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A REMARK ON WEIGHTED REPRESENTATION FUNCTIONS

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Abstract. Let G be a finite abelian group, and k_1, k_2 be two integers. For any subset $A \subset G$, let $r_{k_1,k_2}(A, n)$ denote the number of solutions of $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. In this paper, we generalize a result of Q.-H. Yang and Y.-G. Chen to finite abelian groups. More precisely, we characterize all subsets $A \subset G$ such that $r_{k_1,k_2}(A, n) = r_{k_1,k_2}(G\backslash A, n)$ for all $n \in G$.

1. Introduction

Let $\mathbf{N}$ be the set of nonnegative integers. For any subset $A \subset \mathbf{N}$ and $n \in \mathbf{N}$, let $R_1(A, n)$, $R_2(A, n)$ and $R_3(A, n)$ denote the number of solutions of $n = a + a'$ with $a, a' \in A$, $n = a + a'$ with $a, a' \in A$, $a < a'$, and $n = a + a'$ with $a, a' \in A$, $a \le a'$ respectively. These representation functions are studied by Erdős, Sárközy and Sós in a series of papers [7, 8, 11, 9, 10]. Since then, representation functions have been extensively studied by many authors.

Sárközy asked, for each $i = 1, 2, 3$, whether there exist sets A and B with infinite symmetric difference such that $R_i(A, n) = R_i(B, n)$ for all sufficiently large integers n. Dombi [5] observed that the answer is negative for $i = 1$, and is affirmative for $i = 2$. Chen and Wang [3] gave an example of a subset $A \subset \mathbf{N}$ such that $R_3(A, n) = R_3(\mathbf{N}\backslash A, n)$ for all $n \ge 1$. For $i = 2, 3$, Lev, Sándor and Tang [6, 12, 13] characterized all subsets A with the property that $R_i(A, n) = R_i(\mathbf{N}\backslash A, n)$ for all $n \ge 2N + 1$. Some asymptotic results of the representation functions of these sets are obtained in [1, 2].

For any two positive integers $k_1 \le k_2$, $A \subset \mathbf{N}$ and $n \in \mathbf{N}$, one can also define the weighted representation function $r_{k_1,k_2}(A, n)$ as the number of solutions of the equation $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. If $k_2 \ge k_1 \ge 2$, Cillruelo and Rué [4] proved that $r_{k_1,k_2}(A, n)$ can not be eventually constant. Yang and Chen [14] proved that there

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exists a set $A \subset \mathbf{N}$ such that $r_{k_1,k_2}(A, n) = r_{k_1,k_2}(\mathbf{N} \backslash A, n)$ for all sufficiently large n if and only if $k_1 \mid k_2$ and $k_1 < k_2$.

In a recent paper [15], Yang and Chen studied weighted representation functions on $\mathbf{Z}_m$, the cyclic group of order m. For any two integers k_1, k_2, $A \subset \mathbf{Z}_m$ and $n \in \mathbf{Z}_m$, define $r_{k_1,k_2}(A, n)$ to be the number of solutions of the equation $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. For $d \mid m$, A is said uniformly distributed modulo d if

$$|\{x : x \in A, x \equiv i \pmod{d}\}| = |A|/d$$

for all $i = 0, 1, \cdots, d - 1$. They proved the following theorem.

Theorem A. *Let m, k_1, and k_2 be three integers with $m \geq 2$, $A \subseteq \mathbf{Z}_m$. Then $r_{k_1,k_2}(A, n) = r_{k_1,k_2}(\mathbf{Z}_m \backslash A, n)$ for all $n \in \mathbf{Z}_m$ if and only if $|A| = m/2$ and A is uniformly distributed modulo d'_1 and d'_2, respectively, where $d'_1 = (k_1, m)/(k_1, k_2, m)$ and $d'_2 = (k_2, m)/(k_1, k_2, m)$.*

In this paper, we generalize their results to finite abelian groups. We fix some notation first. Let G be a finite abelian group of order m. For any two integers k_1, k_2, $A \subset G$ and $n \in G$, we define similarly the weighted representation function $r_{k_1,k_2}(A, n)$ to be the number of solutions of the equation $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. For any integer k, let kG denote the subgroup $kG = \{kg : g \in G\}$, and G_k denote the subgroup $G_k = \{g : g \in G, kg = 0\}$. For $i = 1, 2$, let $d_i = (k_i, m)$, $d_3 = (d_1, d_2) = (k_1, k_2, m)$, $d_i = d'_i d_3$. Then d'_1 and d'_2 are coprime. Let $H_i = d'_i G + G_{d_3}$, $i = 1, 2$. For a subgroup $H < G$, we say that A is uniformly distributed modulo H if $|A \cap (g + H)|$ is independent of $g \in G$. The following results are proved.

Theorem 1. *Let G be a finite abelian group of order m, and k_1, k_2 be two integers, $A \subset G$. With other notations introduced as above, then $r_{k_1,k_2}(A, n) = r_{k_1,k_2}(G \backslash A, n)$ for all $n \in G$ if and only if $|A| = m/2$ and A is uniformly distributed modulo H_1 and H_2, respectively.*

Corollary 1. *Notations as in Theorem 1. Then there exists a set $A \subset G$ such that $r_{k_1,k_2}(A, n) = r_{k_1,k_2}(G \backslash A, n)$ for all $n \in G$ if and only if $|H_1|$ and $|H_2|$ are both even.*

Remark. For $G = \mathbf{Z}_m$, $G_{d_3} = \frac{m}{d_3}\mathbf{Z}_m$. Since $d'_i = d_i/d_3$ is a divisor of m/d_3, $\frac{m}{d_3}\mathbf{Z}_m \subset d'_i\mathbf{Z}_m$, hence $H_i = d'_i\mathbf{Z}_m + \frac{m}{d_3}\mathbf{Z}_m = d'_i\mathbf{Z}_m$ for $i = 1, 2$. Theorem 1 is consistent with Theorem A in this case. In general, d'_iG may be a proper subset of H_i. For example, $G = \mathbf{Z}_{60} \oplus \mathbf{Z}_2$, $m = 120$, $k_1 = d_1 = 12$, $k_2 = d_2 = 10$. Then $d_3 = 2$, $d'_1 = 6$, $d'_1G = 6\mathbf{Z}_{60} \oplus 2\mathbf{Z}_2$, $G_{d_3} = 30\mathbf{Z}_{60} \oplus \mathbf{Z}_2$, and G_{d_3} is not a subset of d'_1G, thus d'_1G is a proper subset of $H_1 = d'_1G + G_{d_3}$.

2. Proof of the Results

For any subsets $S, T \subset G$ and $n \in G$, let $r_{k_1,k_2}(S,T,n)$ denote the number of solutions of $n = k_1 s + k_2 t$ with $s \in S$ and $t \in T$. Let $\Phi_i(n) = \{g : g \in G, n - k_{3-i} g \in k_i G\}$ for $i = 1, 2$. We need the following lemma.

Lemma 1. *Let $i = 1, 2$. If $n \notin d_3 G$, then $\Phi_i(n) = \varnothing$. If $n \in d_3 G$, then $\Phi_i(n)$ is a coset of H_i and $|\Phi_i(n)| = |H_i|$. For any $A \subset G$ and $n \in G$,*

$$r_{k_1,k_2}(G,A,n) = |A \cap \Phi_1(n)| \cdot |G_{d_1}|,$$

and

$$r_{k_1,k_2}(A,G,n) = |A \cap \Phi_2(n)| \cdot |G_{d_2}|.$$

Proof. If $\Phi_i(n) \neq \varnothing$, say $g \in \Phi_i(n)$, then $n \in k_{3-i} g + k_i G \subset d_3 G$ since $d_3 \mid (k_1, k_2)$. Thus $n \notin d_3 G$ implies $\Phi_i(n) = \varnothing$.

Suppose $n \in d_3 G$. Since $d_3 = (k_1, k_2, m)$, write $d_3 = k_1 u + k_2 v + m w$ for some $u, v, w \in \mathbf{Z}$. For any $g \in G$,

$$d_3 g = (k_1 u + k_2 v + m v) g = k_1(ug) + k_2(vg) \in k_1 G + k_2 G,$$

therefore $k_1 G + k_2 G \supset d_3 G$. On the other hand, $k_1 G + k_2 G \subset d_3 G$, thus we conclude that $k_1 G + k_2 G = d_3 G$. In particular, $n = k_1 g_1 + k_2 g_2$ for some $g_1, g_2 \in G$, therefore $g_{3-i} \in \Phi_i(n)$ and $\Phi_i(n) \neq \varnothing$. Assume $g \in \Phi_i(n)$, then $h \in \Phi_i(n)$ if and only if $(g-h) k_{3-i} \in k_i G = d_i G$. Since $(k_{3-i}, d_i) = d_3$, it is equivalent to $g - h \in d_i' G + G_{d_3} = H_i$, thus $\Phi_i(n)$ is a coset of H_i. In particular, $|\Phi_i(n)| = |H_i|$.

If $a \in A$ and $g \in G$ satisfy $n = k_1 a + k_2 g$, then $n - k_1 a \in k_2 G$, thus $a \in A \cap \Phi_2(n)$. On the other hand, for any $a \in A \cap \Phi_2(n)$, there exists $g_0 \in G$ such that $n = k_1 a + k_2 g_0$ by the definition of $\Phi_2(n)$. Since

$$\{g : g \in G, n = k_1 a + k_2 g\} = g_0 + G_{k_2} = g_0 + G_{d_2},$$

which is a coset of G_{d_2}, we have $|\{g : g \in G, n = k_1 a + k_2 g\}| = |G_{d_2}|$. Therefore

$$r_{k_1,k_2}(A,G,n) = |\{(a,g) : a \in A, g \in G, k_1 a + k_2 g = n\}|$$

$$= \sum_{a \in A \cap \Phi_2(n)} |\{g : g \in G, n = k_1 a + k_2 g\}| = |A \cap \Phi_2(n)| \cdot |G_{d_2}|.$$

Similarly,

$$r_{k_1,k_2}(G,A,n) = |A \cap \Phi_1(n)| \cdot |G_{d_1}|.$$

This completes the proof of the lemma. ∎

Proof of Theorem 1. Let $B = G \setminus A$. We first note that

$$r_{k_1,k_2}(A,n) = r_{k_1,k_2}(B,n) \tag{1}$$

is equivalent to

$$\big(r_{k_1,k_2}(A,A,n)+r_{k_1,k_2}(B,A,n)\big)+\big(r_{k_1,k_2}(A,A,n)+r_{k_1,k_2}(A,B,n)\big)$$
$$=\big(r_{k_1,k_2}(A,B,n)+r_{k_1,k_2}(B,B,n)\big)+\big(r_{k_1,k_2}(B,A,n)+r_{k_1,k_2}(B,B,n)\big),$$

that is,

$$(2)\qquad r_{k_1,k_2}(G,A,n)+r_{k_1,k_2}(A,G,n)=r_{k_1,k_2}(G,B,n)+r_{k_1,k_2}(B,G,n).$$

By Lemma 1, equality (2) is equivalent to

$$(3)\qquad \sum_{i=1}^{2}|A\cap\Phi_i(n)|\cdot|G_{d_i}|=\sum_{i=1}^{2}|B\cap\Phi_i(n)|\cdot|G_{d_i}|.$$

If $n\notin d_3G$, then $\Phi_i(n)=\varnothing$ by Lemma 1, hence both sides of (3) are zero. Assume now $n\in d_3G$, by Lemma 1, $|\Phi_i(n)|=|H_i|$, hence

$$|A\cap\Phi_i(n)|+|B\cap\Phi_i(n)|=|\Phi_i(n)|=|H_i|.$$

Adding both sides of equation (3), we see that (3) is equivalent to

$$(4)\qquad \sum_{i=1}^{2}|A\cap\Phi_i(n)|\cdot|G_{d_i}|=\frac{1}{2}\sum_{i=1}^{2}|H_i|\cdot|G_{d_i}|$$

for all $n\in d_3G$.

We now prove the sufficiency part. Assume $|A|=m/2$, and A is uniformly distributed modulo H_1 and H_2, respectively. We shall show that equality (4) holds for all $n\in d_3G$. By Lemma 1, $\Phi_i(n)$ is a coset of H_i, therefore $|A\cap\Phi_i(n)|=|H_i|/2$, and

$$\sum_{i=1}^{2}|A\cap\Phi_i(n)|\cdot|G_{d_i}|=\frac{1}{2}\sum_{i=1}^{2}|H_i|\cdot|G_{d_i}|.$$

Next we prove the necessity part. Assume that equalities (1)-(4) are satisfied. Since

$$(5)\qquad |A|^2=\sum_{n\in G}r_{k_1,k_2}(A,n)=\sum_{n\in G}r_{k_1,k_2}(B,n)=|B|^2,$$

we have $|A|=|B|$, hence $|A|=m/2$. Note that the right hand side of equality (4) is fixed. If $n\in k_1G$, then

$$\Phi_1(n)=\{g:g\in G,k_2g\in k_1G\}=H_1,$$

which is independent of $n\in k_1G$. Consequently, $|A\cap\Phi_2(n)|$ is independent of $n\in k_1G$. When n runs through all elements of k_1G, $\Phi_2(n)$ runs through all cosets of H_2 as we immediately see that $g\in\Phi_2(k_1g)$ for all $g\in G$. It follows that $|A\cap(g+H_2)|$

is independent of $g \in G$, hence A is uniformly distributed modulo H_2. By similar arguments, A is also uniformly distributed modulo H_1. ∎

Proof of Corollary 1. Assume there exists a subset $A \subset G$ such that

$$r_{k_1,k_2}(A, n) = r_{k_1,k_2}(G\backslash A, n)$$

for all $n \in G$. By Theorem 1, $|A| = m/2$ and A is uniformly distributed modulo H_1 and H_2 respectively. Therefore $|A \cap H_i| = |H_i|/2$, hence $|H_i|$ is even for $i = 1, 2$.

Conversely, assume $|H_i|$ is even for $i = 1, 2$. Since

$$H_1 + H_2 \supseteq d_1'G + d_2'G = (d_1', d_2', m)G = G,$$

we have $H_1 + H_2 = G$. Let $X_1, X_2, \cdots, X_s$ and $Y_1, Y_2, \cdots, Y_t$ denote the cosets of H_1 and H_2 respectively. Put $H = H_1 \cap H_2$. By Chinese Remainder Theorem, we have $G/H \cong G/H_1 \times G/H_2$. Therefore

$$X_i \cap Y_j, \quad 1 \le i \le s, 1 \le j \le t$$

are all the cosets of H, and $|X_i| = |H_1| = t|H|$, $|Y_j| = |H_2| = s|H|$ for $1 \le i \le s$, $1 \le j \le t$.

Case 1. $|H|$ is even. We take

$$A = \bigcup_{\substack{1\le i\le s\\ 1\le j\le t}} A_{ij},$$

where $A_{ij} \subset X_i \cap Y_j$ is any subset with $|A_{ij}| = |H|/2$. Then

$$|A \cap X_i| = \left|\bigcup_{1\le j\le t} (A \cap X_i \cap Y_j)\right| = \sum_{j=1}^{t} |A_{ij}| = t|H|/2 = |H_1|/2,$$

that is, A is uniformly distributed modulo H_1. Similarly, A is uniformly distributed modulo H_2.

Case 2. $|H|$ is odd. Since $|H_2|$ is even and $|H_2| = s|H|$, we see that s is even. Similarly, t is also even. Write $s = 2k$, $t = 2l$, and we take

$$A = \left(\bigcup_{\substack{1\le i\le k\\ 1\le j\le l}} (X_i \cap Y_j)\right) \bigcup \left(\bigcup_{\substack{k+1\le i\le 2k\\ l+1\le j\le 2l}} (X_i \cap Y_j)\right).$$

For $1 \le i \le k$,

$$|A \cap X_i| = \sum_{j=1}^{l} |X_i \cap Y_j| = l|H| = |X_i|/2.$$

For $k+1 \leq i \leq 2k$,

$$|A \cap X_i| = \sum_{j=l+1}^{2l} |X_i \cap Y_j| = l|H| = |X_i|/2.$$

Thus A is uniformly distributed modulo H_1. Similarly, A is uniformly distributed modulo H_2. By Theorem 1, we have

$$r_{k_1,k_2}(A,n) = r_{k_1,k_2}(G \backslash A, n)$$

for all $n \in G$. This completes the proof of the corollary. ∎

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