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NEW CRITERIA FOR EXPONENTIAL STABILITY OF LINEAR TIME-VARYING DIFFERENTIAL SYSTEMS WITH DELAY

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Abstract. General linear time-varying differential systems with delay are considered. Several explicit criteria for exponential stability are presented. Furthermore, an explicit robust stability bound for systems subject to time-varying perturbations is given. Two examples are given to illustrate the obtained results. To the best of our knowledge, the results of this paper are new.

1. INTRODUCTION

Delay differential equations have numerous applications in science and engineering. They are used as models for of phenomena in the life sciences, physics and technology, chemistry and economics see e.g. [6, 12, 21].

In particular, problems of stability of time-delay systems have been investigated intensively during the past decades, see e.g. [1-5, 9, 13-16, 24-26] and references therein. Recently, the exponential stability of delay systems have attracted much attention from researchers, see e.g. [6, 9, 13-16, 24-26]. In this paper, we first investigate exponential stability of linear differential systems with *time-varying* delay of the form

$$(1)\quad \dot{x}(t) = A_0(t)x(t) + \sum_{k=1}^{m} A_k(t)x(t-h_k(t)) + \int_{-h(t)}^{0} B(t,s)x(t+s)ds, \quad t \geq \sigma.$$

Then we deal with the problem of robust stability of (1) under *time-varying structured perturbations*.

In general, problems of stability of time-varying differential systems with delay are hard. The traditional approaches to stability of time-varying differential systems with delay are Lyapunov's method and its variants (Razumikhin-type theorems, Lyapunov-Krasovskii functional techniques), see e.g. [1, 4, 5, 7, 25, 26]. In contrast to the

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traditional approaches, we present in this paper a novel approach to problems of exponential stability of linear *time-varying* differential systems with delay. Consequently, we get some new explicit criteria for the exponential stability of linear differential systems with time-varying delay (1). Our approach is based on the celebrated Perron-Frobenius theorem and the comparison principle. It is worth noticing that the approach utilized in this paper can be used to study stability of various dynamical systems, see e.g. [16-19].

The organization of the paper is as follows. In the next section, we give some notations and preliminary results which will be used in what follows. The main results are presented in Section 3. We first offer some new explicit criteria for exponential stability of the linear time-varying differential system with delay (1). Then we give an explicit robust stability bound for the system (1) subject to the time-varying structured perturbations. A brief discussion of the obtained results and two illustrative examples are presented.

2. Preliminaries

Let us denote by $\mathbb{N}$, the set of all natural numbers. For given $m \in \mathbb{N}$, let $\underline{m} := \{1, 2, ..., m\}$ and $\underline{m_0} := \{0, 1, 2, ..., m\}$. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ where $\mathbb{C}$ and $\mathbb{R}$ denote the sets of all complex and all real numbers, respectively. For an integer $l, q \geq 1, \mathbb{K}^l$ denotes the l-dimensional vector space over $\mathbb{K}$ and $\mathbb{K}^{l\times q}$ stands for the set of all $l \times q$-matrices with entries in $\mathbb{K}$. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l\times q}$, we write $A \geq B$ iff $a_{ij} \geq b_{ij}$ for $i = 1, ..., l, j = 1, ...q$. In particular, if $a_{ij} > b_{ij}$ for $i = 1, ..., l, j = 1, ...q$, then we write $A \gg B$ instead of $A \geq B$. We denote by $\mathbb{R}_+^{l\times q}$ the set of all nonnegative matrices $A \geq 0$. Similar notations are adopted for vectors.

For $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. Then one has

$$|PQ| \leq |P||Q|, \quad \forall P \in \mathbb{R}^{l\times q}, \forall Q \in \mathbb{R}^{q\times r}.$$

A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{K}^n, |x| \leq |y|$. Every p-norm on $\mathbb{K}^n$ $(\|x\|_p = (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}, 1 \leq p < \infty$ and $\|x\|_\infty = \max_{i=1,2,...,n} |x_i|)$, is monotonic. Throughout the paper, if otherwise not stated, the norm of a matrix $P \in \mathbb{K}^{l\times q}$ is understood as its operator norm associated with a given pair of monotonic vector norms on $\mathbb{K}^l$ and $\mathbb{K}^p$, that is $\|P\| = \max\{\|Py\| : \|y\| = 1\}$. Note that, one has

$$P \in \mathbb{K}^{l\times q}, Q \in \mathbb{R}_+^{l\times q}, |P| \leq Q \Rightarrow \|P\| \leq \||P|\| \leq \|Q\|,$$

see, e.g. [22]. In particular, if $\mathbb{R}^n$ is endowed with $\|\cdot\|_1$ or $\|\cdot\|_\infty$ then $\|A\| = \||A|\|$ for any $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. More precisely, one has

$$\|A\|_1 = \||A|\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|; \quad \|A\|_\infty = \||A|\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|.$$

For any matrix $M \in \mathbb{K}^{n\times n}$ the *spectral abscissa* of M is denoted by $\mu(M) = \max\{\Re\lambda : \lambda \in \sigma(M)\}$, where $\sigma(M) := \{\lambda \in \mathbb{C} : \det(\lambda I_n - M) = 0\}$ is spectrum of M. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz stable if $\mu(M) < 0$.

A matrix $M \in \mathbb{R}^{n\times n}$ is called a *Metzler matrix* if all off-diagonal elements of M are nonnegative. We now summarize some properties of Metzler matrices which will be used in what follows.

Theorem 2.1. [22]. *Suppose that $M \in \mathbb{R}^{n\times n}$ is a Metzler matrix. Then*

(i) (Perron-Frobenius) $\mu(M)$ *is an eigenvalue of* M *and there exists a nonnegative eigenvector* $x \neq 0$ *such that* $Mx = \mu(M)x$.

(ii) *Given* $\alpha \in \mathbb{R}$, *there exists a nonzero vector* $x \geq 0$ *such that* $Mx \geq \alpha x$ *if and only if* $\mu(M) \geq \alpha$.

(iii) $(tI_n - M)^{-1}$ *exists and is nonnegative if and only if* $t > \mu(M)$.

(iv) *Given* $B \in \mathbb{R}_+^{n\times n}, C \in \mathbb{C}^{n\times n}$. *Then*

$$|C| \leq B \Rightarrow \mu(M + C) \leq \mu(M + B).$$

The following is immediate from Theorem 2.1.

Theorem 2.2. *Let $M \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Then the following statements are equivalent*

(i) $\mu(M) < 0$;

(ii) $Mp \ll 0$ *for some* $p \in \mathbb{R}_+^n, p \gg 0$;

(iii) M *is invertible and* $M^{-1} \leq 0$;

(iv) *For given* $b \in \mathbb{R}^n, b \gg 0$ *there exists* $x \in \mathbb{R}_+^n$, *such that* $Mx + b = 0$;

(v) *For any* $x \in \mathbb{R}^n\backslash\{0\}$, *the row vector* $x^T M$ *has at least one negative entry.*

Let $\mathbb{K}^{m\times n}$ be endowed with the norm $\|\cdot\|$ and let J be an interval of $\mathbb{R}$. Denote by $C(J, \mathbb{K}^{m\times n})$, the vector space of all continuous functions on J with values in $\mathbb{K}^{m\times n}$. In particular, $C([\alpha, \beta], \mathbb{K}^{m\times n})$ is a Banach space endowed with the norm $\| \varphi \| := \max_{\theta\in[\alpha,\beta]} \| \varphi(\theta) \|$. In what follows, we write $\mathcal{C}$ instead of $C([-h, 0], \mathbb{R}^n)$ and denote $\mathcal{C}_r := \{\varphi \in \mathcal{C} : \|\varphi\| \leq r\}$, for given $r > 0$. For a matrix function $\varphi(\cdot) : J \to \mathbb{R}^{m\times n}$, we say that $\varphi(\cdot)$ nonnegative and denote it by $\varphi \geq 0$ if $\varphi(\theta) \geq 0$ for all $\theta \in J$.

3. Exponential Stability of Linear Time-varying Differential Systems with Delay

3.1. Explicit criteria for exponential stability

Consider a linear time-varying differential system with delay of the form (1), where

(i) $h(\cdot), h_k(\cdot) : \mathbb{R} \to \mathbb{R}$ ($k \in \underline{m}$) are given continuous functions such that $0 < h(t) \le h;\ 0 < h_k(t) \le h_k,\ \forall t \in \mathbb{R}$, for some positive numbers $h, h_k (k \in \underline{m})$ and $h \ge \max_{k \in \underline{m}} \{h_k\}$;

(ii) $A_k(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}, k \in \underline{m_0}$ and $B(\cdot;\cdot) : \mathbb{R} \times [-h, 0] \to \mathbb{R}^{n\times n}$ are given matrix-valued continuous functions.

It is well-known that for fixed $\sigma \in \mathbb{R}$ and given $\varphi \in \mathcal{C}$, (1) has a unique solution satisfying the initial value condition

$$x(s+\sigma) = \varphi(s), \quad s \in [-h, 0], \tag{2}$$

see e.g. [7]. This solution is denoted by $x(\cdot;\sigma,\varphi)$.

Definition 3.1. The system (1) is said to be exponentially stable if there exist positive numbers K, β such that

$$\|x(t;\sigma,\varphi)\| \le K e^{-\beta(t-\sigma)} \|\varphi\|, \quad \forall t \ge \sigma,$$

for any $\sigma \in \mathbb{R}$ and any $\varphi \in \mathcal{C}$.

With a given matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, we associate the Metzler matrix $M(A) := (\hat{a}_{ij}) \in \mathbb{R}^{n\times n}$, where

$$\hat{a}_{ij} := |a_{ij}|,\ i \ne j,\ i, j \in \underline{n}; \quad \hat{a}_{ii} := a_{ii},\ i \in \underline{n}.$$

We are now in the position to prove the main result of this paper.

Theorem 3.2. *Let* $A_0(t) := (a_{ij}^{(0)}(t)) \in \mathbb{R}^{n\times n}, t \in \mathbb{R}$. *The system* (1) *is exponentially stable provided one of the following conditions holds:*

(i) *there exist* $\beta_1 > 0$ *and* $p \in \mathbb{R}^n_+, p \gg 0$ *such that*

$$\Big(M(A_0(t)) + \sum_{k=1}^{m} |A_k(t)| e^{\beta_1 h_k(t)} + \int_{-h(t)}^{0} |B(t,s)| e^{-\beta_1 s} ds \Big) p \ll -\beta_1 p,\ \forall t \in \mathbb{R}; \tag{3}$$

(ii) *there exist* $\beta_2 > 0$ *and a Hurwitz stable matrix* $B_0 \in \mathbb{R}^{n\times n}$ *such that*

$$M(A_0(t)) + \sum_{k=1}^{m} |A_k(t)| e^{\beta_2 h_k(t)} + \int_{-h(t)}^{0} |B(t,s)| e^{-\beta_2 s} ds \le B_0, \quad \forall t \in \mathbb{R}; \tag{4}$$

(iii) *there exist* $A_0 \in \mathbb{R}^{n\times n}$ *and* $B_0 \in \mathbb{R}^{n\times n}_+$ *such that*

$$M(A_0(t)) \le A_0, \quad \forall t \in \mathbb{R}, \tag{5}$$

$$\sum_{k=1}^{m} |A_k(t)| + \int_{-h(t)}^{0} |B(t,s)| ds \le B_0, \quad \forall t \in \mathbb{R}, \tag{6}$$

and $A_0 + B_0$ *is Hurwitz stable.*

Proof. (i) Let $p := (\alpha_1, \alpha_2, ..., \alpha_n)^T, \alpha_i > 0, \forall i \in \underline{n}$ and let $\varphi \in \mathcal{C}_1$. Choose a positive number $K > 0$ such that $|\varphi(t)| \ll Ke^{-\beta_1 t}p$ for any $t \in [-h, 0]$ and for any $\varphi \in \mathcal{C}_1$. Define $u(t) := Ke^{-\beta_1(t-\sigma)}p, t \in [\sigma - h, +\infty)$. Set $x(t) := x(t;\sigma,\varphi), t \geq \sigma$. Clearly, $|x(t)| \ll u(t), \forall t \in [\sigma - h, \sigma]$. We claim that $|x(t)| \leq u(t), \forall t \in [\sigma, +\infty)$.

Assume on contrary that there exists $t_0 > \sigma$ such that $|x(t_0)| \nleq u(t_0)$. Set $t_1 := \inf\{t \in (\sigma, +\infty) : |x(t)| \nleq u(t)\}$. By continuity, $t_1 > \sigma$ and there is $i_0 \in \underline{n}$ such that

$$|x(t)| \leq u(t), \forall t \in [\sigma, t_1); |x_{i_0}(t_1)| = u_{i_0}(t_1), |x_{i_0}(t)| > u_{i_0}(t), \forall t \in (t_1, t_1+\epsilon), \tag{7}$$

for some $\epsilon > 0$. Let $A_k(t) := (a_{ij}^{(k)}(t)) \in \mathbb{R}^{n\times n},\ t \in \mathbb{R},\ k \in \underline{m}$ and let $B(t,s) := (b_{ij}(t,s)) \in \mathbb{R}^{n\times n},\ t \in \mathbb{R},\ s \in [-h, 0]$. For every $i \in \underline{n}$, we have

$$\frac{d}{dt}|x_i(t)| = sgn(x_i(t))\dot{x}_i(t) \leq a_{ii}^{(0)}(t)|x_i(t)| + \sum_{j=1, j\neq i}^{n} |a_{ij}^{(0)}(t)||x_j(t)|$$
$$+ \sum_{k=1}^{m}\sum_{j=1}^{n} |a_{ij}^{(k)}(t)||x_j(t - h_k(t))| + \sum_{j=1}^{n} \int_{-h(t)}^{0} |b_{ij}(t,s)||x_j(t+s)|ds,$$

for almost any $t \in [\sigma, +\infty)$. It follows that for any $t \in [\sigma, +\infty)$

$$D^+|x_i(t)| := \limsup_{\epsilon\to 0^+} \frac{|x_i(t+\epsilon)| - |x_i(t)|}{\epsilon} = \limsup_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_t^{t+\epsilon} \frac{d}{ds}|x_i(s)|ds \leq a_{ii}^{(0)}(t)|x_i(t)|$$
$$+ \sum_{j=1, j\neq i}^{n} |a_{ij}^{(0)}(t)||x_j(t)| + \sum_{k=1}^{m}\sum_{j=1}^{n} |a_{ij}^{(k)}(t)||x_j(t - h_k(t))|$$
$$+ \sum_{j=1}^{n} \int_{-h(t)}^{0} |b_{ij}(t,s)||x_j(t+s)|ds,$$

where D^+ denotes the Dini upper-right derivative. In particular, it follows that

$$D^+\, |x_{i_0}(t_1)| \overset{(7)}{\leq} a_{i_0 i_0}^{(0)}(t_1)Ke^{-\beta_1(t_1-\sigma)}\alpha_{i_0} + \sum_{j=1, j\neq i_0}^{n} |a_{i_0 j}^{(0)}(t_1)|Ke^{-\beta_1(t_1-\sigma)}\alpha_j$$
$$+ \sum_{k=1}^{m}\sum_{j=1}^{n} |a_{i_0 j}^{(k)}(t_1)|Ke^{-\beta_1(t_1-\sigma)}e^{\beta_1 h_k(t_1)}\alpha_j$$
$$+ \sum_{j=1}^{n} \int_{-h(t_1)}^{0} |b_{i_0 j}(t_1,s)|Ke^{-\beta_1(t_1-\sigma)}e^{-\beta_1 s}\alpha_j ds$$
$$= Ke^{-\beta_1(t_1-\sigma)}\Bigg(a_{i_0 i_0}^{(0)}(t_1)\alpha_{i_0} + \sum_{j=1, j\neq i_0}^{n} |a_{i_0 j}^{(0)}(t_1)|\alpha_j + \sum_{k=1}^{m}\sum_{j=1}^{n} |a_{i_0 j}^{(k)}(t_1)|e^{\beta_1 h_k(t_1)}\alpha_j$$
$$+ \sum_{j=1}^{n} \int_{-h(t_1)}^{0} |b_{i_0 j}(t_1,s)|e^{-\beta_1 s}\alpha_j ds\Bigg) \overset{(3)}{<} -\beta_1 Ke^{-\beta_1(t_1-\sigma)}\alpha_{i_0} = D^+ u_{i_0}(t_1).$$

However, this conflicts with (7). Therefore

$$|x(t;\sigma,\varphi)| \le u(t) = Ke^{-\beta_1(t-\sigma)}p, \quad \forall t \ge \sigma;\ \forall \varphi \in \mathcal{C}_1.$$

By the monotonicity of vector norms, this yields

$$\|x(t;\sigma,\varphi)\| \le K_1 e^{-\beta_1(t-\sigma)}, \quad \forall t \ge \sigma;\ \forall \varphi \in \mathcal{C}_1,$$

for some $K_1 > 0$. By the linearity of (1),

$$\frac{1}{\|\varphi\|}\|x(t;\sigma,\varphi)\| = \|x(t;\sigma,\frac{\varphi}{\|\varphi\|})\| \le K_1 e^{-\beta_1(t-\sigma)},\ \forall t \ge \sigma,\ \forall \varphi \in \mathcal{C}, \varphi \ne 0.$$

Therefore,

$$\|x(t;\sigma,\varphi)\| \le K_1 e^{-\beta_1(t-\sigma)}\|\varphi\|,\ \forall t \ge \sigma,\ \forall \varphi \in \mathcal{C}.$$

Thus, (1) is exponentially stable.

(ii) It remains to show that (ii) implies (i). Since B_0 is a Hurwitz stable Metzler matrix, there exists $p \in \mathbb{R}^n_+, p \gg 0$ so that $B_0 p \ll 0$, by Theorem 2.2. By continuity, this implies

$$B_0 p \ll -\eta p, \tag{8}$$

for some sufficiently small $\eta > 0$. Let β be as in (ii) and let $\beta_0 := \min\{\beta, \eta\} > 0$. Clearly,

$$\begin{aligned}&\left(M(A_0(t)) + \sum_{k=1}^m |A_k(t)|e^{\beta_0 h_k(t)} + \int_{-h(t)}^0 |B(t,s)|e^{-\beta_0 s}ds\right)\\ \le &\left(M(A_0(t)) + \sum_{k=1}^m |A_k(t)|e^{\beta h_k(t)} + \int_{-h(t)}^0 |B(t,s)|e^{-\beta s}ds\right) \overset{(4)}{\le} B_0, \quad \forall t \in \mathbb{R}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(M(A_0(t)) + \sum_{k=1}^m |A_k(t)|e^{\beta_0 h_k(t)} + \int_{-h(t)}^0 |B(t,s)|e^{-\beta_0 s}ds\right)p\\ &\le B_0 p \overset{(8)}{\ll} -\eta p \le -\beta_0 p,\ \forall t \in \mathbb{R}.\end{aligned}$$

Thus, (i) holds.

(iii) We show that (i) holds. Since $A_0 + B_0$ is a Hurwitz stable Metzler matrix, there exists $p \in \mathbb{R}^n_+, p \gg 0$ so that $(A_0 + B_0)p \ll 0$, by Theorem 2.2 (ii). By continuity, this yields

$$(A_0 + e^{\beta h}B_0)p \ll -\beta p, \tag{9}$$

for some sufficiently small $\beta > 0$. Then (3) follows from (5)-(6) and (9). This completes the proof. ■

The following is immediate from Theorem 3.2.

Corollary 3.3. *Suppose there exist* $A_0 := (a_{ij}^{(0)}) \in \mathbb{R}^{n\times n}$ *and* $A_k \in \mathbb{R}_+^{n\times n}, k \in \underline{m}$ *and a continuous matrix-valued function* $C(\cdot) : [-h, 0] \to \mathbb{R}_+^{n\times n}$ *so that* (5) *holds and*

$$(10) \qquad |A_k(t)| \le A_k, \forall t \in \mathbb{R}, k \in \underline{m}\,; \qquad |B(t,s)| \le C(s),\ \forall t \in \mathbb{R}, \forall s \in [-h, 0].$$

If $\sum_{k=0}^{m} A_k + \int_{-h}^{0} C(s)ds,$ *is Hurwitz stable then* (1) *is exponentially stable.*

Remark 3.4. (A discussion of the obtained results).

(i) In the well-known book [7, page 145], it has been shown that the scalar differential equation with delay

$$\dot{x}(t) = -a(t)x(t) - \sum_{k=1}^{m} b_k(t)x(t - h_k(t)),$$

is exponentially stable for all bounded continuous functions $a(\cdot), b_k(\cdot), h_k(\cdot) \in C(\mathbb{R}, \mathbb{R})$, $k \in \underline{m}$, provided $a(t) \ge \delta > 0, \sum_{k=1}^{m} |b_k(t)| \le \theta\delta, 0 < \theta < 1, 0 \le h_k(t) \le h,$ for all $t \in \mathbb{R}$. The proof given in [7] relies completely upon a Razumikhin-type theorem. However, this is immediate from Corollary 3.3.

A similar result has been found in [11, Example 5.1, page 74]. More precisely, the differential equation

$$\dot{x}(t) = -ax(t) + b(t)x(t - h),$$

where $a, h > 0$ and $b(\cdot) \in C(\mathbb{R}, \mathbb{R})$ such that $\sup_{t\ge t_0} |b(t)| < a$, is exponentially stable. One again, this assertion follows from Corollary 3.3.

On the other hand, based on a generalized Halanay inequality, it has been showed in [10] that the equation

$$(11) \qquad \dot{x}(t) = -a(t)x(t) - b(t)\int_{t-\tau}^{t} x(s)ds,$$

where $a(\cdot), b(\cdot) \in C(\mathbb{R}, \mathbb{R})$, is exponentially stable provided there exist positive numbers a, η such that $0 < a(t) \le a, t \in \mathbb{R}$ and

$$(12) \qquad \inf_{t\in\mathbb{R}} \frac{a(t) - \tau|b(t)|}{1 + \frac{3}{2}\tau^2|b(t)|} \ge \eta > 0.$$

We show that this follows from Theorem 3.2. Clearly, $e^t < 1 + \frac{3}{2}t,\ t \in (0, \beta)$, for sufficiently small $\beta > 0$. Let $0 < \beta_1 < \min\{\frac{\beta}{\tau}, \eta\}$. It follows from (12) that

$$-a(t) + \int_{-\tau}^{0} |b(t)|e^{-\beta_1 s}ds \le -a(t) + \tau|b(t)|e^{\beta_1\tau}$$
$$< -a(t) + \tau|b(t)|(1 + \frac{3}{2}\beta_1\tau) \overset{(12)}{\le} -\beta_1, \quad \forall t \in \mathbb{R}.$$

Thus, (i) of Theorem 3.2 holds and (11) is exponentially stable.

(ii) Generally speaking, a dynamical system is called positive if for any nonnegative initial condition, the corresponding solution of the system is also nonnegative, see [6, 13-20]. Positive dynamical systems play an important role in modelling of dynamical phenomena whose variables are restricted to be nonnegative. They are often encountered in applications, for example, networks of reservoirs, industrial processes involving chemical reactors, heat exchangers, distillation columns, storage systems, hierarchical systems, compartmental systems used for modelling transport and accumulation phenomena of substances, see e.g. [6].

It is well-known that a linear time-invariant differential system of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - h_k) + \int_{-h}^{0} C(s) x(t+s) ds, \tag{13}$$

is positive if, and only if, $A_0 \in \mathbb{R}^{n\times n}$ is a Metzler matrix and $A_k \in \mathbb{R}_+^{n\times n}$ for all $k \in \underline{m}$ and $C(s) \in \mathbb{R}_+^{n\times n}$ for all $s \in [-h, 0]$, see e.g. [17]. Then, roughly speaking, Corollary 3.3 means that the linear time-varying differential systems with delay (1) is "bounded above" (in some sense) by the positive system (13) and then (1) is exponentially stable provided (13) is exponentially stable. This is a nice surprise because it is very similar to the well-known Weierstrass M-test in the theory of infinite series of functions (see e.g. [2]).

3.2. Stability of perturbed systems

In this subsection, we deal with the problem of stability of (1) subject to time-varying structured perturbations.

Suppose all of the hypotheses of Corollary 3.3 are satisfied and thus (1) is exponentially stable. Consider a perturbed system of the form

$$\begin{aligned} \dot{x}(t) &= (A_0(t) + D_0(t)\Delta_0(t)E_0(t))x(t) \\ &+ \sum_{k=1}^{m} A_k(t) x(t - h_k(t)) + \sum_{k=1}^{m} D_k(t)\Delta_k(t)E_k(t) x(t - \tau_k(t)) \\ &+ \int_{-h(t)}^{0} (B(t,s) + D(t,s)\delta(t,s)E(t,s)) x(t+s) ds, \quad t \geq \sigma, \end{aligned} \tag{14}$$

where

(i) $\tau_k(\cdot) : \mathbb{R} \to \mathbb{R}$ is a given continuous function for each $k \in \underline{m}$ such that $0 < \tau_k(t) \leq \tau_k \leq h, \forall t \in \mathbb{R}$, for some $\tau_k \in \mathbb{R}_+$;

(ii) $D_k(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times l_k}, E_k(\cdot) : \mathbb{R} \to \mathbb{R}^{q_k \times n}, k \in \underline{m_0}$ and $D(\cdot;\cdot) : \mathbb{R} \times [-h, 0] \to \mathbb{R}^{n\times l}, E(\cdot;\cdot) : \mathbb{R} \times [-h, 0] \to \mathbb{R}^{q\times n}$ are given matrix-valued continuous functions;

(iii) $\Delta_k(\cdot) : \mathbb{R} \to \mathbb{R}^{l_k \times q_k}, k \in \underline{m_0}$ and $\delta(\cdot;\cdot) : \mathbb{R} \times [-h, 0] \to \mathbb{R}^{l\times q}$ are *unknown* matrix-valued continuous functions.

We show that there exists a positive number r such that an arbitrary perturbed equation of the form (14) *remains exponentially stable whenever the size of perturbations is less than r.*

Theorem 3.5. *Assume that all of the hypotheses of Corollary 3.3 are satisfied. Suppose the exist $D_k \in \mathbb{R}_+^{n\times l_k}, E_k \in \mathbb{R}_+^{q_k\times n}, \Delta_k \in \mathbb{R}_+^{l_k\times q_k}$ for $k \in \underline{m_0}$ and $D_{m+1} \in \mathbb{R}_+^{n\times l}, E_{m+1} \in \mathbb{R}_+^{q\times n},\ \delta_{m+1}(\cdot) \in C([-h,0],\mathbb{R}_+^{l\times q})$ such that*

$$|D_k(t)| \le D_k, \quad |E_k(t)| \le E_k, \quad |\Delta_k(t)| \le \Delta_k, \quad \forall t \in \mathbb{R}, \forall k \in \underline{m_0}, \tag{15}$$

and

$$\begin{aligned} &|D(t,s)| \le D_{m+1}, \quad |E(t,s)| \le E_{m+1}, \quad |\delta(t,s)| \le \delta_{m+1}(s), \\ &\forall t \in \mathbb{R}, \forall s \in [-h,0]. \end{aligned} \tag{16}$$

Then the perturbed equation (14) *remains exponentially stable provided*

$$\begin{aligned} &\sum_{k=0}^{m} \|\Delta_k\| + \int_{-h}^{0} \|\delta_{m+1}(s)\| ds \\ &< \frac{1}{\max_{i,j\in\{0,1,\dots,m+1\}} \|E_i(A_0 + \sum_{k=1}^m A_k + \int_{-h}^0 C(s)ds)^{-1} D_j\|}. \end{aligned} \tag{17}$$

Remark 3.6. The problem of robust stability of the linear time-invariant differential system with discrete delays

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t-h_k), \quad t \ge 0, \tag{18}$$

under the time-invariant structured perturbations

$$A_k \rightsquigarrow A_k + D_k \Delta_k E_k, \quad k \in \underline{m_0}, \tag{19}$$

has been addressed in [8, 14, 20, 23]. Stability bounds for (18) subject to the time-invariant structured perturbations (19) can be found in the mentioned papers. However, the problem of robust stability of (18) subject to time-varying structured perturbations is still open and a result like Theorem 3.5 cannot be found in the literature.

Proof of Theorem 3.5. It follows from (15) that

$$|D_k(t)\Delta_k(t)E_k(t)| \le D_k \Delta_k E_k, \quad \forall t \in \mathbb{R}, \forall k \in \underline{m_0}.$$

Furthermore, (10) and (16) imply that

$$|B(t,s) + D(t,s)\delta(t,s)E(t,s)| \le C(s) + D_{m+1}\delta_{m+1}(s)E_{m+1}, \quad \forall t \in \mathbb{R}, \forall s \in [-h,0].$$

Let $D_0\Delta_0E_0 := (m_{ij}^{(0)})$ and let $D_0(t)\Delta_0(t)E_0(t) := (m_{ij}^{(0)}(t))$ and $A_0(t) := (a_{ij}^{(0)}(t))$, $t \in \mathbb{R}$. Thus, $A_0(t)+D_0(t)\Delta_0(t)E_0(t) = (a_{ij}^{(0)}(t)+m_{ij}^{(0)}(t))$. It follows from (5) that

$$a_{ii}^{(0)}(t) + m_{ii}^{(0)}(t) \le a_{ii}^{(0)} + m_{ii}^{(0)}, \quad \forall t \in \mathbb{R}; \forall i \in \underline{n},$$

and

$$|a_{ij}^{(0)}(t) + m_{ij}^{(0)}(t)| \le a_{ij}^{(0)} + m_{ij}^{(0)}, \quad \forall t \in \mathbb{R}; \forall i, j \in \underline{n}, i \neq j.$$

By Corollary 3.3, (14) is exponentially stable if

$$M_* := A_0 + D_0\Delta_0E_0 + \sum_{k=1}^{m}(A_k + D_k\Delta_kE_k) + \int_{-h}^{0}(C(s) + D_{m+1}\delta_{m+1}(s)E_{m+1})ds,$$

is Hurwitz stable.

Assume on the contrary that $\mu_0 := \mu(M_*) \ge 0$. By the Perron-Frobenius theorem (Theorem 2.1 (i)), there exists $x_0 \in \mathbb{R}_+^n, x_0 \neq 0$ such that

$$\left(A_0+D_0\Delta_0E_0+\sum_{k=1}^{m}(A_k+D_k\Delta_kE_k)+\int_{-h}^{0}(C(s)+D_{m+1}\delta_{m+1}(s)E_{m+1})ds\right)x_0=\mu_0x_0.$$

By assumption, $\mu(\sum_{k=0}^{m} A_k+\int_{-h}^{0} C(s)ds) < 0$. Thus $(\mu_0I_n-\sum_{k=0}^{m} A_k-\int_{-h}^{0} C(s)ds)$ is invertible and this implies

$$\begin{aligned}(20)\qquad &\left(\mu_0I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds\right)^{-1}\left(\sum_{k=0}^{m} D_k\Delta_kE_kx_0\right.\\ &\left.+D_{m+1}\int_{-h}^{0}\delta_{m+1}(s)dsE_{m+1}x_0\right) = x_0.\end{aligned}$$

Let i_0 be an index such that $\|E_{i_0}x_0\| = \max_{i\in\{0,1,\ldots,m+1\}}\|E_ix_0\|$. It follows from (20) that $\|E_{i_0}x_0\| > 0$. Multiply both sides of (20) from the left by E_{i_0} to get

$$\begin{aligned}&E_{i_0}\left(\mu_0I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds\right)^{-1}\left(\sum_{k=0}^{m} D_k\Delta_kE_kx_0\right.\\ &\left.+D_{m+1}\int_{-h}^{0}\delta_{m+1}(s)dsE_{m+1}x_0\right) = E_{i_0}x_0.\end{aligned}$$

This gives

$$\sum_{k=0}^{m}\|E_{i_0}(\mu_0I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1}D_k\|\|\Delta_k\|\|E_kx_0\|+$$

$$\|E_{i_0}(\mu_0 I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} D_{m+1}\| \int_{-h}^{0} \|\delta_{m+1}(s)\| ds \|E_{m+1}x_0\| \geq \|E_{i_0}x_0\|.$$

Therefore,

$$\max_{i,j\in\{0,1,\dots,m+1\}} \|E_i(\mu_0 I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} D_j\| (\sum_{k=0}^{m} \|\Delta_k\| + \int_{-h}^{0} \|\delta_{m+1}(s)\| ds) \|E_{i_0}x_0\| \geq \|E_{i_0}x_0\|,$$

or equivalently,

$$\max_{i,j\in\{0,1,\dots,m+1\}} \|E_i(\mu_0 I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} D_j\| (\sum_{k=0}^{m} \|\Delta_k\| + \int_{-h}^{0} \|\delta_{m+1}(s)\| ds) \geq 1. \tag{21}$$

On the other hand, the resolvent identity gives

$$\begin{aligned}&(0I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} - (\mu_0 I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} \\ &= (\mu_0 - 0)(0I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1}(\mu_0 I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1}.\end{aligned}$$

Since $\mu_0 \geq 0$, Theorem 2.1 (iii) implies that

$$(-\sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} \geq (\mu_0 I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} \geq 0.$$

This yields,

$$E_i(-\sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} D_j \geq E_i(\mu_0 I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} D_j \geq 0,$$

for any $i, j \in \underline{(m+1)_0}$. By monotonicity of an operator norm associated with a given pair of monotonic vector norms, we have

$$\|E_i(-\sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} D_j\| \geq \|E_i(\mu_0 I_n - \sum_{k=0}^{m} A_k - \int_{-h}^{0} C(s)ds)^{-1} D_j\|, \tag{22}$$

for any $i, j \in \underline{(m+1)_0}$. Finally, (21) and (22) imply that

$$\sum_{k=0}^{m}\|\Delta_k\| + \int_{-h}^{0}\|\delta_{m+1}(s)\|ds$$
$$\geq \frac{1}{\max_{i,j\in\{0,1,\dots,m+1\}}\|E_i(A_0+\sum_{k=1}^{m}A_k+\int_{-h}^{0}C(s)ds)^{-1}D_j\|}.$$

However, this conflicts with (17). This completes the proof.

Let $A \in \mathbb{R}^{n\times n}$ be given. Consider a linear differential system with time-varying delay of the form

$$\dot{x}(t) = (A + A_0(t))x(t) + \sum_{k=1}^{m} A_k(t)x(t-h_k(t)) + \int_{-h(t)}^{0} B(t,s)x(t+s)ds, \tag{23}$$

where $A_k(\cdot) \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ $(k \in \underline{m_0})$ and $h(\cdot), h_k(\cdot) \in C(\mathbb{R}, \mathbb{R})$ with $0 < h(t), h_k(t) \leq h, \forall t \in \mathbb{R}$ $(k \in \underline{m})$ and $B(\cdot;\cdot) : \mathbb{R} \times [-h, 0] \to \mathbb{R}^{n\times n}$, is a continuous function.

Corollary 3.7. Assume that $M(A)$ is Hurwitz stable. Then (23) is exponentially stable provided there exist $A_k \in \mathbb{R}^{n\times n}_+, k \in \underline{m_0}$ and $C(\cdot) \in C([-h, 0], \mathbb{R}^{n\times n}_+)$ such that

$$|A_k(t)| \leq A_k, \forall t \in \mathbb{R}, k \in \underline{m_0}; \quad |B(t,s)| \leq C(s), \forall t \in \mathbb{R}, \forall s \in [-h, 0],$$

and

$$\sum_{k=0}^{m}\|A_k\| + \int_{-h}^{0}\|C(s)\|ds < \frac{1}{\|M(A)^{-1}\|}.$$

4. Illustrative Examples

We illustrate the obtained results by a couple of examples.

Example 4.1. Consider the linear time-varying differential system with delay

$$\begin{aligned}\dot{x}(t) &= (\ln(1+\frac{10}{9}\sin^2 t) - 2e^{t^2+2})x(t)\\ &\quad +(\frac{9}{4}e^{-2-t^2}\sin t - e^{-t^2}\cos t)x(t+\cos t - 1)\\ &\quad -\frac{7}{2}\int_{\sin t-1}^{0} e^{2s-t^2}\cos(s+t)x(t+s)ds.\end{aligned} \tag{24}$$

Clearly, (24) is of the form (1) with $a_0(t) := \ln(1+\frac{10}{9}\sin^2 t) - 2e^{t^2+2}, t \in \mathbb{R}$ and $a_1(t) := \frac{9}{4}e^{-2-t^2}\sin t - e^{-t^2}\cos t, t \in \mathbb{R}$ and $b(t,s) := -\frac{7}{2}e^{2s-t^2}\cos(s+t), t \in \mathbb{R}, s \in [-2, 0]$.

Let $\beta = 1, p = 1$. It is clear that

$$a_0(t) + |a_1(t)|e^{1-\cos t} + \int_{\sin t-1}^{0}|b(t,s)|e^{-s}ds = \ln(1+\frac{10}{9}\sin^2 t) - 2e^{t^2+2}$$

$$+|\frac{9}{4}e^{-2-t^2}\sin t - e^{-t^2}\cos t|e^{1-\cos t} + \frac{7}{2}\int_{\sin t-1}^{0}|e^{2s-t^2}\cos(s+t)|e^{-s}ds$$
$$\leq \frac{10}{9} - 2e^2 + \frac{9}{4} + e^2 + \frac{7}{2}(1-e^{-2}) < 0, \forall t \in \mathbb{R}.$$

Therefore, (24) is exponentially stable, by Theorem 3.2.

Example 4.2. Consider a linear time-varying differential system with delay in $\mathbb{R}^2$ given by

$$(25)\quad \dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h_1(t)) + \int_{-h(t)}^{0} B(t,s)x(t+s)ds, \quad t \geq \sigma \geq 0,$$

where $h_1(\cdot), h(\cdot) \in C(\mathbb{R},\mathbb{R})$ with $0 < h_1(t), h(t) \leq h, \forall t \in \mathbb{R}$ and

$$A_0(t) := \begin{pmatrix} -6(1+t^2) & e^{-t^2}\sin t \\ \cos t & \frac{1-t^2}{1+t^2} - 7 \end{pmatrix}; \; A_1(t) := \begin{pmatrix} 0 & -\frac{2t}{1+t^2} \\ e^{-\sin^2 t} & 0 \end{pmatrix}, \quad t \in \mathbb{R},$$

and

$$B(t,s) := \begin{pmatrix} 0 & e^{\frac{s}{2}+st^2}\cos st \\ -e^{\frac{s}{2}}\sin st & 0 \end{pmatrix}, \; t \in \mathbb{R}, s \in [-h, 0].$$

Let us define

$$A_0 := \begin{pmatrix} -6 & 1 \\ 1 & -6 \end{pmatrix}; \quad A_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad C(s) := \begin{pmatrix} 0 & e^{\frac{s}{2}} \\ e^{\frac{s}{2}} & 0 \end{pmatrix}, \quad s \in [-h, 0].$$

Note that $A_0(t)$ and A_0 satisfy (5) and $|A_1(t)| \leq A_1, \forall t \in \mathbb{R}$; $|B(t,s)| \leq C(s), \quad \forall t \in \mathbb{R}, \forall s \in [-h, 0]$. It is easy to check that $M := A_0 + A_1 + \int_{-h}^{0} C(s)ds = \begin{pmatrix} -6 & 2(2-e^{-\frac{h}{2}}) \\ 2(2-e^{-\frac{h}{2}}) & -6 \end{pmatrix}$ and $\mu(M) < 0$. Thus (25) is exponentially stable, by Corollary 3.3.

Consider the perturbed equation

$$(26)\quad \begin{aligned} \dot{x}(t) &= (A_0(t) + D_0(t)\Delta_0(t)E_0(t))x(t) + A_1(t)x(t-h_1(t)) \\ &+ \int_{-h(t)}^{0}(B(t,s) + D(t,s)\delta(t,s)E(t,s))x(t+s)ds, \quad t \geq \sigma \geq 0, \end{aligned}$$

where

$$D_0(t) := \begin{pmatrix} \sin t \\ 0 \end{pmatrix}; E_0(t) := \begin{pmatrix} -\ln(1+\cos^2 t) & 0 \\ 0 & \frac{(1-t)^2}{2(1+t^2)} \end{pmatrix}, \quad t \in \mathbb{R},$$

and

$$D(t,s) := \begin{pmatrix} 0 \\ \cos(s-t) \end{pmatrix}; \quad E(t,s) := \begin{pmatrix} -e^{st^2} & 0 \\ 0 & -\frac{2st}{1+(st)^2} \end{pmatrix}, \quad t \in \mathbb{R}, s \in [-h,0],$$

and

$$\Delta_0(t) := \begin{pmatrix} a\cos t & be^{-t^2} \end{pmatrix}, \ \delta(t,s) := \begin{pmatrix} ce^{st^2+s} & -de^s(1+\sin st) \end{pmatrix}, \ t \in \mathbb{R}, s \in [-h,0],$$

with $a, b, c, d \geq 0$ are parameters.

Note that for any $t \in \mathbb{R},\ s \in [-h,0]$, we have

$$|D_0(t)| \leq D_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |D(t,s)| \leq D := \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad |E_0(t)| \leq E_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$|E(t,s)| \leq E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad |\Delta_0(t)| \leq \Delta_0 := \begin{pmatrix} a & b \end{pmatrix};$$

$$|\delta(t,s)| \leq \delta(s) := \begin{pmatrix} ce^s & 2de^s \end{pmatrix},$$

and

$$E_0 M^{-1} D_0 = E M^{-1} D_0 = \begin{pmatrix} -\frac{3}{18-2(2-e^{-\frac{h}{2}})^2} \\ -\frac{2-e^{-\frac{h}{2}}}{18-2(2-e^{-\frac{h}{2}})^2} \end{pmatrix};$$

$$E_0 M^{-1} D = E M^{-1} D = \begin{pmatrix} -\frac{2-e^{-\frac{h}{2}}}{18-2(2-e^{-\frac{h}{2}})^2} \\ -\frac{3}{18-2(2-e^{-\frac{h}{2}})^2} \end{pmatrix}.$$

Let $\mathbb{R}^2$ be endowed with 1-norm. By Theorem 3.5, (26) is exponentially stable provided

$$\max\{a,b\} + (1-e^{-h})\max\{c, 2d\} < 2(1+e^{-\frac{h}{2}}).$$

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