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H-SEMI-SLANT SUBMERSIONS FROM ALMOST QUATERNIONIC HERMITIAN MANIFOLDS

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Abstract. As a generalization of semi-slant submersions, h-slant submersions, and h-semi-invariant submersions, we introduce the notions of h-semi-slant submersions and almost h-semi-slant submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds. We obtain characterizations and investigate the integrability of distributions, the geometry of fibers, and the harmonicity of such maps. We also find a condition for such maps to be totally geodesic. Moreover, we give some examples of such maps.

1. INTRODUCTION

Given a C^∞-submersion F from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N), according to the conditions on the map $F : (M, g_M) \mapsto (N, g_N)$, we obtain the following:

a Riemannian submersion ([10, 15, 9]), an almost Hermitian submersion [23], an invariant submersion [22], an anti-invariant submersion [19], a slant submersion ([7, 20]), a semi-invariant submersion [21], a semi-slant submersion [18], a quaternionic submersion [11], a h-slant submersion and an almost h-slant submersion [16], a h-semi-invariant submersion and an almost h-semi-invariant submersion [17], etc.

As we know, Riemannian submersions were independently introduced by B. O'Neill [15] and A. Gray [10] in 1960s. In particular, by using the notion of almost Hermitian submersions, B. Watson [23] gave some differential geometric properties among fibers, base manifolds, and total manifolds. After that, there are lots of results on this topic.

It is well-known that Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([5, 24]), Kaluza-Klein theory ([4, 12]), Supergravity and superstring theories ([13, 14]), etc. And the quaternionic Kähler manifolds have applications in physics as the target spaces for nonlinear $\sigma-$models with supersymmetry [8].

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The paper is organized as follows. In section 2 we remind some notions, which are needed in the following sections. In section 3 we give the definitions of h-semi-slant submersions and almost h-semi-slant submersions and obtain some properties on them: the characterizations of such maps, the harmonicity of such maps, the conditions for such maps to be totally geodesic, the integrability of distributions, the geometry of fibers, etc. In section 4 we obtain some examples of h-semi-slant submersions and almost h-semi-slant submersions.

2. Preliminaries

Let (M, g_M) and (N, g_N) be Riemannian manifolds, where g_M and g_N are Riemannian metrics on C^∞-manifolds M and N, respectively.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a C^∞-map.

We call the map F a *C^∞-submersion* if F is surjective and the differential $(F_*)_p$ has maximal rank for any $p \in M$.

Then the map F is said to be a *Riemannian submersion* ([15], [9]) if F is a C^∞-submersion and

$$(F_*)_p : ((\ker(F_*)_p)^\perp, (g_M)_p) \mapsto (T_{F(p)}N, (g_N)_{F(p)})$$

is a linear isometry for any $p \in M$, where $(\ker(F_*)_p)^\perp$ is the orthogonal complement of the space $\ker(F_*)_p$ in the tangent space T_pM to M at p.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a Riemannian submersion.

For any vector field $U \in \Gamma(TM)$, we have

$$U = \mathcal{V}U + \mathcal{H}U,$$

where $\mathcal{V}U \in \Gamma(\ker F_*)$ and $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$.

Define the (O'Neill) tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

for vector fields $E, F \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of g_M ([15], [9]).

Define $\widehat{\nabla}_X Y := \mathcal{V}\nabla_X Y$ for $X, Y \in \Gamma(\ker F_*)$.

Let (M, g_M, J) be an almost Hermitian manifold, where J is an almost complex structure on M.

A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *semi-slant submersion* if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$ [18].

We call the angle θ a *semi-slant angle*.

Let M be a $4m-$dimensional C^∞-manifold and let E be a rank 3 subbundle of End(TM) such that for any point $p \in M$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call E an *almost quaternionic structure* on M and (M, E) an *almost quaternionic manifold* [1].

Moreover, let g be a Riemannian metric on M such that for any point $p \in M$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{1}$$

$$g(J_\alpha X, J_\alpha Y) = g(X, Y) \tag{2}$$

for all vector fields $X, Y \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call (M, E, g) an *almost quaternionic Hermitian manifold* [11].

Conveniently, the above basis $\{J_1, J_2, J_3\}$ satisfying (1) and (2) is said to be a *quaternionic Hermitian basis*.

Let (M, E, g) be an almost quaternionic Hermitian manifold.

We call (M, E, g) a *quaternionic Kähler manifold* if there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1, 2, 3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}$$

for any vector field $X \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3 [11].

If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on M (i.e., $\nabla J_\alpha = 0$ for $\alpha \in \{1, 2, 3\}$, where ∇ is the Levi-Civita connection of the metric g), then (M, E, g) is said to be a *hyperkähler manifold*. Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on M and g a *hyperkähler metric* [2].

Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds.

A map $F : M \mapsto N$ is called a $(E_M, E_N)-$*holomorphic map* if given a point $x \in M$, for any $J \in (E_M)_x$ there exists $J' \in (E_N)_{F(x)}$ such that

$$F_* \circ J = J' \circ F_*.$$

A Riemannian submersion $F : M \mapsto N$ which is a $(E_M, E_N)-$holomorphic map is called a *quaternionic submersion* [11].

Moreover, if (M, E_M, g_M) is a quaternionic Kähler manifold (or a hyperkähler manifold), then we say that F is a *quaternionic Kähler submersion* (or a *hyperkähler submersion*) [11].

Then it is well-known that any quaternionic Kähler submersion is a harmonic map [11].

Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is said to be an *almost h-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$ [16].

We call such a basis $\{I, J, K\}$ an *almost h-slant basis*.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$, and $\theta = \theta_I(X) = \theta_J(X) = \theta_K(X)$ [16].

We call such a basis $\{I, J, K\}$ a *h-slant basis* and the angle θ a *h-slant angle*.

And a Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-semi-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1 \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ R(\mathcal{D}_1) = \mathcal{D}_1,\ R(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$ [17].

We call such a basis $\{I, J, K\}$ a *h-semi-invariant basis*.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-semi-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R,\ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,\ R(\mathcal{D}_2^R) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$ [17].

We call such a basis $\{I, J, K\}$ an *almost h-semi-invariant basis*.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F : (M, g_M) \mapsto (N, g_N)$ a C^∞-map.

The *second fundamental form* of F is given by

$$(\nabla F_*)(X,Y) := \nabla^F_X F_* Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where ∇^F is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N [6].

Recall that F is said to be *harmonic* if $trace(\nabla F_*) = 0$ and F is called a *totally geodesic* map if $(\nabla F_*)(X,Y) = 0$ for $X, Y \in \Gamma(TM)$ [6].

Throughout this paper, we will use the above notations.

3. H-Semi-slant Submersions

Definition 3.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-semi-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1 \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ \ R(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in U$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We call such a basis $\{I, J, K\}$ a *h-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *h-semi-slant angles*.

Furthermore, if we have

$$\theta = \theta_I = \theta_J = \theta_K,$$

then we call the map $F : (M, E, g_M) \mapsto (N, g_N)$ a *strictly h-semi-slant submersion*, $\{I, J, K\}$ a *strictly h-semi-slant basis*, and the angle θ a *strictly h-semi-slant angle*.

Definition 3.2. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-semi-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$.

We call such a basis $\{I, J, K\}$ an *almost h-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-semi-slant angles*.

Remark 3.3. Obviously, almost h-semi-invariant submersions and h-semi-invariant submersions are almost h-semi-slant submersions with almost h-semi-slant angles $\theta_I = \theta_J = \theta_K = \frac{\pi}{2}$ and h-semi-slant submersions with h-semi-slant angles $\theta_I = \theta_J = \theta_K = \frac{\pi}{2}$, respectively [17]. As we know, the fibers of h-semi-invariant submersions from hyperkähler manifolds onto Riemannian manifolds are quaternionic CR-submanifolds ([3, 17]).

Remark 3.4. Clearly, almost h-slant submersions are h-semi-slant submersions with $\ker F_* = \mathcal{D}_2$ [16]. Like Remark 2.2 of [18], there are some similarities and differences between almost h-slant submersions and almost h-semi-slant submersions. For the sufficient conditions for such maps to be harmonic, almost h-slant submersions have more nice form than almost h-semi-slant submersions. But almost h-semi-slant submersions contain much more information than almost h-slant submersions. (i.e., the mean curvature vector field of fibers, the geometry of distributions, etc.)

Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an almost h-semi-slant submersion.

Given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \; R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$.

Then for $X \in \Gamma(\ker F_*)$, we have

$$X = P_R X + Q_R X,$$

where $P_R X \in \Gamma(\mathcal{D}_1^R)$ and $Q_R X \in \Gamma(\mathcal{D}_2^R)$.

For $X \in \Gamma(\ker F_*)$, we get

$$RX = \phi_R X + \omega_R X,$$

where $\phi_R X \in \Gamma(\ker F_*)$ and $\omega_R X \in \Gamma((\ker F_*)^\perp)$.

For $Z \in \Gamma((\ker F_*)^\perp)$, we obtain

$$RZ = B_R Z + C_R Z,$$

where $B_R Z \in \Gamma(\ker F_*)$ and $C_R Z \in \Gamma((\ker F_*)^\perp)$.

Then

$$(\ker F_*)^\perp = \omega_R \mathcal{D}_2^R \oplus \mu_R,$$

where μ_R is the orthogonal complement of $\omega_R \mathcal{D}_2^R$ in $(\ker F_*)^\perp$ and is R-invariant. Furthermore,

$$\phi_R \mathcal{D}_1^R = \mathcal{D}_1^R, \; \omega_R \mathcal{D}_1^R = 0, \; \phi_R \mathcal{D}_2^R \subset \mathcal{D}_2^R, \; B_R((\ker F_*)^\perp) = \mathcal{D}_2^R$$

$$\phi_R^2 + B_R \omega_R = -id, \; C_R^2 + \omega_R B_R = -id, \; \omega_R \phi_R + C_R \omega_R = 0, \; B_R C_R + \phi_R B_R = 0.$$

Then it is easy to have

Lemma 3.5. *Let F be an almost h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Then we get*

(1)
$$\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \omega_R Y = \phi_R \widehat{\nabla}_X Y + B_R \mathcal{T}_X Y$$
$$\mathcal{T}_X \phi_R Y + \mathcal{H}\nabla_X \omega_R Y = \omega_R \widehat{\nabla}_X Y + C_R \mathcal{T}_X Y$$
for $X, Y \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$.

(2)
$$\mathcal{V}\nabla_Z B_R W + \mathcal{A}_Z C_R W = \phi_R \mathcal{A}_Z W + B_R \mathcal{H}\nabla_Z W$$
$$\mathcal{A}_Z B_R W + \mathcal{H}\nabla_Z C_R W = \omega_R \mathcal{A}_Z W + C_R \mathcal{H}\nabla_Z W$$
for $Z, W \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$.

(3)
$$\widehat{\nabla}_X B_R Z + \mathcal{T}_X C_R Z = \phi_R \mathcal{T}_X Z + B_R \mathcal{H}\nabla_X Z$$
$$\mathcal{T}_X B_R Z + \mathcal{H}\nabla_X C_R Z = \omega_R \mathcal{T}_X Z + C_R \mathcal{H}\nabla_X Z$$
for $X \in \Gamma(\ker F_*)$, $Z \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$.

Theorem 3.6. *Let F be a h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-slant basis. Then the following conditions are equivalent:*

(a) *the complex distribution $\mathcal{D}_1$ is integrable.*
(b) $Q_I(\widehat{\nabla}_X \phi_I Y - \widehat{\nabla}_Y \phi_I X) = 0$ *and* $\mathcal{T}_X \phi_I Y = \mathcal{T}_Y \phi_I X$ *for* $X, Y \in \Gamma(\mathcal{D}_1)$.
(c) $Q_J(\widehat{\nabla}_X \phi_J Y - \widehat{\nabla}_Y \phi_J X) = 0$ *and* $\mathcal{T}_X \phi_J Y = \mathcal{T}_Y \phi_J X$ *for* $X, Y \in \Gamma(\mathcal{D}_1)$.
(d) $Q_K(\widehat{\nabla}_X \phi_K Y - \widehat{\nabla}_Y \phi_K X) = 0$ *and* $\mathcal{T}_X \phi_K Y = \mathcal{T}_Y \phi_K X$ *for* $X, Y \in \Gamma(\mathcal{D}_1)$.

Proof. Given $X, Y \in \Gamma(\mathcal{D}_1)$ and $R \in \{I, J, K\}$, we obtain

$$\begin{aligned} R[X, Y] &= R(\nabla_X Y - \nabla_Y X) = \nabla_X RY - \nabla_Y RX \\ &= \widehat{\nabla}_X \phi_R Y - \widehat{\nabla}_Y \phi_R X + \mathcal{T}_X \phi_R Y - \mathcal{T}_Y \phi_R X. \end{aligned}$$

Since $\mathcal{D}_1$ is R-invariant, we have

$$a) \Leftrightarrow b), \quad a) \Leftrightarrow c), \quad a) \Leftrightarrow d).$$

Therefore, we get the result. ∎

Theorem 3.7. *Let F be a h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-slant basis. Then the following conditions are equivalent:*

(a) *the slant distribution* $\mathcal{D}_2$ *is integrable.*
(b) $P_I(\widehat{\nabla}_X \phi_I Y - \widehat{\nabla}_Y \phi_I X + \mathcal{T}_X \omega_I Y - \mathcal{T}_Y \omega_I X) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_2)$.
(c) $P_J(\widehat{\nabla}_X \phi_J Y - \widehat{\nabla}_Y \phi_J X + \mathcal{T}_X \omega_J Y - \mathcal{T}_Y \omega_J X) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_2)$.
(d) $P_K(\widehat{\nabla}_X \phi_K Y - \widehat{\nabla}_Y \phi_K X + \mathcal{T}_X \omega_K Y - \mathcal{T}_Y \omega_K X) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_2)$.

Proof. Given $X, Y \in \Gamma(\mathcal{D}_2)$, $Z \in \Gamma(\mathcal{D}_1)$, and $R \in \{I, J, K\}$, we obtain

$$\begin{aligned} g_M([X,Y], RZ) &= -g_M(R[X,Y], Z) = -g_M(\nabla_X RY - \nabla_Y RX, Z) \\ &= -g_M(\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \phi_R Y + \mathcal{T}_X \omega_R Y + \mathcal{H}\nabla_X \omega_R Y - \widehat{\nabla}_Y \phi_R X \\ &\quad - \mathcal{T}_Y \phi_R X - \mathcal{T}_Y \omega_R X - \mathcal{H}\nabla_Y \omega_R X, Z) \\ &= -g_M(\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \omega_R Y - \widehat{\nabla}_Y \phi_R X - \mathcal{T}_Y \omega_R X, Z). \end{aligned}$$

Since $[X, Y] \in \Gamma(\ker F_*)$, we have

$$(a) \Leftrightarrow (b), \quad (a) \Leftrightarrow (c), \quad (a) \Leftrightarrow (d).$$

Therefore, the result follows. ∎

Proposition 3.8. *Let* F *be an almost h-semi-slant submersion from an almost quaternionic Hermitian manifold* (M, E, g_M) *onto a Riemannian manifold* (N, g_N)*. Then we get*

$$\phi_R^2 X = -\cos^2 \theta_R X \quad \textit{for } X \in \Gamma(\mathcal{D}_2^R) \textit{ and } R \in \{I, J, K\},$$

where $\{I, J, K\}$ *is an almost h-semi-slant basis with the almost h-semi-slant angles* $\{\theta_I, \theta_J, \theta_K\}$*.*

Proof. Since

$$\cos\theta_R = \frac{g_M(RX, \phi_R X)}{|RX| \cdot |\phi_R X|} = \frac{-g_M(X, \phi_R^2 X)}{|X| \cdot |\phi_R X|}$$

and $\cos\theta_R = \dfrac{|\phi_R X|}{|RX|}$, we obtain

$$\cos^2 \theta_R = -\frac{g_M(X, \phi_R^2 X)}{|X|^2} \quad \text{for } X \in \Gamma(\mathcal{D}_2^R).$$

Hence,

$$\phi_R^2 X = -\cos^2 \theta_R X \quad \text{for } X \in \Gamma(\mathcal{D}_2^R).$$ ∎

Remark 3.9. In particular, it is easy to see that the converse of Proposition 3.8 is also true.

Assume that the almost h-semi-slant angle $\theta_R \in [0, \frac{\pi}{2})$ for some $R \in \{I, J, K\}$ and define an endomorphism $\widehat{R}$ of $\ker F_*$ by

$$\widehat{R} := RP_R + \frac{1}{\cos\theta_R}\phi_R Q_R.$$

Then,

$$\widehat{R}^2 = -id \quad \text{on } \ker F_*. \tag{3}$$

Remark 3.10. Let F be an almost h-semi-slant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Assume that $\dim M = 4m$, $\dim N = n$, and $\{\theta_I, \theta_J, \theta_K\} \cap [0, \frac{\pi}{2}) \neq \varnothing$. From (3), we obtain

$$\dim(\ker(F_*)_p) = 2k \text{ and } \dim((\ker(F_*)_p)^\perp) = 4m - 2k \quad \text{for } p \in M,$$

where k is a non-negative integer.

Hence, n should be even.

Theorem 3.11. *Let F be an almost h-semi-slant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N) such that $\{\theta_I, \theta_J, \theta_K\} \cap [0, \frac{\pi}{2}) \neq \varnothing$, where $\{\theta_I, \theta_J, \theta_K\}$ are almost h-semi-slant angles. Then N is an even-dimensional manifold.*

Proposition 3.12. *Let F be an almost h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Then the following conditions are equivalent:*

(a) *the distribution* $(\ker F_*)^\perp$ *defines a totally geodesic foliation.*

(b) $\phi_I(\mathcal{V}\nabla_X B_I Y + \mathcal{A}_X C_I Y) + B_I(\mathcal{A}_X B_I Y + \mathcal{H}\nabla_X C_I Y) = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.

(c) $\phi_J(\mathcal{V}\nabla_X B_J Y + \mathcal{A}_X C_J Y) + B_J(\mathcal{A}_X B_J Y + \mathcal{H}\nabla_X C_J Y) = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.

(d) $\phi_K(\mathcal{V}\nabla_X B_K Y + \mathcal{A}_X C_K Y) + B_K(\mathcal{A}_X B_K Y + \mathcal{H}\nabla_X C_K Y) = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $X, Y \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$, we get

$$\begin{aligned}\nabla_X Y &= -R\nabla_X RY = -R(\mathcal{V}\nabla_X B_R Y + \mathcal{A}_X B_R Y + \mathcal{A}_X C_R Y + \mathcal{H}\nabla_X C_R Y)\\ &= -(\phi_R\mathcal{V}\nabla_X B_R Y + \omega_R\mathcal{V}\nabla_X B_R Y + B_R\mathcal{A}_X B_R Y + C_R\mathcal{A}_X B_R Y + \phi_R\mathcal{A}_X C_R Y\\ &\quad + \omega_R\mathcal{A}_X C_R Y + B_R\mathcal{H}\nabla_X C_R Y + C_R\mathcal{H}\nabla_X C_R Y).\end{aligned}$$

Thus,

$$\nabla_X Y \in \Gamma((\ker F_*)^\perp) \Leftrightarrow \phi_R(\mathcal{V}\nabla_X B_R Y + \mathcal{A}_X C_R Y) + B_R(\mathcal{A}_X B_R Y + \mathcal{H}\nabla_X C_R Y) = 0.$$

Hence, we have

$$a) \Leftrightarrow b), \quad a) \Leftrightarrow c), \quad a) \Leftrightarrow d).$$

Therefore, we get the result. ∎

In a similar way, we have

Proposition 3.13. *Let F be an almost h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Then the following conditions are equivalent:*

(a) *the distribution* $\ker F_*$ *defines a totally geodesic foliation.*

(b) $\omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0$ *for* $X, Y \in \Gamma(\ker F_*)$.

(c) $\omega_J(\widehat{\nabla}_X \phi_J Y + \mathcal{T}_X \omega_J Y) + C_J(\mathcal{T}_X \phi_J Y + \mathcal{H}\nabla_X \omega_J Y) = 0$ *for* $X, Y \in \Gamma(\ker F_*)$.

(d) $\omega_K(\widehat{\nabla}_X \phi_K Y + \mathcal{T}_X \omega_K Y) + C_K(\mathcal{T}_X \phi_K Y + \mathcal{H}\nabla_X \omega_K Y) = 0$ *for* $X, Y \in \Gamma(\ker F_*)$.

Proposition 3.14. *Let F be a h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-slant basis. Then the following conditions are equivalent:*

(a) *the distribution $\mathcal{D}_2$ defines a totally geodesic foliation.*

(b)

$$P_I(\phi_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + B_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y)) = 0$$
$$\omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

(c)

$$P_J(\phi_J(\widehat{\nabla}_X \phi_J Y + \mathcal{T}_X \omega_J Y) + B_J(\mathcal{T}_X \phi_J Y + \mathcal{H}\nabla_X \omega_J Y)) = 0$$
$$\omega_J(\widehat{\nabla}_X \phi_J Y + \mathcal{T}_X \omega_J Y) + C_J(\mathcal{T}_X \phi_J Y + \mathcal{H}\nabla_X \omega_J Y) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

(d)

$$P_K(\phi_K(\widehat{\nabla}_X \phi_K Y + \mathcal{T}_X \omega_K Y) + B_K(\mathcal{T}_X \phi_K Y + \mathcal{H}\nabla_X \omega_K Y)) = 0$$
$$\omega_K(\widehat{\nabla}_X \phi_K Y + \mathcal{T}_X \omega_K Y) + C_K(\mathcal{T}_X \phi_K Y + \mathcal{H}\nabla_X \omega_K Y) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

Proof. Given $X, Y \in \Gamma(\mathcal{D}_2)$ and $R \in \{I, J, K\}$, we get

$$\begin{aligned}
\nabla_X Y = -R\nabla_X RY &= -R(\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \phi_R Y + \mathcal{T}_X \omega_R Y + \mathcal{H}\nabla_X \omega_R Y) \\
&= -(\phi_R \widehat{\nabla}_X \phi_R Y + \omega_R \widehat{\nabla}_X \phi_R Y + B_R \mathcal{T}_X \phi_R Y + C_R \mathcal{T}_X \phi_R Y + \phi_R \mathcal{T}_X \omega_R Y \\
&\quad + \omega_R \mathcal{T}_X \omega_R Y + B_R \mathcal{H}\nabla_X \omega_R Y + C_R \mathcal{H}\nabla_X \omega_R Y).
\end{aligned}$$

Thus,

$$\nabla_X Y \in \Gamma(\mathcal{D}_2) \Leftrightarrow \begin{cases} P_R(\phi_R(\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \omega_R Y) + B_R(\mathcal{T}_X \phi_R Y + \mathcal{H}\nabla_X \omega_R Y)) = 0, \\ \omega_R(\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \omega_R Y) + C_R(\mathcal{T}_X \phi_R Y + \mathcal{H}\nabla_X \omega_R Y) = 0. \end{cases}$$

Hence, we have

$$a) \Leftrightarrow b), \quad a) \Leftrightarrow c), \quad a) \Leftrightarrow d).$$

Therefore, the result follows. ■

Similarly, we get

Proposition 3.15. *Let F be a h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-slant basis. Then the following conditions are equivalent:*

(a) *the distribution $\mathcal{D}_1$ defines a totally geodesic foliation.*

(b)

$$Q_I(\phi_I \widehat{\nabla}_X \phi_I Y + B_I \mathcal{T}_X \phi_I Y) = 0 \text{ and } \omega_I \widehat{\nabla}_X \phi_I Y + C_I \mathcal{T}_X \phi_I Y = 0$$

for $X, Y \in \Gamma(\mathcal{D}_1)$.

(c)

$$Q_J(\phi_J \widehat{\nabla}_X \phi_J Y + B_J \mathcal{T}_X \phi_J Y) = 0 \text{ and } \omega_J \widehat{\nabla}_X \phi_J Y + C_J \mathcal{T}_X \phi_J Y = 0$$

for $X, Y \in \Gamma(\mathcal{D}_1)$.

(d)

$$Q_K(\phi_K \widehat{\nabla}_X \phi_K Y + B_K \mathcal{T}_X \phi_K Y) = 0 \text{ and } \omega_K \widehat{\nabla}_X \phi_K Y + C_K \mathcal{T}_X \phi_K Y = 0$$

for $X, Y \in \Gamma(\mathcal{D}_1)$.

Now, we obtain a condition for such maps to be totally geodesic.

Theorem 3.16. *Let F be an almost h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Then the following conditions are equivalent:*

(a) *F is a totally geodesic map.*

(b)

$$\omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0$$
$$\omega_I(\widehat{\nabla}_X B_I Z + \mathcal{T}_X C_I Z) + C_I(\mathcal{T}_X B_I Z + \mathcal{H}\nabla_X C_I Z) = 0$$

for $X, Y \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$.

(c)

$$\omega_J(\widehat{\nabla}_X \phi_J Y + \mathcal{T}_X \omega_J Y) + C_J(\mathcal{T}_X \phi_J Y + \mathcal{H}\nabla_X \omega_J Y) = 0$$

$$\omega_J(\widehat{\nabla}_X B_J Z + \mathcal{T}_X C_J Z) + C_J(\mathcal{T}_X B_J Z + \mathcal{H}\nabla_X C_J Z) = 0$$

for $X, Y \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$.

(d)

$$\omega_K(\widehat{\nabla}_X \phi_K Y + \mathcal{T}_X \omega_K Y) + C_K(\mathcal{T}_X \phi_K Y + \mathcal{H}\nabla_X \omega_K Y) = 0$$

$$\omega_K(\widehat{\nabla}_X B_K Z + \mathcal{T}_X C_K Z) + C_K(\mathcal{T}_X B_K Z + \mathcal{H}\nabla_X C_K Z) = 0$$

for $X, Y \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$.

Proof. Since F is a Riemannian submersion, we get

$$(\nabla F_*)(Z_1, Z_2) = 0 \quad \text{for } Z_1, Z_2 \in \Gamma((\ker F_*)^\perp).$$

Given $X, Y \in \Gamma(\ker F_*)$, we have

$$\begin{aligned}(\nabla F_*)(X,Y) &= -F_*(\nabla_X Y) = F_*(I\nabla_X(\phi_I Y + \omega_I Y)) \\ &= F_*(\phi_I \widehat{\nabla}_X \phi_I Y + \omega_I \widehat{\nabla}_X \phi_I Y + B_I \mathcal{T}_X \phi_I Y + C_I \mathcal{T}_X \phi_I Y + \phi_I \mathcal{T}_X \omega_I Y \\ &\quad + \omega_I \mathcal{T}_X \omega_I Y + B_I \mathcal{H}\nabla_X \omega_I Y + C_I \mathcal{H}\nabla_X \omega_I Y).\end{aligned}$$

Thus,

$$(\nabla F_*)(X,Y) = 0 \Leftrightarrow \omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0.$$

For $X \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$, since $(\nabla F_*)(X,Z) = (\nabla F_*)(Z,X)$, it is sufficient to consider the following:

$$\begin{aligned}(\nabla F_*)(X,Z) &= -F_*(\nabla_X Z) = F_*(I\nabla_X(B_I Z + C_I Z)) \\ &= F_*(\phi_I \widehat{\nabla}_X B_I Z + \omega_I \widehat{\nabla}_X B_I Z + B_I \mathcal{T}_X B_I Z + C_I \mathcal{T}_X B_I Z + \phi_I \mathcal{T}_X C_I Z \\ &\quad + \omega_I \mathcal{T}_X C_I Z + B_I \mathcal{H}\nabla_X C_I Z + C_I \mathcal{H}\nabla_X C_I Z).\end{aligned}$$

Thus,

$$(\nabla F_*)(X,Z) = 0 \Leftrightarrow \omega_I(\widehat{\nabla}_X B_I Z + \mathcal{T}_X C_I Z) + C_I(\mathcal{T}_X B_I Z + \mathcal{H}\nabla_X C_I Z) = 0.$$

Hence,

$$(a) \Leftrightarrow (b).$$

Similarly, we get

$$(a) \Leftrightarrow (c) \quad \text{and} \quad (a) \Leftrightarrow (d).$$

Therefore, the result follows. ■

Let F be an almost h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Given a complex structure $R \in \{I, J, K\}$, we can choose a local orthonormal frame $\{v_1, \cdots, v_l\}$ of $\mathcal{D}_2^R$ and a local orthonormal frame $\{e_1, \cdots, e_{2k}\}$ of $\mathcal{D}_1^R$ such that $e_{2i} = Re_{2i-1}$ for $1 \leq i \leq k$. If $\mathcal{D}_1^R$ is integrable, then we easily obtain

$$F_*(\nabla_{Re_{2i-1}} Re_{2i-1}) = -F_*(\nabla_{e_{2i-1}} e_{2i-1}) \quad \text{for } 1 \leq i \leq k$$

so that we have

$$trace(\nabla F_*) = 0 \Leftrightarrow \sum_{j=1}^{l} F_*(\nabla_{v_j} v_j) = 0.$$

Theorem 3.17. *Let F be an almost h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Then each of the following conditions implies that F is a harmonic map:*

(a) $\mathcal{D}_1^I$ *is integrable and* $trace(\nabla F_*) = 0$ *on* $\mathcal{D}_2^I$.
(b) $\mathcal{D}_1^J$ *is integrable and* $trace(\nabla F_*) = 0$ *on* $\mathcal{D}_2^J$.
(c) $\mathcal{D}_1^K$ *is integrable and* $trace(\nabla F_*) = 0$ *on* $\mathcal{D}_2^K$.

Corollary 3.18. *Let F be an almost h-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Assume that* $\ker F_* = \mathcal{D}_1^R$ *for some* $R \in \{I, J, K\}$. *Then F is a harmonic map.*

Let $F : (M, g_M) \mapsto (N, g_N)$ be a Riemannian submersion. The map F is called a Riemannian submersion *with totally umbilical fibers* if

(4) $$\mathcal{T}_X Y = g_M(X, Y) H \quad \text{for } X, Y \in \Gamma(\ker F_*),$$

where H is the mean curvature vector field of the fiber.

Lemma 3.19. *Let F be an almost h-semi-slant submersion with totally umbilical fibers from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Then we obtain*

$$H \in \Gamma(\omega_R \mathcal{D}_2^R) \quad \textit{for } R \in \{I, J, K\}.$$

Proof. Given $X, Y \in \Gamma(\mathcal{D}_1^R)$, $W \in \Gamma(\mu_R)$, and $R \in \{I, J, K\}$, we get

$$\mathcal{T}_X RY + \widehat{\nabla}_X RY = \nabla_X RY = R\nabla_X Y = B_R \mathcal{T}_X Y + C_R \mathcal{T}_X Y + \phi_R \widehat{\nabla}_X Y + \omega_R \widehat{\nabla}_X Y$$

so that

$$g_M(\mathcal{T}_X RY, W) = g_M(C_R \mathcal{T}_X Y, W).$$

By (4), we easily have

$$g_M(X, RY) g_M(H, W) = -g_M(X, Y) g_M(H, RW).$$

Interchanging the role of X and Y, we get

$$g_M(Y, RX) g_M(H, W) = -g_M(Y, X) g_M(H, RW).$$

Using the above two equations, we obtain

$$g_M(X, Y) g_M(H, RW) = 0,$$

which implies $H \in \Gamma(\omega_R \mathcal{D}_2^R)$, since μ_R is R-invariant.

Therefore, we have the result. ■

4. Examples

Note that given an Euclidean space $\mathbb{R}^{4m}$ with coordinates $(x_1, x_2, \cdots, x_{4m})$, we can canonically choose complex structures I, J, K on $\mathbb{R}^{4m}$ as follows:

$$I(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+2}}, I(\tfrac{\partial}{\partial x_{4k+2}}) = -\tfrac{\partial}{\partial x_{4k+1}}, I(\tfrac{\partial}{\partial x_{4k+3}}) = \tfrac{\partial}{\partial x_{4k+4}}, I(\tfrac{\partial}{\partial x_{4k+4}}) = -\tfrac{\partial}{\partial x_{4k+3}},$$
$$J(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+3}}, J(\tfrac{\partial}{\partial x_{4k+2}}) = -\tfrac{\partial}{\partial x_{4k+4}}, J(\tfrac{\partial}{\partial x_{4k+3}}) = -\tfrac{\partial}{\partial x_{4k+1}}, J(\tfrac{\partial}{\partial x_{4k+4}}) = \tfrac{\partial}{\partial x_{4k+2}},$$
$$K(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+4}}, K(\tfrac{\partial}{\partial x_{4k+2}}) = \tfrac{\partial}{\partial x_{4k+3}}, K(\tfrac{\partial}{\partial x_{4k+3}}) = -\tfrac{\partial}{\partial x_{4k+2}}, K(\tfrac{\partial}{\partial x_{4k+4}}) = -\tfrac{\partial}{\partial x_{4k+1}}$$

for $k \in \{0, 1, \cdots, m-1\}$.

Then we easily check that $(I, J, K, \langle\ ,\ \rangle)$ is a hyperkähler structure on $\mathbb{R}^{4m}$, where $\langle\ ,\ \rangle$ denotes the Euclidean metric on $\mathbb{R}^{4m}$. Throughout this section, we will use these notations.

Example 4.1. Let F be an almost h-slant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Then the map $F : (M, E, g_M) \mapsto (N, g_N)$ is a h-semi-slant submersion with $\mathcal{D}_2 = \ker F_*$. [16].

Example 4.2. Let F be an almost h-semi-invariant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Then the map $F : (M, E, g_M) \mapsto (N, g_N)$ is an almost h-semi-slant submersion with the almost h-semi-slant angles $\theta_I = \theta_J = \theta_K = \frac{\pi}{2}$. [17].

Example 4.3. Let (M, E, g) be an almost quaternionic Hermitian manifold. Let $\pi : TM \mapsto M$ be the natural projection. Then the map π is a strictly h-semi-slant submersion such that $\mathcal{D}_1 = \ker \pi_*$ and the strictly h-semi-slant angle $\theta = 0$ [11].

Example 4.4. Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds. Let $F : M \mapsto N$ be a quaternionic submersion. Then the map F is a strictly h-semi-slant submersion such that $\mathcal{D}_1 = \ker F_*$ and the strictly h-semi-slant angle $\theta = 0$ [11].

Example 4.5. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^3$ by

$$F(x_1, \cdots, x_8) = (x_5 \sin\alpha - x_7 \cos\alpha, x_6, x_8),$$

where α is constant. Then the map F is a strictly h-semi-slant submersion such that

$$\mathcal{D}_1 =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} > \text{ and } \mathcal{D}_2 =< \cos\alpha \frac{\partial}{\partial x_5} + \sin\alpha \frac{\partial}{\partial x_7} >$$

with the strictly h-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 4.6. Let (M, I, J, K, g_M) be a $4m-$dimensional hyperkähler manifold and (N, g_N) a $(4m-1)-$dimensional Riemannian manifold. Let $\widehat{F} : (M, I, J, K, g_M) \mapsto (N, g_N)$ be a Riemannian submersion.

Define a map $F : (M, I, J, K, g_M) \times \mathbb{R}^{4k} \mapsto (N, g_N)$ by

$$F(x, y) = \widehat{F}(x) \quad \text{for } x \in M \text{ and } y \in \mathbb{R}^{4k}.$$

Then the map F is a strictly h-semi-slant submersion such that

$$\mathcal{D}_1 = 0 \times \mathbb{R}^{4k} \text{ and } \mathcal{D}_2 = \ker \widehat{F}_* \times 0$$

with the strictly h-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 4.7. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^4$ by

$$F(x_1, \cdots, x_{12}) = (\frac{x_5 - x_7}{\sqrt{2}}, x_8, \frac{x_9 - x_{11}}{\sqrt{2}}, x_{10}).$$

Then the map F is a h-semi-slant submersion such that

$$\mathcal{D}_1 =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} > \text{ and } \mathcal{D}_2 =< \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{11}} >$$

with the h-semi-slant angles $\{\theta_I = \frac{\pi}{4}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{4}\}$.

Example 4.8. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^2$ by

$$F(x_1, \cdots, x_{12}) = (x_5 \cos\alpha - x_7 \sin\alpha, x_6 \sin\beta - x_8 \cos\beta),$$

where α and β are constant. Then the map F is a h-semi-slant submersion such that

$$\mathcal{D}_1 =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >$$

and

$$\mathcal{D}_2 =< \sin\alpha\frac{\partial}{\partial x_5} + \cos\alpha\frac{\partial}{\partial x_7}, \cos\beta\frac{\partial}{\partial x_6} + \sin\beta\frac{\partial}{\partial x_8} >$$

with the h-semi-slant angles $\{\theta_I, \theta_J = \frac{\pi}{2}, \theta_K\}$ such that $\cos\theta_I = |\sin(\alpha+\beta)|$ and $\cos\theta_K = |\cos(\alpha+\beta)|$.

Example 4.9. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^6$ by

$$F(x_1, \cdots, x_{12}) = (x_3, \cdots, x_8).$$

Then the map F is an almost h-semi-slant submersion such that

$$\begin{aligned}
&\mathcal{D}_1^I =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >, \\
&\mathcal{D}_1^J = \mathcal{D}_1^K =< \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >, \\
&\mathcal{D}_2^I = 0, \quad \mathcal{D}_2^J = \mathcal{D}_2^K =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} > .
\end{aligned}$$

with the almost h-semi-slant angles $\{\theta_I = 0, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$.

By Corollary 3.18, F is also harmonic.

Example 4.10. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^4$ by

$$F(x_1, \cdots, x_{12}) = (x_7, x_5, x_1, x_2).$$

Then the map F is an almost h-semi-slant submersion such that

$$\begin{aligned}
&\mathcal{D}_1^I =< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >, \\
&\mathcal{D}_1^J =< \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >, \\
&\mathcal{D}_1^K =< \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >, \\
&\mathcal{D}_2^I =< \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} >, \quad \mathcal{D}_2^J =< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >, \\
&\mathcal{D}_2^K =< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} >
\end{aligned}$$

with the almost h-semi-slant angles $\{\theta_I = \frac{\pi}{2}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$.

Example 4.11. Let $\widehat{F}$ be an almost h-slant submersion from an almost quaternionic Hermitian manifold (M_1, E_1, g_{M_1}) onto a Riemannian manifold (N, g_N) and (M_2, E_2, g_{M_2}) an almost quaternionic Hermitian manifold.

Define a map $F : (M_1, E_1, g_{M_1}) \times_f (M_2, E_2, g_{M_2}) \mapsto (N, g_N)$ by

$$F(x, y) = \widehat{F}(x) \quad \text{for } x \in M_1 \text{ and } y \in M_2,$$

where $(M_1, E_1, g_{M_1}) \times_f (M_2, E_2, g_{M_2})$ is the warped product of (M_1, E_1, g_{M_1}) and (M_2, E_2, g_{M_2}) with the warping function $f : M_1 \mapsto \mathbb{R}^+$. i.e., $g = g_{M_1} + f^2 g_{M_2}$.

Then the map F is a h-semi-slant submersion such that

$$\mathcal{D}_1 = TM_2 \text{ and } \mathcal{D}_2 = \ker \widehat{F}_*$$

with the h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$, where $\{I, J, K\}$ is an almost h-slant basis of the map $\widehat{F}$ with the slant angles $\{\theta_I, \theta_J, \theta_K\}$ [16].

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