times \Lambda \to Y$ be two mappings defined by

$$\varphi(x, y, \mu) = (-\frac{3}{2} - \mu^2, (\mu^2 + 1)x),$$

and

$$\psi(x, \mu) = (-9\mu - 7, 2).$$

Define $A : \Lambda \to 2^Y$ by $A(\mu) = [-1, 1]$.

Obviously, $A(\cdot)$ is a continuous set-valued mapping from Λ to R with nonempty compact values, and conditions (ii) (iii) and (v) of Theorem 4.2 are satisfied.

Let $f = (0, 2) \in C^* \setminus \{0\}$, it follows from a direct computation that $V_f(\mu) = [0, 1]$ for any $\mu \in \Lambda$. Hence, for any $x \in A(\mu) \setminus V_f(\mu)$, there exists $y = 0 \in V_f(\mu)$ such that,

$$\begin{aligned} & \varphi(x, y, \mu) + \varphi(y, x, \mu) + B(0, d^r(x, y)) \\ = \; & (-\frac{3}{2} - \mu^2, (\mu^2 + 1)x) + (-\frac{3}{2} - \mu^2, (\mu^2 + 1)y) + B(0, d^r(x, y)) \\ = \; & (-3 - 2\mu^2, (\mu^2 + 1)x) + B(0, |x - 0|^r) \\ \subset \; & -C. \end{aligned}$$

Thus, the condition (iv) of Theorem 4.2 is also satisfied. By Theorem 4.2, $V_s(\cdot)$ is u.s.c on Λ.

However, the condition that φ is a C-strictly monotone mapping is violated. Indeed, for any $\mu \in \Lambda$ and $x \in A(\mu)\setminus V_f(\mu)$, there exist $y = -x \in V_f(\mu)$ such that

$$\varphi(x, y, \mu) + \varphi(y, x, \mu) = (-3 - 2\mu^2, 0) \notin -\text{int}\, C,$$

which implies that $\varphi(\cdot, \cdot, \cdot)$ is not $\mathbb{R}^2_+$-strictly monotone on $A(\Lambda) \times A(\Lambda) \times \Lambda$. Moreover, $\varphi(x, x, \mu) \notin C, (\forall x \in A(\mu))$. Then, Theorem 4.2 of [12], Theorem 3.3 of [15] are not applicable.

When $r = 1$, we can easily obtain the following corollary.

Corollary 4.1. *Let $f \in C^* \setminus \{0\}$. Suppose that C has a base and the following conditions are satisfied:*

(i) *$A(\cdot)$ is continuous with compact values on Λ;*

(ii) *For each $\mu \in \Lambda$ and $x \in A(\mu)$, $\psi(\cdot, \mu) + \varphi(x, \cdot, \mu)$ is C-convexlike on $A(\mu)$;*

(iii) *For each $\mu \in \Lambda$, $\varphi(\cdot, \cdot, \mu)$ is continuous on $B \times B$, $\psi(\cdot, \mu)$ is continuous on B;*

(*iv*) *For each* $\mu \in \Lambda, x \in A(\mu) \setminus V_f(\mu)$, *there exist* $y \in V_f(\mu)$ *such that*

$$\varphi(x, y, \mu) + \varphi(y, x, \mu) + B(0, d(x, y)) \subset -C;$$

(*v*) $M = \{\varphi(x, y, \mu) : x, y \in A(\mu)\}$ *and* $\psi(A(\mu), \mu)$ *are bounded subsets of* Y *for each* $\mu \in \Lambda$.

Then, $V_s(\cdot)$ is u.s.c on Λ.

We can find the assumption (iv) in Corollary 4.1 is essential. Now, we give the following example to illustrate it.

Example 4.2. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+$, $\Lambda = [-1, 0]$ be a subset of Z. Let $\varphi : X \times X \times \Lambda \to Y$ and $\psi : X \times \Lambda \to Y$ be two mappings defined by

$$\varphi(x, y, \mu) = (\frac{1}{2}(y - x), (1 + \mu^2)\mu x(y - x)),$$

and

$$\psi(x, \mu) = (\mu, 2 + \mu).$$

And define $A : \Lambda \to 2^Y$ by $A(\mu) = [0, 2]$.

Obviously, $A(\cdot)$ is a continuous set-valued mapping from Λ to R with nonempty compact values, and conditions (ii) (iii) and (v) of Corollary 4.1 are satisfied.

It follows from a direct computation that

$$V_s(\mu) = \begin{cases} \{0\}, & \text{if } \mu = 0, \\ \{0, 2\}, & \text{if } \mu \in [-1, 0). \end{cases}$$

Hence, $V_s(\cdot)$ is even not u.s.c at $\mu = 0$. The reason is that the assumption (iv) is violate. In fact, for any $x, y \in A(\mu) = [0, 2], x \neq y$, we have

$$\begin{aligned} & \varphi(x, y, \mu) + \varphi(y, x, \mu) + B(0, d(x, y)) \\ = \; & (\frac{1}{2}(y - x), (1 + \mu^2)\mu x(y - x)) + (\frac{1}{2}(x - y), (1 + \mu^2)\mu y(x - y)) + B(0, d(x, y)) \\ = \; & (0, -(1 + \mu^2)\mu(y - x)^2) + B(0, d(x, y)) \\ \not\subset \; & -C. \end{aligned}$$

Now, we show that $V_s(\cdot)$ is not u.s.c at $\mu = 0$. Indeed, there exists open set $V = (-1, 1)$ such that

$$V_s(0) \subset V,$$

however, for any $\delta > 0$, there exists $\tilde{\mu} \in B(0, \delta) \cap [-1, 0)$ such that

$$V_s(\tilde{\mu}) = \{0, 2\} \not\subset V = (-1, 1).$$

By Definition 2.5, $V_s(\cdot)$ is not u.s.c at $\mu = 0$. Hence, the assumption (iv) in Corollary 4.1 is essential.

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