TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 6, pp. 1897-1908, December 2014
DOI: 10.11650/tjm.18.2014.4120
This paper is available online at http://journal.taiwanmathsoc.org.tw

CONVEXITY AND GLOBAL WELL-POSEDNESS IN SET-OPTIMIZATION

Giovanni P. Crespi, Daishi Kuroiwa and Matteo Rocca

Abstract. Well-posedness for vector optimization problems has been extensively studied. More recently, some attempts to extend thee results to set-valued optimization have been proposed, mainly applying some scalarization. In this paper we propose a new definition of global well-posedness for set-optimization problems.

Using an embedding technique proposed by Kuroiwa and Nuriya (2006), we prove well-posedness property of a class of generalized convex set-valued maps.

1. INTRODUCTION

The notion of well-posedness has been deeply studied in scalar and vector optimization. Especially, for vector optimization, two main classes of definitions have been identified in [18]. Usually a notion of well-posed vector optimization problem is said to be pointwise if it involves a single value in the solution set. Instead global notions consider the solution set as a whole.
In [19] the notion of well-posedness and sensitivity analysis have been studied in the framework of Asplund spaces. By means of coderivatives of set-valued maps, necessary and sufficient conditions for well-posedness properties for set-valued functions are proved. More recently, the notion of well-posedness has been proposed also for optimization problems with set-valued objective map. In [9] a pointwise notion has been proposed, while [21] introduces some global notion. Both papers focus on the so called set-optimization approach as introduced by Kuroiwa and Nuriya in [13], that involves ordering relations among sets.

In this paper we introduce a well-posedness notion which slightly generalizes the one in [21] and we investigate well-posedness properties of convex and generalized convex set-valued maps. We define a new class of quasiconvex set-valued maps that

Received November 29, 2013, accepted March 21, 2014.
Communicated by Boris Mordukhovich.
2010 *Mathematics Subject Classification*: 90C29, 90C48, 49J24, 90C31.
Key words and phrases: Set-optimization, Global well-posedness, Generalized convexity.

guarantees well-posedness of the set-optimization problem. The class is broader than the one of convex functions and strictly included in that of quasiconvex maps proposed by Kuroiwa in [10] and studied also in [3]. The proofs are based on the embedding approach introduced by Kuroiwa in [13], that allows to study set-optimization problems thorough a suitable vector optimization problem. Therefore this paper generalizes to set-valued maps the well-posedness properties of generalized convex vector functions proved in [4, 5, 6].

The paper is organized as follows. Section 2 presents the basic notations and some results on the embedding technique. Section 3 introduces the class of quasiconvex set-valued maps that is studied in Section 4 in connection with the global well-posedness of the set-optimization problem. Finally Section 5 is devoted to concluding remarks.

2. Setting

Let $K \subseteq \mathbb{R}^m$ be a pointed closed convex cone with nonempty interior. The usual order relation in $\mathbb{R}^m$ requires that

$$x \leq_K y \quad \text{if} \quad y - x \in K$$
$$x <_K y \quad \text{if} \quad y - x \in \operatorname{int} K.$$

To deal with set-optimization problems, six types of order relations among sets have been introduced in [11, 12]. Among them, in this paper, we focus only on the following notion. Let $A, B \in 2^{\mathbb{R}^m}$ be compact and convex sets. Then

$$A \leq^l_K B \quad \text{if} \quad A + K \supseteq B.$$

and

$$A <^l_K B \quad \text{if } \exists r > 0 \text{ such that} \quad A + K \supseteq B + r\,\mathcal{B}_m,$$

where $\mathcal{B}_m := \{y \in \mathbb{R}^m \mid \|y\| \leq 1\}$ is the unit ball in $\mathbb{R}^m$. If no confusion occurs, the subscript m is omitted and $\mathcal{B}$ denotes the unit ball in the appropriate space. One can easily note that $A <^l_K B$ if and only if $B \subseteq A + \operatorname{int} K$ (see e.g. [15]). We denote by K^+ the positive polar cone of K, that is the set

$$K^+ := \{l \in \mathbb{R}^m \mid \langle l, k\rangle \geq 0, \ \forall k \in K\}$$

Let $X \subseteq \mathbb{R}^n$ be a closed convex set. In the sequel we deal with the set-optimization problem

$$(\text{P}(F, K)) \qquad \min_{x \in X} F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^m}$$

where F is a set-valued function with $F(x)$ nonempty, compact and convex for all $x \in X$. According to the order given by K we can have different solution concepts (see e.g. [11, 12]).

Definition 1. A vector $x^0 \in X$ is a weak minimizer of P(F, K) when

(1) $$F(x) \not<_K^l F(x^0), \ \ \forall x \in X.$$

The set of all weak minimizers of P(F, K) is denoted by $\mathrm{WEff}\,(F, X)$.
A vector $x^0 \in X$ is a minimizer of P(F, K) when

(2) $$F(x) \leq_K^l F\left(x^0\right) \Longrightarrow F\left(x^0\right) \leq_K^l F(x), \quad \forall x \in X$$

For the vector-valued optimization problem

(VP(f, K)) $$\min_{x \in X} f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

Letting $F(x) = \{f(x)\}$ for all $x \in X$, Definition 1 reduces to the classical notions of weak efficient solution and efficient solution. Therefore, if no confusion occurs on the valuedness of the objective function, we may refer to $\mathrm{WEff}\,(f, X)$ as the set of all weak efficient solutions of Problem VP(f, K). Throughout the paper, lower case letters are devoted to vector-valued functions and capital letters to set-valued ones. Moreover we assume that $\mathrm{WEff}\,(F, X)$ is nonempty.

Let χ be the family of all compact convex subsets in $\mathbb{R}^m$. Following the approach in [20, 13] any two couples $(A, B); (C, D) \in \chi^2$ are equivalent if $A+D+K = B+C+K$. When this occurs we write $(A, B) \equiv (C, D)$. The equivalence family of the couple (A, B) is defined by the set

$$[A, B] := \left\{(C, D) \in \chi^2 \mid (A, B) \equiv (C, D)\right\}.$$

In [13] it has been introduced the vector space $\left(\chi^2 / \equiv, +, \cdot\right)$, where

- $[A, B] + [C, D] = [A + C, B + D]$;
- $\lambda \cdot [A, B] = \begin{cases} [\lambda A, \lambda B], & \lambda \geq 0 \\ [-\lambda A, -\lambda B], & \lambda < 0 \end{cases}$.

Given a compact base W of K, the embedding space $\left(\chi^2 / \equiv, +, \cdot\right)$ is normed (see e.g. [13, 14]), introducing

(3) $$\|\, [A, B] \,\| := \sup_{w \in W} \left|\inf\langle w, A\rangle - \inf\langle w, B\rangle\right|.$$

A partial order in $\chi^2 / \equiv$ can be introduced through the pointed, closed and convex cone

$$\mu(K) := \left\{[A, B] \in \chi^2 / \equiv \ \mid B \leq_K^l A\right\}$$

depending on the ordering cone K on $\mathbb{R}^m$. The interior of $\mu(K)$ is defined as

$$\mathrm{int}\,\mu(K) := \left\{[A, B] \in \chi^2 / \equiv \ \mid B <_K^l A\right\}.$$

Therefore we can define order relations in the vector space $\chi^2/\equiv$ by

$$[A, B] \leq_{\mu(K)} [C, D] \text{ if } [C, D] - [A, B] \in \mu(K)$$

and

$$[A, B] <_{\mu(K)} [C, D] \text{ if } [C, D] - [A, B] \in \operatorname{int} \mu(K)$$

Problem P(F, K) can be embedded into the vector optimization problem on $\left(\chi^2/\equiv, +, \cdot\right)$

$$(\text{VP}(f, \mu(K))) \qquad \min_{x \in X} f : X \to \chi^2/\equiv$$

where, for all $x \in X$,

$$f(x) = (\varphi \circ F)(x) = [F(x), \{0\}]\,. \tag{4}$$

In [11, 15] we find the following result.

Theorem 1. *Let $F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^m}$ and let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be defined by (4). Then*

- (i) *$x^0 \in X$ is a minimizer of P(F, K) if and only if it is an efficient solution of VP$(f, \mu(K))$;*
- (ii) *$x^0 \in X$ is a weak minimizer of P(F, K) if and only if it is a weak efficient solution of VP$(f, \mu(K))$.*

Moreover we have also that continuity and convexity are preserved by embedding. We recall that $F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is K-convex when, for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ we have

$$\lambda F(x_1) + (1 - t) F(x_2) \subseteq F(\lambda x_1 + (1 - t) x_2) + K \tag{5}$$

In [14] it has been remarked that clearly (5) holds if and only if

$$\varphi \circ F(tx_1 + (1 - t) x_2) \leq_{\mu(K)} t(\varphi \circ F(x_1)) + (1 - t)(\varphi \circ F(x_2))$$

that is if and only if the vector valued function $f(x) = [F(x), \{0\}]$ is $\mu(K)$-convex.

Continuity for set-valued functions has been defined e.g. in [1]. A set-valued function F is upper semicontinuous (usc) at $x^0 \in X$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$F(x) \subseteq F\left(x^0\right) + \varepsilon \mathcal{B}, \quad \forall x \in \left(x^0 + \delta \mathcal{B}\right) \cap X$$

Analogously we can define lower semicontinuity (lsc) at $x^0 \in X$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$F\left(x^0\right) \subseteq F(x) + \varepsilon \mathcal{B}, \quad \forall x \in \left(x^0 + \delta \mathcal{B}\right) \cap X$$

A set-valued function is Hausdorff continuous at $x^0 \in X$ if and only if it is both upper and lower semicontinuous.

Proposition 1. *The set-valued function* $F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^m}$ *is such that* $F + K$ *is Hausdorff continuous if and only if the vector valued embedded function* $f = (\varphi \circ F) : X \to \chi^2 / \equiv$ *is continuous.*

Proof. Assume $f = \varphi \circ F$ is continuous at $x^0 \in X$. Given a compact base W of K^+, continuity of f at x^0 means that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\left\| \left[F(x), F\left(x^0\right) \right] \right\| < \varepsilon$, $\forall x \in \left(x^0 + \delta \mathcal{B}\right) \cap X$. This is equivalent to

$$(6) \quad -\varepsilon < \inf \langle w, F(x) \rangle - \inf \langle w, F\left(x^0\right) \rangle < \varepsilon, \quad \forall w \in W, \quad \forall x \in \left(x^0 + \delta \mathcal{B}\right) \cap X.$$

Let $p \in \operatorname{int} K$ and consider the compact base $W = \{ l \in K^+ : \langle l, p \rangle = 1 \}$. We first prove lower semicontinuity of $F + K$. Indeed (6) implies

$$\begin{aligned} &\inf \langle w, F(x) \rangle < \inf \langle w, F\left(x^0\right) \rangle + \varepsilon \\ = &\inf \langle w, F\left(x^0\right) + \varepsilon p \rangle, \forall w \in W, \forall x \in \left(x^0 + \delta \mathcal{B}\right) \cap X \end{aligned}$$

Hence, see e.g. [13]

$$\begin{aligned} & F\left(x^0\right) + \varepsilon p \subseteq F(x) + K \\ \text{i.e.} \quad & F\left(x^0\right) \subseteq F(x) + K - \varepsilon p \subseteq F(x) + K + \varepsilon \mathcal{B} \end{aligned}$$

Therefore $F + K$ is lsc at x^0. By a similar argument we prove upper semicontinuity and then Hausdorff continuity of $F + K$. To prove the reverse implication, let a norm in $\mathbb{R}^m$ be defined as the Minkowski functional of the set $\tilde{\mathcal{B}} = \text{conv}\left((-W) \cup W\right)$ (every norm in $\mathbb{R}^m$ is topologically equivalent). Hence $\forall \varepsilon > 0$, the set $\varepsilon \tilde{\mathcal{B}}$ is a neighborhood of 0. Hausdorff continuity of $F + K$ implies that $F\left(x^0\right) + K \subseteq F(x) + K + \varepsilon \tilde{\mathcal{B}}$. Then $\forall w \in W$, we have

$$\begin{aligned} \inf \langle w, F\left(x^0\right) + K \rangle &\geq \inf \langle w, F(x) + K + \varepsilon \tilde{\mathcal{B}} \rangle \\ &\geq \inf \langle w, F(x) + K \rangle + \inf \langle w, \varepsilon \tilde{\mathcal{B}} \rangle \\ &\geq \inf \langle w, F(x) + K \rangle - \varepsilon. \end{aligned}$$

Therefore $\inf \langle w, F(x) + K \rangle - \inf \langle w, F\left(x^0\right) + K \rangle \leq \varepsilon$. Analogously we can prove the inequality $\inf \langle w, F\left(x^0\right) + K \rangle - \inf \langle w, F(x) + K \rangle \geq -\varepsilon$ and therefore the continuity of f at x^0. ■

In general, continuity of $\varphi \circ F$ does not guarantee Hausdorff continuity of F, as the following example shows.

Example 1. Let $K := [0, +\infty) \subseteq \mathbb{R}$ and $F : \mathbb{R} \to 2^{\mathbb{R}}$ be defined as

$$F(x) = \begin{cases} \left[-x^2, x^2\right] & \text{if } x \in (0, 1] \\ [0, 1] & \text{if } x = 0 \end{cases}$$

Then $\|F(x_n) - F(0)\| \to 0$ for all $x_n \downarrow 0$. Indeed, it is enough to take $W = \{1\}$ as a basis for $\mathbb{R}_+$ and for all $x_n \downarrow 0$ $\inf w\, F(x_n) = \inf F(x_n) = -(x_n)^2$, where $w = 1$. Moreover $\inf w\, F(0) = 0$ and hence $\|[F(x_n), F(0)]\| \to 0$, as $n \to +\infty$. However function F is not Hausdorff continuous at zero, while $F + K$ is Hausdorff continuous.

3. Quasiconvexity

Both for scalar and vector optimization, well-posedness is strictly related to convexity and quasi-convexity. Some notion of quasiconvexity for set-valued functions F have been provided in the literature. The following we quote from [3].

Definition 2. A set-valued function $F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is said K-quasiconvex on X when the level sets $F^{-1}(y - K) := \{x \in X \,:\, y \in F(x) + K\}$ are convex for all $y \in \mathbb{R}^m$.

Quasiconvexity of a vector valued function $f : X \to \mathbb{R}^m$ is a special case of Definition 2. In this paper we propose the following alternative definition.

Definition 3. A set-valued function $F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is said K-quasiconvex in the embedding sense when the vector-valued function $f : X \to \chi^2/\equiv$ defined by (4) is $\mu(K)$-quasiconvex.

Definition 3 can be stated also through convexity of appropriate level sets. According to the embedding technique used, a level set can be defined as the set

$$\{x \in X \,:\, F(x) + B + K \supseteq A\} = \left\{x \in X \,:\, F(x) + B \leq^l_K A\right\} \tag{7}$$

for some $A, B \subseteq \mathbb{R}^m$ compact and convex.

Proposition 2. *A set-valued function $F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is K-quasiconvex in the embedding sense if and only if $\forall A, B \subseteq \mathbb{R}^m$, with A, B compact and convex subsets, the level set (7) is convex.*

Proof. Level sets as in (7) are equivalent to the following:

$$\left\{x \in X \,:\, [F(x), \{0\}] \leq_{\mu(K)} [A, B]\right\} = \left\{x \in X : f(x) \leq_{\mu(K)} [A, B]\right\} \tag{8}$$

where $f(x)$ is defined by (4). For any compact convex sets $A, B \in 2^{\mathbb{R}^m}$, $[A, B] \in \chi^2/\equiv$ is an element of the image space of f. In [17], a vector-valued function is said to be quasiconvex when its level sets are convex. Since (8) are level sets for the vector-valued function $\varphi \circ F$, with respect to the ordering cone $\mu(K)$, the proof follows straightforward. ∎

Remark 1. Since Definition 3 provide a characterization of quasiconvexity for set-valued maps through quasiconvexity of vector-valued ones, it is easy to see that any K-convex set-valued map is also K-quasiconvex in the embedding sense. Indeed K-convexity implies $\mu(K)$-convexity of f, defined in (4), which, in turns, implies $\mu(K)$-quasiconvexity of f.

The next example shows that this new class of generalized convex set-valued functions is not empty and broader than the class of K-convex set-valued functions.

Example 2. Let $X = [0, 1]$, $K = \mathbb{R}_+$ and $F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}}$ be defined as $F(x) = \left[-x^2; x^2\right]$. The function is not K-convex. Indeed let $x_1 = 0$ and $x_2 = 1$. Therefore since $F(0) = \{0\}$ and $F(1) = [-1; 1]$, we have

$$\frac{1}{2}F(0) + \frac{1}{2}F(1) = \left[-\frac{1}{2}; \frac{1}{2}\right]$$

but $F\left(\frac{1}{2}\right) = \left[-\frac{1}{4}; \frac{1}{4}\right]$, proving that $\frac{1}{2}F(0) + \frac{1}{2}F(1) \not\subseteq F\left(\frac{1}{2}\right) + K$.
However, F is K-quasiconvex in the embedding sense. Indeed $\forall \overline{x},\ \hat{x} \in X$ with $\overline{x} < \hat{x}$ it holds that $F(\overline{x}) + K \subseteq F(\hat{x}) + K$. Therefore, $\forall A, B \subseteq \mathbb{R}$, compact and convex such that $A \subseteq F(x_i) + B + K$, $i = 1, 2$, $x_1 < x_2$, it holds that $F(x_1) + K \subseteq F(\lambda x_1 + (1 - \lambda) x_2) + K$, for all $\lambda \in [0, 1]$, since $x_1 \leq \lambda x_1 + (1 - \lambda) x_2$,. Hence

$$A \subseteq F(x_1) + K + B \subseteq F(\lambda x_1 + (1 - \lambda) x_2) + K + B \quad \forall \lambda \in [0, 1]$$

proving the convexity of sets (7). Finally the same function also fulfills Definition 2. Indeed, this is easily seen since $\forall x_1 < x_2 \in X$ it holds that $F(x_1) + K \subseteq F(x_2) + K$.

Proposition 3. *If $F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is K-quasiconvex in the embedding sense, then F is K-quasiconvex.*

Proof. The implication easily follows assuming $A = \{y\}$ and $B = \{0\}$ in the definition of K-quasiconvexity in the embedding sense. ∎

However the converse is not necessarily true, even assuming continuity, as the next example shows.

Example 3. Let $K = \mathbb{R}^2_+$ and $F : [0, 1] \subseteq \mathbb{R} \to 2^{\mathbb{R}^2}$ be defined as follows.

$$F(x) = \begin{cases} \{(x - 1/2,\ 1/2)\} & \text{if } x \in [0, 1/2] \\ \{(0,\ 1 - x)\} & \text{if } x \in (1/2, 1] \end{cases}.$$

Function F is Hausdorff continuous and K-quasiconvex, but not K-quasiconvex in the embedding sense. Indeed, if B is the segment joining points $(1/2,\ -1/2)$ and $(-1/2,\ 1/2)$, and $A = \{(-1/4,\ 3/4)\}$, we have $A \subseteq F(0) + B + K$ and $A \subseteq F(1) + B + K$. However $A \not\subseteq F(1/2) + B + K$.

4. Global Well-posedness and Convexity

We recall the notion of global well-posedness for vector optimization, starting from the definition of minimizing sequence.

Definition 4. Let $p \in \operatorname{int} K$ be given. A minimizing sequence for (w.r.t. p) $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a sequence $\{x^n\} \subseteq X$ for which $\exists \varepsilon_n \downarrow 0$ s.t. $(f(X) - f(x^n)) \cap (-\operatorname{int} K - \varepsilon_n p) = \emptyset$.

Definition 4 does not depend on the choice of $p \in \operatorname{int} K$. To show this we first need the following Lemma we quote from [4] without proof.

Lemma 1. *Let p^1 and p^2 be vectors in $\operatorname{int} K$. Then, there exist positive constants α and β, such that for every $\varepsilon > 0$ it holds:*

$$-K - \alpha\varepsilon p^2 \subseteq -K - \varepsilon p^1 \subseteq -K - \beta\varepsilon p^2 .$$

Lemma 2. *Let p^1, $p^2 \in \operatorname{int} K$. A sequence $\{x^n\} \subseteq X$ is minimizing for problem $\mathrm{P}(F, K)$ w.r.t. p^1 if and only if it is a minimizing sequence w.r.t. p^2.*

Proof. By assumption there exists $\varepsilon_n \downarrow 0$ s.t.

$$(f(X) - f(x^n)) \cap \left(-\operatorname{int} K - \varepsilon_n p^1\right) = \emptyset.$$

Hence, by Lemma 1 there exists $\alpha > 0$ s.t.

$$(f(X) - f(x^n)) \cap \left(-\operatorname{int} K - \alpha\varepsilon_n p^2\right) = \emptyset$$

proving the sequence is minimizing also w.r.t. p^2. ∎

According to this notion, in [5] the following definition of global well-posedness has been introduced.

Definition 5. Problem $\mathrm{VP}(f, K)$ is (globally) well-posed when for every minimizing sequence there exists a subsequence $\{x^{n_k}\}$ such that $\operatorname{dist}(x^{n_k}, \mathrm{WEff}(f, X)) \to 0$.

Motivated by the previous notions, we can introduce the following definition of (globally) minimizing sequence for problem $\mathrm{P}(F, K)$.

Definition 6. Let $p \in \operatorname{int} K$. A sequence $\{x^n\} \in X$ is a *minimizing sequence* for $\mathrm{P}(F, K)$ when $\exists \varepsilon_n \downarrow 0$ s.t.

$$F(x^n) \not\subseteq F(x) + \operatorname{int} K + \varepsilon_n p, \quad \forall x \in X \tag{9}$$

i.e.

$$F(x^n) - \varepsilon_n p \not<^l_K F(x), \quad \forall x \in X$$

Therefore the following definition of global well-posedness is straightforward.

Definition 7. Problem $\mathrm{P}(F,K)$ is (globally) well-posed when every minimizing sequence $\{x^n\}$ admits a subsequence $\{x^{n_k}\}$ s.t. $\operatorname{dist}(x^{n_k}, \mathrm{WEff}\,(F,X)) \to 0$.

In [21] a first approach to extend global well-posedness to set-optimization has been proposed. Definition 7 is slightly more general since it does not require that minimizing sequences converge to some specific weak efficient solution but just that the distance between the minimizing sequence and the set $\mathrm{WEff}\,(F,K)$ converges to 0.

Proposition 4. *Problem P(F,K) is globally well-posed if and only if problem VP$(f,\mu(K))$ with objective function $f : X \subseteq \mathbb{R}^n \to \chi^2/\equiv$ defined by (4), is well-posed according to Definition 5.*

Proof. In view of Theorem 1, it is enough to prove that $\{x^n\}$ is a minimizing sequence for $\mathrm{P}(F,K)$ if and only if it is so for $\mathrm{VP}(f,\mu(K))$.
Indeed from (9) we obtain

$$F(x^n) - \varepsilon_n p \not\subseteq F(x) + \operatorname{int} K, \quad \forall x \in X.$$

Hence

$$[F(x),\{0\}] \not<_{\mu(K)} [F(x^n) - \varepsilon_n p, \{0\}], \quad \forall x \in X$$

or, equivalently, $\forall x \in X$

$$\begin{aligned}
[F(x),\{0\}] \quad &\not<_{\mu(K)} \quad [F(x^n),\{0\}] + [\{-\varepsilon_n p\},\{0\}],\ i.e.\\
[F(x),\{0\}] \quad &\notin \quad [F(x^n),\{0\}] - \operatorname{int}\mu(K) - \varepsilon_n [\{p\},\{0\}].
\end{aligned}$$

The proof is complete, observing that $[\{p\},\{0\}] \in \operatorname{int}\mu(K)$.
Conversely, assume that $\{x^n\}$ is a minimizing sequence for $\mathrm{VP}(f,\mu(K))$. Then, for some $[P,Q] \in \operatorname{int}\mu(K)$ we have

$$(\varphi \circ F)(x) \not<_{\mu(K)} (\varphi \circ F)(x^n) - \varepsilon_n\,[P,Q], \quad \forall x \in X$$

By Lemma 2, we can choose $[P,Q] = [\{p\},\{0\}] \in \operatorname{int}\mu(K)$, with $p \in \operatorname{int} K$. Hence, for all $x \in X$ we have

$$[F(x),\{0\}] \notin [F(x^n),\{0\}] - \operatorname{int}\mu(K) - \varepsilon_n [\{p\},\{0\}]$$

from which the conclusion easily follows. ∎

The embedding technique allows us to prove Theorem 2 below, that relates well-posedness to quasiconvexity. In order to prove it, we need the following result from [5].

Theorem 1. *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a continuous and K-quasiconvex function. Moreover assume that, for every $y \in \mathbb{R}^m$, the level set*

$$f^{-1}(y - K) := \{x \in X \mid y \in f(x) + K\}$$

is bounded and $\mathrm{WEff}(f, X)$ is bounded. Then problem VP(f, K) is globally well-posed.

Theorem 2. *Let $F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be K-quasiconvex in the embedding sense and such that $F + K$ is Hausdorff continuous. Moreover, assume that, for every $A, B \subseteq \mathbb{R}^m$ compact and convex, the set $\{x \in X \mid A \subseteq F(x) + B + K\}$ is bounded and $\mathrm{WEff}(F, X)$ is nonempty and bounded. Then problem P(F, K) is globally well-posed.*

Proof. Embedding problem P(F, K) in the vector valued problem VP$(f, \mu(K))$ we obtain from (4) the objective function $f(x) = [F(x), \{0\}]$. According to Propositions 1 and 2, f is continuous and $\mu(K)$-quasiconvex. Moreover, level sets of f, w.r.t. $\mu(K)$, that is the sets $\{x \in X : y \in f(x) + \mu(K)\}$, coincide with sets $\{x \in X \mid A \subseteq F(x) + B + K\}$, for all $y = [A, B] \in \chi^2/\equiv$ and, therefore are bounded. Finally, Theorem 1 guarantees that $\mathrm{WEff}(f, X) = \mathrm{WEff}(F, X)$ and hence it is bounded. We can finally apply Theorem 1 to problem VP$(f, \mu(K))$ to get the thesis. ■

In [6] it has been proved that Theorem 1 holds with weaker assumptions for K-convex vector-valued functions.

Theorem 3. *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a K-convex function and assume $\mathrm{WEff}(f, X)$ is nonempty and bounded. Then problem VP$(f, \mu(K))$ is well-posed.*

Corollary 1. *Let $F : X \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be K-convex. Moreover assume that $\mathrm{WEff}(F, X)$ is nonempty and bounded. Then problem P(F, K) is globally well-posed.*

Proof. Function f defined by (4) is $\mu(K)$-convex, according to Proposition 1. The set $\mathrm{WEff}(\varphi \circ F, X) = \mathrm{WEff}(F, X)$ is bounded. Hence we can apply Theorem 3 to problem VP$(f, \mu(K))$ to prove the thesis. ■

5. Concluding Remarks

In this paper we introduced a notion of global well-posedness for set-optimization that most intuitively extend the notion introduced in [4] for vector optimization. The definition is motivated by the embedding technique popularized by [13], that allows to study problem P(F, K) through an equivalent vector-valued problem in the embedding space.

We proved that a certain class of generalized convex set-valued functions implies well-posedness of the set-optimization problem. This class, broader than the class of K-convex functions, is not equivalent to the class of K-quasiconvex set-valued function studied in [3]. We leave as an open question whether Theorem 2 can be extended to this larger class.

Acknowledgment

The authors are thankful to the anonymous referee, whose comments helped to improve the results in the paper.

References

1. J. P. Aubin and H. Frankowska, *Set-valued Analysis*, Birkhüser, Boston, MA, USA, 1990.
2. M. S. Bazaraa and C. M. Shetty, *Foundations of Optimization*, Springer, Berlin, 1976.
3. J. Benoist and N. Popovici, Characterization of convex and quasiconvex set-valued maps, *Math. Meth. Oper. Res.*, **57** (2003), 427-435.
4. G. P. Crespi, A. Guerraggio and M. Rocca, Well-posedness in vector optimization problems and variational inequalities, *J. Optim. Theory Appl.*, **132(1)** (2007), 213-226.
5. G. P. Crespi, M. Papalia and M. Rocca, Extended Well-Posedness of Quasiconvex Vector Optimization Problems, *J. Optim. Theory Appl.*, **141** (2009), 285-297.
6. G. P. Crespi, M. Papalia and M. Rocca, Extended well-posedness of vector optimization problems: the convex case, *Taiwanese J. Math.*, **15(4)** (2011), 1545-1559.
7. A. L. Dontchev and T. Zolezzi, *Well-posed Optimization Problems*, Springer, Berlin, 1993.
8. M. Durea, Optimality condition for weak and firm efficiency in set-valued optimization, *J. Math. Anal. Appl.*, **344** (2008), 1018-1028.
9. G. Gutiérrez, E. Miglierina, E. Molho and V. Novo, Pointwise well-posedness in set optimization with cone proper sets, *Nonlinear Anal.*, **75** (2012), 1822-1833.
10. D. Kuroiwa, Convexity for set-valued maps, *Appl. Math. Letters*, **9(2)** (1996), 97-101.
11. D. Kuroiwa, On set-valued optimization, *Nonlinear Anal.*, **47** (2001), 1395-1400.
12. D. Kuroiwa, Existence theorems of set optimization with set-valued maps, *J. Inf. and Optim. Sci.*, **1** (2003), 73-84
13. D. Kuroiwa and T. Nuriya, A generalized embedding vector space in set optimization, in: *Proceedings of the Forth International Conference on Nonlinear and Convex Analysis*, 2006, pp. 297-304.
14. D. Kuroiwa, On derivatives and convexity of set-valued maps and optimality conditions in set optimization, *J. Nonlinear Convex Anal.*, **10** (2009), 41-50.

15. D. Kuroiwa, Set optimization with DC objective function, in *Nonlinear and Convex Analysis*, S. Akashi, D. S. Kim, T. H. Kim, G. M. Lee, W. Takahashi, T. Tanaka eds, 2011, pp. 291-297.

16. P. Loridan, Well-posedness in vector optimization, *Math. Appl.*, **331** (1995), 171-192.

17. D. T. Luc, *Theory of Vector Optimization*, Springer, Berlin, 1989.

18. E. Miglierina, E. Molho and M. Rocca, Well-posedness and scalarization in vector optimization, *J. Optim. Theory Appl.*, **126** (2005), 391-409.

19. Mordukhovich and S. Boris, *Variational Analysis and Generalized Differentiation.* I, Grundlehren der Mathematischen Wissenschaften, [Fundamental Principles of Mathematical Sciences], 330, Springer-Verlag, Berlin, 2006.

20. H. Radström, An embending theorem for spaces of convex sets, *Proceedings of the American Mathematical Society*, **3(1)** (1952), 165-169.

21. W. Y. Zhang, S. J. Li and K. L. Teo, Well-posedness for set optimization problems, *Nonlinear Anal.*, **71** (2009), 3769-3778.

Giovanni P. Crespi
University of Valle d'Aosta
Department of Economic and Political Sciences
University of Valle d'Aosta
Loc. Grand Chemin 73/75
11020 Saint-Christophe (AO)
Italy
E-mail: g.crespi@univda.it

Daishi Kuroiwa
Department of Mathematics and Computer Sciences
Shimane University
Matsue 690-8504
Japan
E-mail: kuroiwa@math.shimane-u.ac.jp

Matteo Rocca
University of Insubria
Department of Economics
via Ravasi 2, 21100 Varese
Italia
E-mail: mrocca@eco.uninsubria.it