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BICOVARIANT DIFFERENTIAL CALCULI ON A WEAK HOPF ALGEBRA

Haixing Zhu*, Shuanhong Wang and Juzhen Chen

Abstract. Let H be a weak Hopf algebra with bijective antipode. In this paper we follow Woronowicz's fundamental method to characterize bicovariant differential calculi on H. We show that there exists a 1-1 correspondence between bicovariant differential calculi and some right ideals of H contained in $ker\varepsilon_s$ such that these ideals are right H-comodules with coadjoint maps, where ε_s is the source map of H. This is a generalization of well-known Woronowicz's theorem about bicovariant differential calculi on quantum groups.

1. INTRODUCTION

Noncommutative geometry is the study of noncommutative algebras as if they were algebras of functions on spaces which was initiated in [4] by Connes. An interesting example of such noncommutative geometry was provided in the framework of quantum groups (i.e., noncommutative and noncocommutative Hopf algebras). Based on the ideas of Connes, Woronowicz [23] used the axiomatic method to introduce first order differential calculi and investigated bicovariant differential calculi on quantum groups. He showed that there exists a 1-1 correspondence between bicovariant differential calculi and some special right ideals of a quantum group, see [23]. Using this correspondence, the classifications of bicovariant differential calculi of some quantum groups have been carried out, for example, see [11, 12, 15, 20].

At the same time, when considering the solution of a quantum dynamical Yang-Baxter equation, Felder [9] used the Faddeev-Reshetikhin-Takhtajan method to obtain a certain algebra F_U called the dynamical quantum group. However, F_U is not a Hopf algebra, but a Hopf algebroid in [13]. In general, to any dynamical twist in [1], one can associate a Hopf algebroid, see [6, 8, 24]. In particular, for every dynamical twist of a Hopf algebra H, by [6] one can obtain a dynamical quantum group F_H, which is called a weak Hopf algebra (special Hopf algebroid).

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*Corresponding author.

Following Woronowicz's axiomatic method, the notion of a differential calculus can be easily defined on a dynamical quantum group. However, this kind of differential calculi is a bit different from the case of Hopf algebras (for example, if we consider its high order differential calculi, the tensor product must be over some subalgebra, not its base field k, see [3]). So it is very necessary to investigate these differential calculi. In this paper, we mainly focus on differential calculi on weak Hopf algebras. Although a weak Hopf algebra is just a special Hopf algebroid, our theory is enough to cover many important cases such as dynamical quantum groups [6], compact face algebras [10] and weak Hopf $*$-algebras.

In [3], Chen and Wang generalized results in [18, 19] to weak Hopf algebras. For example, they gave noncommutative differential calculus on weak smash product, and also studied connections and high order differential calculi. Here our main goal is to characterize bicovariant differential calculi of a weak Hopf algebra by a fundamental method in [23].

Let H be a weak Hopf algebra with the target subalgebra H_t and the source subalgebra H_s. We show that a bicovariant differential calculi on H must be both H_t- and H_s-bilinear. This unexpected fact is a bit different from the case of Hopf algebras. However, it explains why the high order differential calculi on a weak Hopf algebra must be over some minimal weak Hopf algebra H_{min}, see [3]. Although H_t- and H_s-linearities appear, we still find some 1-1 correspondence between bicovariant differential calculi and some special ideals. Similar to quantum groups, this result provides us a possibility of the classifications of bicovariant differential calculi on a compact face algebra and a dynamical quantum group obtained by a dynamical twist of a Hopf algebra.

This paper is organized as follows. In Section 2 we recall the basic definitions and results about weak Hopf algebras, weak Hopf bimodules and first order differential calculi. Section 3 is devoted to some special Hopf bimodules. The investigation of bicovariant differential calculi are carried out in Section 4. We first characterize the H_t and H_s-linearities of first order differential calculi by several necessary and sufficient conditions. Next we show that a left first order differential calculus is H_t-bilinear while a right first order differential calculus is H_s-bilinear. Finally, we study bicovariant differential calculi on weak Hopf algebras. Here is our main result:

Let H be a weak Hopf algebra with bijective antipode. Let ε_s be the source map of H. Then there exists a 1-1 correspondence between bicovariant differential calculi on H and some right ideals of H contained in $ker\varepsilon_s$ such that these ideals are right H-comodules with coadjoint maps.

It constitutes a generalization of the well-known Woronowicz's theorem about bicovariant differential calculi on quantum groups, see [23, Thm 1.8].

2. Basic Definitions and Results

Throughout this paper, k is a fixed field. Unless otherwise stated, unadorned tensor products will be over k. For a k-coalgebra, the coproduct will be denoted by Δ. We adopt a Sweedler's like notation e.g., $\Delta(a) = a_{(1)} \otimes a_{(2)}$, see [21].

2.1. Weak Hopf algebras

For the basic definitions and properties of weak Hopf algebras, the reader is referred to [2]. A weak Hopf algebra H is an k-algebra (H, m, μ) and a k-coalgebra (H, Δ, ε) such that the following axioms hold:

- $\Delta(hk) = \Delta(h)\Delta(k)$,
- $\Delta^2(1) = 1_1 \otimes 1_2 1_{(1')} \otimes 1_{2'} = 1_1 \otimes 1_{1'} 1_2 \otimes 1_{2'}$,
- $\varepsilon(hkl) = \varepsilon(hk_1)\varepsilon(k_2 l) = \varepsilon(hk_2)\varepsilon(k_1 l)$,
- There exists a k-linear map $S : H \longrightarrow H$, called the antipode, satisfying
$$h_1 S(h_2) = \varepsilon(1_1 h)1_2, \qquad S(h_1)h_2 = 1_1 \varepsilon(h 1_2),$$
$$S(h) = S(h_1)h_2 S(h_3),$$

for all $h, k, l \in H$. We have idempotent maps $\varepsilon_t, \varepsilon_s$: $H \longrightarrow H$ defined by

$$\varepsilon_t(h) = \varepsilon(1_1 h)1_2, \qquad \varepsilon_s(h) = 1_1 \varepsilon(h 1_2).$$

Here ε_t (ε_s) is called the target map (source map), and its imagine H_t (H_s) is called the target (source space), which can also be described as follows:

$$H_t = \{h \in H \mid \varepsilon_t(h) = h\} = \{h \in H \mid \Delta(h) = 1_1 h \otimes 1_2 = h 1_1 \otimes 1_2\},$$
$$H_s = \{h \in H \mid \varepsilon_s(h) = h\} = \{h \in H \mid \Delta(h) = 1_1 \otimes h 1_2 = 1_1 \otimes 1_2 h\}.$$

Let H be a weak Hopf algebra. There are the following equations:

$$h_1 \otimes h_2 S(h_3) = 1_1 h \otimes 1_2, \tag{2.1}$$

$$S(h_1)h_2 \otimes h_3 = 1_1 \otimes h 1_2, \tag{2.2}$$

$$h_1 \otimes S(h_2)h_3 = h 1_1 \otimes S(1_2), \tag{2.3}$$

$$h_1 S(h_2) \otimes h_3 = S(1_1) \otimes 1_2 h, \tag{2.4}$$

$$\varepsilon(g \varepsilon_t(h)) = \varepsilon(gh) = \varepsilon(\varepsilon_s(g) h), \tag{2.5}$$

$$y 1_1 \otimes S(1_2) = 1_1 \otimes S(1_2) y, \tag{2.6}$$

$$z S(1_1) \otimes 1_2 = S(1_1) \otimes 1_2 z, \tag{2.7}$$

for all $g, h \in H, y \in H_s$ and $z \in H_t$. Moreover, S restricts to an anti-algebra isomorphism $H_t \longrightarrow H_s$.

Remark.

(1) H is an ordinary Hopf algebra if and only if $\Delta(1) = 1 \otimes 1$ if and only if ε is a homomorphism if and only if $H_t = H_s = k$.

(2) The paragroups [17], the generalized Kac algebras in [25] and the face algebras [10] are the important subclasses of weak Hopf algebras, respectively.

(3) By [6] for every dynamical twist of a Hopf algebra H, one can obtain a dynamical quantum group F_H, which is actually a weak Hopf algebra.

(4) There exists closely relations between representation categories of some special weak Hopf algebras and (multi)fusion categories, for example, see [7].

(5) A weak Hopf algebra is a special Hopf algebroid, see [6, Sec. 2]. A finite dimensional weak Hopf algebra is self-dual and its antipode is automatically bijective.

In the sequel, a weak Hopf algebra always means a weak Hopf algebra with bijective antipode.

2.2. Weak Hopf bimodules

Let H be a weak Hopf algebra. Following [22, 23] a left weak Hopf bimodule M over H is an H-bimodule M with a left coaction denoted by $\rho_L : M \longrightarrow H \otimes M$, such that is ρ_L is an H-bimodule map:

$$\Delta(h)\rho_L(m) = \rho_L(hm) \quad \text{and} \quad \rho_L(m)\Delta(h) = \rho_L(mh).$$

A right weak Hopf bimodule M over H is a bimodule M with a right coaction denoted by $\rho_L : M \longrightarrow M \otimes H$, such that is ρ_R is an H-bimodule map:

$$\Delta(h)\rho_R(m) = \rho_R(hm) \quad \text{and} \quad \rho_R(m)\Delta(h) = \rho_R(mh).$$

A weak Hopf bimodule M over H is both a left weak Hopf bimodule and a right weak Hopf bimodule with left and right coactions, denoted by ρ_L and ρ_R, such that M is also a bicomodule. We shall also use Sweedler's notations, namely, $\rho_R(v) = v_{(0)} \otimes v_{(1)}$ and $\rho_L(v) = v_{(-1)} \otimes v_{(0)}$ for all $v \in V$.

In particular, a vector space M is a right-right weak Hopf module if M is both a right H-module and a right H-comodule such that

$$\rho_R(m \cdot h) = m_{(0)} \cdot h_{(1)} \otimes m_{(1)} h_{(2)},$$

for all $h \in H$ and $m \in M$. Similarly, one can define a left-left (left-right or right-left) weak Hopf module. Let M be a left-left weak Hopf module. Let

$$^{coH}M = \left\{ m \in M \mid \rho_L(m) = 1_{(1)} \otimes 1_{(2)} \cdot m \right\}$$

be the vector space of left coinvariants. It follows from [2] that ^{coH}M is a left subcomodule of M and $M \cong H \otimes_{H_s} {}^{coH}M$. Similarly, one can define N^{coH} to be the vector space of right coinvariants of a right-right weak Hopf module N and have $N^{coH} \otimes_{H_t} H \cong N$.

Let M be a left (right) weak Hopf bimodule over H. Assume that N is a left (right) weak Hopf sub-bimodule of N. It is not hard to see that the factor space M/N is naturally a left (right) weak Hopf bimodule over H.

2.3. First order differential calculi

Let A be an algebra with unity. Let Γ be a bimodule over A and $d : A \longrightarrow \Gamma$ be a k-linear map. By [23, Defn 1.1] (Γ, d) is called a first order differential calculus over A if

(1) $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$;

(2) the map $A \otimes A \longrightarrow \Gamma, \quad a \otimes b \longmapsto ad(b)$ is surjective.

Two first order differential calculi (Γ, d) and (Γ', d') over A are said to be identical if there exists a bimodule isomorphism $i : \Gamma \longrightarrow \Gamma'$ such that

$$i(d(a)) = d'(a) \;\text{ for all } a \in A.$$

Denote by A^2 the vector space $\{a \otimes b \in A \otimes A |\ ab = 0\}$. Then A^2 is an A-bimodule with the following structure

$$c(a \otimes b) = ca \otimes b, \quad (a \otimes b)c = a \otimes bc,$$

for any $a \otimes b \in A^2$ and $c \in A$. Define $D(b) = 1 \otimes b - b \otimes 1$ for all $b \in A$. Then (A^2, D) is a first order differential calculus over A. There exists the following lemma (see [23, Prop 1.1]):

(Γ, d) is a first order differential calculus over A if and only if there exists a sub-bimodule $N \subset A^2$ such that $\Gamma = A^2/N$ and $d = \pi \circ D$, where π is the canonical epimorphism $A^2 \longrightarrow \Gamma$.

3. Weak Hopf Bimodules

Let H be a weak Hopf algebra. In this section we will discuss some special weak Hopf bimodules needed in next section.

3.1. Left weak Hopf bimodules

First consider a k-linear map:

$$\rho_L : \; H \otimes H \longrightarrow H \otimes H \otimes H, \; a \otimes b \longmapsto a_1 b_1 \otimes a_2 \otimes b_2.$$

Lemma 3.1.1. *Let H be a weak Hopf algebra. Then*
$\rho_L(a \otimes b) = a_1 b_1 \otimes a_2 S(1_1) \otimes 1_2 b_2 = \rho_L(aS(1_1) \otimes 1_2 b), \;\; \forall\ a \otimes b \in H \otimes H.$

Proof. For all $a \otimes b \in H \otimes H$, we have

$$\begin{aligned}\rho_L(a\otimes b) &= a_1S(1_2)1'_1b_1\otimes a_2S(1_1)\otimes 1'_2b_2\\ &= a_1S(1_2)1'_1b_1\otimes a_2S(1_1)\otimes S^{-1}(S(1'_2))b_2\\ &\overset{(2.6)}{=} a_11'_1b_1\otimes a_2S(1_1)\otimes S^{-1}(S(1'_2)S(1_2))b_2\\ &= a_11'_1b_1\otimes a_2S(1_1)\otimes 1_21'_2b_2\\ &= a_1b_1\otimes a_2S(1_1)\otimes 1_2b_2.\end{aligned}$$

Moreover,

$$\begin{aligned}\rho_L(aS(1_1)\otimes 1_2b) &= (aS(1_1))_1(1_2b)_1\otimes (aS(1_1))_2\otimes (1_2b)_2\\ &= a_1S(1_1)1_2b_1\otimes a_2\otimes b_2 = a_1b_1\otimes a_2\otimes b_2.\end{aligned}$$

■

Similar to [23], consider the following two maps:

$$R_1:\ H\otimes H\longrightarrow H\otimes H,\ a\otimes b\longmapsto ab_1\otimes b_2,$$

$$R_2:\ H\otimes H\longrightarrow H\otimes H,\ a\otimes b\longmapsto aS(b_1)\otimes b_2,$$

for all $a\otimes b\in H\otimes H$. Now we give some relationship between R_1 and R_2.

Lemma 3.1.2. *Let H be a weak Hopf algebra. Then*

$$R_1R_2R_1 = R_1,\qquad R_2R_1R_2 = R_2.$$

Proof. We compute as follows

$$\begin{aligned}R_1R_2R_1(a\otimes b) &= R_1R_2(ab_1\otimes b_2) = R_1(ab_1S(b_2)\otimes b_3)\\ &= ab_1S(b_2)b_3\otimes b_4 \overset{(2.4)}{=} R_1(a\otimes b),\end{aligned}$$

for all $a\otimes b\in H\otimes H$. Similarly $R_2R_1R_2 = R_2$ holds. ■

For the sake of convenience we introduce another two maps:

$$P_1 = R_2R_1 : H\otimes H\longrightarrow H\otimes H,\ a\otimes b\longmapsto aS(1_1)\otimes 1_2b,$$

$$P_2 = R_1R_2 : H\otimes H\longrightarrow H\otimes H,\ a\otimes b\longmapsto a1_1\otimes b1_2.$$

It is easy to see $P_1^2 = P_1$ and $P_2^2 = P_2$. The following lemma is clear:

Lemma 3.1.3. *Let H be a weak Hopf algebra. Then*

(1) *R_1 is a bijective map from $P_1(H\otimes H)$ to $R_1(H\otimes H)$ with the inverse R_2;*

(2) *R_2 is a bijective map from $P_2(H\otimes H)$ to $R_2(H\otimes H)$ with the inverse R_1.*

Lemma 3.1.4. *Let H be a weak Hopf algebra. Then $P_1(H\otimes H)$ is a maximal left H-comodule with the structure map ρ_L defined as above.*

Proof. Note that for all $a\otimes b\in H\otimes H$,

$$(1\otimes\rho_L)\circ\rho_L(a\otimes b) = a_1b_1\otimes a_2b_2\otimes a_3\otimes b_3 = (\Delta\otimes 1)\circ\rho_L(a\otimes b).$$

Lemma 3.1.1 implies that $(1 \otimes \rho_L) \circ \rho_L(aS(1_1) \otimes 1_2 b) = (\Delta \otimes 1) \circ \rho_L(aS(1_1) \otimes 1_2 b)$.

$$\begin{aligned}
& (\varepsilon \otimes 1) \circ \rho_L(aS(1_1) \otimes 1_2 b) \\
= \;& \varepsilon(a_1 b_1) a_2 S(1_1) \otimes 1_2 b_2 \\
= \;& \varepsilon(a_1 \varepsilon_t(b_1)) a_2 S(1_1) \otimes 1_2 b_2 \\
\overset{(2.4)}{=} \;& \varepsilon(a_1 S(1'_1)) a_2 S(1_1) \otimes 1_2 1'_2 b \\
= \;& \varepsilon(a_1 \varepsilon_t(1'_1)) a_2 1'_2 S(1_1) \otimes 1_2 b \\
= \;& \varepsilon(a_1 1'_1) a_2 1'_2 S(1_1) \otimes 1_2 b \\
= \;& aS(1_1) \otimes 1_2 b.
\end{aligned}$$

Suppose that $(H \otimes H)'$ is a subspace of $H \otimes H$ such that $((H \otimes H)', \rho_L)$ is a left H-comodule. By Lemma 3.1.1 we have for any $c \otimes d \in (H \otimes H)'$,

$$c \otimes d = (\varepsilon \otimes 1) \circ \rho_L(c \otimes d) = \varepsilon(c_1 d_1) c_2 S(1_1) \otimes 1_2 d_2 = cS(1_1) \otimes 1_2 d \in P_1(H \otimes H).$$

Thus $(H \otimes H)'$ is a subcomodule of $P_1(H \otimes H)$. ■

Furthermore, $P_1(H \otimes H)$ is a maximal left Hopf bimodule.

Proposition 3.1.5. *Let H be a weak Hopf algebra. Then $P_1(H \otimes H)$ is a maximal left Hopf bimodule with the following structure:*

$$\begin{aligned}
&\rho_L(a \otimes b) = a_1 b_1 \otimes a_2 \otimes b_2, \\
&c(a \otimes b) = ca \otimes b, \\
&(a \otimes b)c = a \otimes bc,
\end{aligned}$$

for all $a \otimes b \in P_1(H \otimes H)$ and $c \in H$.

Proof. This proof follows Lemma 3.1.4. ■

Proposition 3.1.6. *Let H be a weak Hopf algebra. Then*

$$\{\ S(c_1) \otimes c_2 \mid \forall\ c \in H\ \} = {}^{coH}P_1(H \otimes H).$$

Proof. For any $S(c_1) \otimes c_2$, we have

$$S(c_1) \otimes c_2 = S(1_1 c_1) \otimes 1_2 c_2 = S(c_1) S(1_1) \otimes 1_2 c_2 \in P_1(H \otimes H).$$

And $S(c_1) \otimes c_2$ is contained in ${}^{coH}P_1(H \otimes H)$ by the calculation:

$$\begin{aligned}
\rho_L(S(c_1)\otimes c_2) &= S(c_2)c_3\otimes S(c_1)\otimes c_4\\
&= \varepsilon_s(c_2)\otimes S(c_1)\otimes c_3\\
&\overset{(2.3)}{=} S(1_2)\otimes S(c_1 1_1)\otimes c_2\\
&= 1_1\otimes 1_2 S(c_1)\otimes c_2\\
&= 1_1\otimes 1_2\cdot(S(c_1)\otimes c_2).
\end{aligned}$$

Conversely, for any $aS(1_1)\otimes 1_2 b\in {}^{coH}P_1(H\otimes H)$, we can get

$$\rho_L(aS(1_1)\otimes 1_2 b) = 1'_1\otimes 1'_2 aS(1_1)\otimes 1_2 b.$$

Applying $1\otimes\varepsilon\otimes 1$ to two sides of the above,

$$aS(1_1)(1_2b)_1\otimes(1_2b)_2 = 1'_1\varepsilon(1'_2 aS(1_1))\otimes 1_2 b.$$

Applying R_2 to two sides, by Lemma 3.1.3, on one hand,

$$aS(1_1)\otimes 1_2 b = 1'_1\varepsilon(1'_2 aS(1_1))S((1_2b)_1)\otimes(1_2b)_2.$$

On the other hand,

$$\begin{aligned}
& 1'_1\varepsilon(1'_2 aS(1_1))S((1_2b)_1)\otimes(1_2b)_2\\
=\;& 1'_1\varepsilon(1'_2 aS(1_1))S(1_2b_1)\otimes b_2\\
=\;& S(1'_2)\varepsilon(S(1'_1)aS(1_1))S(1_2b_1)\otimes b_2\\
\overset{(2.7)}{=}\;& S(1_2 1''_2 b_1 1'_2\varepsilon(S(1'_1)aS(1''_1)S(1_1)))\otimes b_2\\
=\;& S(1_2 1''_2 b_1 1'_2\varepsilon(1_1 S^{-1}(aS(1''_1))1'_1))\otimes b_2\\
=\;& S(\varepsilon_t(S^{-1}(aS(1''_1))1'_1)1''_2 b_1 1'_2)\otimes b_2\\
=\;& S((\varepsilon_t(S^{-1}(aS(1_1))1'_1)1_2 b 1'_2)_1)\otimes((\varepsilon_t(S^{-1}(aS(1_1))1'_1)1_2 b 1'_2)_2.
\end{aligned}$$

Let $c' = \varepsilon_t(S^{-1}(aS(1_1))1'_1)1_2 b 1'_2$. Then

$$aS(1_1)\otimes 1_2 b = S(c'_1)\otimes c'_2\in\{\,S(c_1)\otimes c_2\mid\forall\,c\in H\,\}. \qquad\blacksquare$$

Proposition 3.1.7. *Let H be a weak Hopf algebra and M be a left weak Hopf bimodule over H. Set*

$$\begin{aligned}
M' &= \{m\in M\mid m_{(-1)}\otimes m_0 = 1_1\otimes m1_2\},\\
M'' &= \{m\in M\mid m_{(-1)}\otimes m_0 = 1_1 1'_1\otimes 1_2 m 1'_2\}.
\end{aligned}$$

Then $\quad M''\supset M'\supset {}^{coH}M.$

Proof. Since M is a left weak Hopf bimodule, for any $m \in M$,

$$\rho(m) = \rho(1m1) = 1_1 m_{(-1)} 1'_1 \otimes 1_2 m_0 1'_2.$$

So $M' \subset M''$ and $M'' \supset {}^{coH}M$. By [2] the map

$$p : M \longrightarrow {}^{coH}M,\ p(m) = S(m_{(-1)})m_0$$

is a projection. For any $m \in {}^{coH}M$, there exists $m' \in M$ such that $m = S(m'_{(-1)})m'_0$. So we have

$$\begin{aligned} m_{(-1)} \otimes m_0 &= 1_1 \otimes 1_2 m = 1_1 \otimes 1_2 S(m'_{(-1)})m'_0 \\ &= 1_1 \otimes 1_2 S(1'_1)S(m'_{(-1)})m'_0 1'_2 \\ &\overset{(2.7)}{=} 1_1 \otimes S(1'_1)S(m'_{(-1)})m'_0 1'_2 1_2 \\ &= 1_1 \otimes S(m'_{(-1)})m'_0 1_2 \\ &= 1_1 \otimes m1_2. \end{aligned}$$

Thus $M' \supset {}^{coH}M$. ∎

Corollary 3.1.8. *Let H be a weak Hopf algebra. Then*

$$P_1(H \otimes H)'' = P_1(H \otimes H)' = {}^{coH}P_1(H \otimes H).$$

Proof. Similar to the proof of proposition 3.1.6, for any $aS(1_1) \otimes 1_2 b \in P_1(H \otimes H)''$, we have

$$\begin{aligned} &1'_1 1''_1 \varepsilon(1'_2 aS(1_1))S((1_2 b1''_2)_1) \otimes (1_2 b1''_2)_2 \\ &= 1'_1 1''_1 \varepsilon(1'_2 aS(1_1))S(1_2 b_1 1''_2) \otimes b_2 \\ &= 1'_1 1''_1 S(1''_2)\varepsilon(1'_2 aS(1_1))S(1_2 b_1) \otimes b_2 \\ &= S(1'_2)\varepsilon(S(1'_1)aS(1_1))S(1_2 b_1) \otimes b_2 \\ &= S((\varepsilon_t(S^{-1}(aS(1_1))1'_1)1_2 b1'_2)_1) \otimes ((\varepsilon_t(S^{-1}(aS(1_1))1'_1)1_2 b1'_2)_2. \end{aligned}$$

Let $c' = \varepsilon_t(S^{-1}(aS(1_1))1'_1)1_2 b1'_2$. We can get

$$aS(1_1) \otimes 1_2 b = S(c'_1) \otimes c'_2 \in \{\ S(c_1) \otimes c_2 \mid \forall c \in H\ \}.$$

By proposition 3.1.6, $aS(1_1) \otimes 1_2 b \in {}^{coH}P_1(H \otimes H)$. So $P_1(H \otimes H)'' \subset {}^{coH}P_1(H \otimes H)$. It follows from proposition 3.1.7 that $P_1(H \otimes H)'' = P_1(H \otimes H)' = {}^{coH}P_1(H \otimes H)$. ∎

Corollary 3.1.9. *Let H be a weak Hopf algebra and M be a left weak Hopf sub-bimodule in $P_1(H\otimes H)$. Then $M''=M'={}^{coH}M$.*

Proof. For any $aS(1_1)\otimes 1_2b\in M''\subset P_1(H\otimes H)''$, if

$$\rho_L(aS(1_1)\otimes 1_2b)=1'_11''_1\otimes 1'_2aS(1_1)\otimes 1_2b1''_2,$$

similar to corollary 3.1.8, there exists an element c' in H such that

$$c'=\varepsilon_t(S^{-1}(aS(1_1))1'_1)1_2b1'_2$$

and $aS(1_1)\otimes 1_2b=S(c'_1)\otimes c'_2$. We obtain

$$\begin{aligned}\rho_L(aS(1_1)\otimes 1_2b) &= 1'_11''_1\otimes 1'_2aS(1_1)\otimes 1_2b1''_2\\ &= 1'_11''_1\otimes 1'_2S(c'_1)\otimes c'_21''_2\\ &= 1'_11''_1\otimes 1'_2S(1_1)S(c'_1)\otimes c'_21_21''_2\\ &\overset{(2.7)}{=} 1'_1\otimes 1'_2S(c'_1)\otimes c'_2\\ &= 1'_1\otimes 1'_2aS(1_1)\otimes 1_2b.\end{aligned}$$

This implies that $aS(1_1)\otimes 1_2b\in {}^{coH}M$. So $M''\subset {}^{coH}M$. By proposition 3.1.6, $M''=M'={}^{coH}M$. ■

3.2. Right weak Hopf bimodules

In the subsection we will write down similar results in the case of right weak Hopf bimodules. Some necessary details will also be given for the sake of completeness.

Now consider:

$$\rho_R:\ A\otimes A\longrightarrow A\otimes A\otimes A,\ a\otimes b\longmapsto a_1\otimes b_1\otimes a_2b_2.$$

Lemma 3.2.1. *Let H be a weak Hopf algebra. Then there exists an equation:*

$$\rho_R(a\otimes b)=a_11_1\otimes S(1_2)b_1\otimes a_2b_2=\rho_R(a1_1\otimes S(1_2)b),\ \forall\, a,b\in H.$$

For any $a\otimes b\in H\otimes H$, define two maps

$$S_1(a\otimes b)=b_1\otimes ab_2,\qquad S_2(a\otimes b)=bS^{-1}(a_2)\otimes a_1.$$

Lemma 3.2.2. *Let H be a weak Hopf algebra. Then*

$$S_1S_2S_1=S_1,\qquad S_2S_1S_2=S_2.$$

Proof. For all $a\otimes b\in H\otimes H$, we compute as follows

$$\begin{aligned} S_1S_2S_1(a\otimes b) &= S_1S_2(b_1\otimes ab_2) = S_1(ab_3S^{-1}(b_2)\otimes b_1) \\ &= b_1\otimes ab_3S^{-1}(b_3)b_2) = b_1\otimes aS^{-1}(S(b_2)b_3S(b_4))) \\ &= b_1\otimes ab_2 = S_1(a\otimes b). \end{aligned}$$

Similarly, $S_2S_1S_2 = S_2$ holds. ■

By lemma 3.2.2, we can define two maps

$$P_3 = S_2S_1 : A\otimes A \longrightarrow A\otimes A,\ a\otimes b \longmapsto a1_1\otimes S(1_2)b\ ,$$
$$P_4 = S_1S_2 : A\otimes A \longrightarrow A\otimes A,\ a\otimes b \longmapsto 1_1a\otimes 1_2b.$$

It is clear that $P_3^2 = P_3$ and $P_4^2 = P_4$.

Lemma 3.2.3. *Let H be a weak Hopf algebra. Then*

(1) S_1 is a bijective map from $P_3(H\otimes H)$ to $S_1(H\otimes H)$ with the inverse S_2;

(2) S_2 is a bijective map from $P_4(H\otimes H)$ to $S_2(H\otimes H)$ with the inverse S_1.

Lemma 3.2.4. *Let H be a weak Hopf algebra. Then $P_3(H\otimes H)$ is a maximal right H-comodule with the structure map ρ_R defined as above.*

Proposition 3.2.5. *Let H be a weak Hopf algebra. Then $P_3(H\otimes H)$ is a maximal right Hopf bimodule with the following structure:*

$$\rho_R(a\otimes b) = a_1\otimes b_1\otimes a_2b_2,$$
$$c(a\otimes b) = ca\otimes b,$$
$$(a\otimes b)c = a\otimes bc.$$

for any $a\otimes b\in P_3(H\otimes H)$ and $c\in H$.

Proposition 3.2.6. *Let H be a weak Hopf algebra. Then*

$$\{\ S^{-1}(c_2)\otimes c_1 \mid \forall\ c\in H\ \} = P_3(H\otimes H)^{coH}.$$

Proof. For any $S^{-1}(c_2)\otimes c_1$, we have

$$S^{-1}(c_2)\otimes c_1 = S^{-1}(c_2)S^{-1}(1_2)\otimes 1_1c_1 = S^{-1}(c_2)1_1\otimes S(1_2)c_1\in P_3(H\otimes H).$$

And $S^{-1}(c_2)\otimes c_1$ is contained in $P_3(H\otimes H)^{coH}$ by the calculation:

$$\begin{aligned} \rho_R(S^{-1}(c_2)\otimes c_1) &= S^{-1}(c_3)_1\otimes c_1\otimes S^{-1}(c_3)_2c_2 \\ &= S^{-1}(c_4)\otimes c_1\otimes S^{-1}(c_3)c_2 \\ &= S^{-1}(c_4)\otimes c_1\otimes S^{-1}(S(c_2)c_3) \\ &\overset{(2.3)}{=} S^{-1}(c_2)\otimes c_11_1\otimes S^{-1}(S(1_2)) \\ &= S^{-1}(c_2)\otimes c_11_1\otimes 1_2 \\ &= (S^{-1}(c_2)\otimes c_1)1_1\otimes 1_2. \end{aligned}$$

Conversely, for any $a1_1 \otimes S(1_2)b \in P_3(H \otimes H)^{coH}$, we can get

$$\rho_R(a1_1 \otimes S(1_2)b) = a1_1 \otimes S(1_2)b1'_1 \otimes 1'_2.$$

Applying $S_2 \circ (\varepsilon \otimes 1 \otimes 1)$ to two sides of the above, we have

$$a1_1 \otimes S(1_2)b = S_2(\varepsilon(a1_1)S(1_2)b1'_1 \otimes 1'_2).$$

Note that

$$\begin{aligned}
& S_2(\varepsilon(a1_1)S(1_2)b1'_1 \otimes 1'_2) \\
= \ & S_2(\varepsilon(a1''_1S(1_1))S(1_2)S(1''_2)b1'_1 \otimes 1'_2) \\
= \ & S_2(\varepsilon(a1''_11_2)1_1S(1''_2)b1'_1 \otimes 1'_2) \\
= \ & S_2(\varepsilon_s(a1_1)S(1_2)b1'_1 \otimes 1'_2) \\
= \ & 1'_2S^{-1}((\varepsilon_s(a1_1)S(1_2)b1'_1)_2) \otimes (\varepsilon_s(a1_1)S(1_2)b1'_1)_1 \\
= \ & 1'_2S^{-1}((\varepsilon_s(a1_1)S(1_2)b)_21'_1) \otimes (\varepsilon_s(a1_1)S(1_2)b)_1 \\
= \ & 1'_2S^{-1}(1'_1)S^{-1}((\varepsilon_s(a1_1)S(1_2)b)_2) \otimes (\varepsilon_s(a1_1)S(1_2)b)_1 \\
= \ & S^{-1}((\varepsilon_s(a1_1)S(1_2)b)_2) \otimes (\varepsilon_s(a1_1)S(1_2)b)_1,
\end{aligned}$$

let $c' = \varepsilon_s(a1_1)S(1_2)b$, then

$$a1_1 \otimes S(1_2)b = S^{-1}(c'_2) \otimes c'_1 \in \{\ S^{-1}(c_2) \otimes c_1 \mid \forall c \in H\ \}. \qquad \blacksquare$$

Proposition 3.2.7. *Let H be a weak Hopf algebra and N be a right weak Hopf bimodule over H. Set*

$$\begin{aligned}
N' &= \{m \in N | m_{(-1)} \otimes m_0 = 1_1m \otimes \cdot 1_2\}, \\
N'' &= \{m \in N | m_{(-1)} \otimes m_0 = 1_1m1'_1 \otimes 1_21'_2\}.
\end{aligned}$$

Then $\quad N'' \supset N' \supset N^{coH}$.

Proof. By [2] the map

$$p : N \longrightarrow N^{coH},\ p(m) = m_0S(m_{(1)})$$

is a projection. The rest is similar to the proof of proposition 3.1.7. $\blacksquare$

Corollary 3.2.8. *Let H be a weak Hopf algebra. Then*

$$P_3(H \otimes H)'' = P_3(H \otimes H)' = P_3(H \otimes H)^{coH}.$$

Proof. Similar to corollary 3.1.8. $\blacksquare$

Corollary 3.1.9. *Let H be a weak Hopf algebra and N be a right weak Hopf sub-bimodule in $P_3(H \otimes H)$. Then* $N'' = N' = N^{coH}$.

Proof. Similar to corollary 3.1.9. ∎

3.3. Weak Hopf bimodules

In the subsection we will investigate weak Hopf bimodules coming from $H \otimes H$.

Lemma 3.3.1. *Let H be a weak Hopf algebra. Then* $P_1 \circ P_3 = P_3 \circ P_1$.

Proof. For all $a \otimes b \in H \otimes H$, we have

$$P_1 \circ P_3(a \otimes b) = a1_1 S(1'_1) \otimes 1'_2 S(1_2)b = aS(1'_1)1_1 \otimes S(1_2)1'_2 b = P_3 \circ P_1(a \otimes b). \quad ∎$$

Let $P = P_1 \circ P_3$. Then $P^2 = P$.

Lemma 3.3.2. *Let H be a weak Hopf algebra. Then $P(H \otimes H)$ is a maximal weak Hopf bimodule with the following structure:*

$$\rho_L(a \otimes b) = a_1 b_1 \otimes a_2 \otimes b_2,$$
$$\rho_R(a \otimes b) = a_1 \otimes b_1 \otimes a_2 b_2,$$
$$c(a \otimes b) = ca \otimes b,$$
$$(a \otimes b)c = a \otimes bc.$$

for any $a \otimes b \in P(H \otimes H)$ and $c \in H$.

Proof. We first prove that $P(H \otimes H)$ is a left weak Hopf bimodule with the structure map ρ_L. It only needs to check that $(P(H \otimes H), \rho_L)$ is a left H-subcomodule, namely, $\rho_L(P(H \otimes H)) \subset H \otimes P(H \otimes H)$. Note that $a \otimes b = P(a \otimes b)$ for all $a \otimes b \in P(H \otimes H)$. We have

$$\begin{aligned} \rho_L(a \otimes b) &= \rho_L(a1_1 S(1'_1) \otimes 1'_2 S(1_2)b) \\ &= (a1_1)_1 (S(1_2)b)_1 \otimes (a1_1)_2 S(1'_1) \otimes 1'_2 (S(1_2)b)_2 \\ &= a_1 b_1 \otimes a_2 1_1 S(1'_1) \otimes 1'_2 S(1_2) b_2 \in H \otimes P(H \otimes H). \end{aligned}$$

Similarly, $P(H \otimes H)$ is a right weak Hopf bimodule with the structure map ρ_R.

Next we check $(1 \otimes \rho_R) \circ \rho_L = (\rho_L \otimes 1) \circ \rho_R$. By Lemma 3.1.1 and 3.2.1,

$$\begin{aligned} &(1 \otimes \rho_R) \circ \rho_L(a \otimes b) \\ &= a_1 b_1 \otimes \rho_R(a_2 \otimes b_2) = a_1 b_1 \otimes a_2 \otimes b_2 \otimes a_3 b_3 \\ &= \rho_L(a_1 \otimes b_1) \otimes a_2 b_2 = (\rho_L \otimes 1) \circ \rho_R(a \otimes b). \end{aligned}$$

Now let M be any subspace of $H \otimes H$. Assume that M with the structure as stated is a weak Hopf bimodule. We prove $M \subset P(H \otimes H)$ as a weak Hopf bimodule. For any $c \otimes d \in M$, we have

$$\begin{aligned} c \otimes d &= (\varepsilon \otimes 1) \circ \rho_L(c \otimes d) \\ &= cS(1_1) \otimes 1_2 d \\ &= (1 \otimes \varepsilon) \circ \rho_R(cS(1_1) \otimes 1_2 d) \\ &= cS(1_1)1'_1 \otimes S(1'_2)1_2 d \in P(H \otimes H). \end{aligned}$$

Thus $M \subset P(H \otimes H)$. ■

Lemma 3.3.3. *Let M be a left weak Hopf sub-bimodule of $P_1(H \otimes H)$ and N be a right weak Hopf sub-bimodule of $P_3(H \otimes H)$. Then*

(1) If M_1 is a left weak Hopf sub-bimodule of M, then $P_3(M_1)$ is a left weak Hopf sub-bimodule of $P_3(M)$;

(2) If N_1 is a right weak Hopf sub-bimodule of N, then $P_1(N_1)$ is a right weak Hopf sub-bimodule of $P_1(N)$.

Proof. Assume that M_1 is a left weak Hopf sub-bimodule of M. Note that $aS(1) \otimes 1_2 b = a \otimes b$ for all $a \otimes b \in M_1$. We have

$$\begin{aligned} \rho_L(P_3(a \otimes b)) &= \rho_L(a1_1S(1'_1) \otimes 1'_2S(1_2)b) \\ &= a_1b_1 \otimes a_2 1_1 S(1'_1) \otimes 1'_2 S(1_2) b_2 \\ &= a_1 b_1 \otimes P_3(a_2 S(1'_1) \otimes 1'_2 b_2) \in H \otimes P_3(M_1) \subset H \otimes P_3(M). \end{aligned}$$

So $P_3(M_1)$ is a left weak Hopf sub-comodule of $P_3(M)$. The rest is easy.

The second statement can be similarly proved. ■

4. Bicovariant Differential Calculi

Let H be a weak Hopf algebra with bijective antipode S. Let (Γ, d) be a first order differential calculus over H. We first investigate the linearities of the map d.

Lemma 4.0.1. *Let (Γ, d) be a first order differential calculus over H. Then for $a, b \in H$, the following are equivalent:*

$$\begin{aligned} &(1)\ \ d(S(1_1))1_2 = 0, \\ &(2)\ \ S(1_1)d(1_2) = 0, \\ &(3)\ \ 1_1 1'_1 \otimes d(1_2)1'_2 = 0, \\ &(4)\ \ 1_1 1'_1 \otimes 1_2 d(1'_2) = 0, \\ &(5)\ \ 1_1 \otimes d(1_2) = 0, \end{aligned}$$

$$\begin{aligned} &(6)\quad d(H_t) = 0, \\ &(7)\quad d(\varepsilon_t(a)b) = \varepsilon_t(a)d(b), \\ &(8)\quad d(a\varepsilon_t(b)) = d(a)\varepsilon_t(b). \end{aligned}$$

Proof. Since $d(1) = d(1) + d(1)$ holds, then $d(1) = 0$. Note that $d(1) = d(S(1_1)1_2) = d(S(1_1))1_2 + S(1_1)d(1_2)$. We easily see that (1) is equivalent to (2).

For any $a, b \in H$, we have $d(\varepsilon_t(a)b) = \varepsilon_t(a)d(b) + d(\varepsilon_t(a))b$, which implies that $(6) \Longrightarrow (7)$ is clear. If (7) holds, we get $d(\varepsilon_t(a))b = 0$. So (6) is obtained by taking $b = 1$. Similarly, (6) is equivalent to (8).

$(2) \Longleftrightarrow (6)$: If (6) is true, so is (2) since $1_1 \otimes 1_2 \in H_s \otimes H_t$. Note that

$$\varepsilon_t(a)S(1_1)d(1_2) \overset{(2.7)}{=} S(1_1)d(1_2\varepsilon_t(a)) = S(1_1)d(1_2)\varepsilon_t(a) + S(1_1)1_2d(\varepsilon_t(a)).$$

If $S(1_1)d(1_2) = 0$, we have

$$0 = \varepsilon_t(a)S(1_1)d(1_2) = 0 + S(1_1)1_2d(\varepsilon_t(a)).$$

So $d(\varepsilon_t(a)) = 0$. Thus (6) holds.

$(5) \Longleftrightarrow (6)$: If $d(H_t) = 0$, then $1_1 \otimes d(1_2) = 1_1 \otimes d(\varepsilon_t(1_2)) = 0$. Conversely, if (5) is true, then

$$d(\varepsilon_t(a)) = \varepsilon(1_1 a)d(1_2) = (\varepsilon \otimes 1)[(1_1 \otimes d(1_2))(a \otimes 1)] = 0.$$

$(3) \Longleftrightarrow (6)$: It is easy to see that (6) implies (3). Assume that (3) holds, i.e., $1_1 1'_1 \otimes d(1_2)1'_2 = 0$. We have

$$0 = \varepsilon(1_1 1'_1)d(1_2)1'_2 = d(\varepsilon_t(1_1))1_2 = d(S(1_1))1_2,$$

which shows that (1) holds and so does (2). The statement (6) follows from (2).

Finally, we turn to check $(4) \Longleftrightarrow (6)$: It is obvious that (6) implies (4). Suppose that $1_1 1'_1 \otimes 1_2 d(1'_2) = 0$. We get

$$0 = \varepsilon(1_1 1'_1)1_2 d(1'_2) = \varepsilon_t(1_1)d(1_2) = S(1_1)d(1_2).$$

This means that (2) holds and so does (6). ∎

Lemma 4.0.2. (Γ, d) *be a first order differential calculus over* H. *Then for* $a, b \in H$, *the following are equivalent:*

$$\begin{aligned} &(1)\quad d(1_1)S(1_2) = 0, \\ &(2)\quad 1_1 d(S(1_2)) = 0, \\ &(3)\quad d(1_1)1'_1 \otimes 1_2 1'_2 = 0, \\ &(4)\quad 1_1 d(1'_1) \otimes 1_2 1'_2 = 0, \\ &(5)\quad d(1_1) \otimes 1_2 = 0, \\ &(6)\quad d(H_s) = 0, \\ &(7)\quad d(\varepsilon_s(a)b) = \varepsilon_s(a)d(b), \\ &(8)\quad d(a\varepsilon_s(b)) = d(a)\varepsilon_s(b). \end{aligned}$$

Proof. Note that $d(1) = d(1_1S(1_2)) = d(1_1)S(1_2) + 1_1d(S(1_2))$, $(1) \Longleftrightarrow (2)$ holds. Similar to Lemma 4.1, $(6) \Longleftrightarrow (7) \Longleftrightarrow (8)$ can be easily checked.

$(2) \Longleftrightarrow (6)$: If (6) is true, clearly, so is (2). If $1_1d(S(1_2)) = 0$, for any $a \in H$, we have

$$\varepsilon_s(a)1_1d(S(1_2)) \overset{(2.6)}{=} 1_1d(S(1_2)\varepsilon_s(a)) = 1_1d(S(1_2))\varepsilon_s(a) + 1_1S(1_2)d(\varepsilon_s(a)) = d(\varepsilon_s(a)),$$

Note that $\varepsilon_s(a)1_1d(S(1_2)) = 0$. So (6) holds.

$(5) \Longleftrightarrow (6)$: If $d(H_s) = 0$, then $d(1_1) \otimes 1_2 = 0$. Conversely, if (5) is true, then

$$d(\varepsilon_s(a)) = \varepsilon(a1_2)d(1_1) = (1 \otimes \varepsilon)[(1 \otimes a)(d(1_1) \otimes 1_2)] = 0.$$

$(3) \Longleftrightarrow (6)$: That (6) implies (3) is easy. If (3) holds, we have

$$0 = d(1_1)1'_1\varepsilon(1_21'_2) = d(1_1)S(1_2).$$

This means that (1) holds. So does (2). Thus (6) follows from (2).

Similarly, $(4) \Longleftrightarrow (6)$ holds. ∎

The statements (7) and (8) in Lemma 4.0.1 (4.0.2) are equivalent to say that the linear map d is left and right $H_t(H_s)$-linear, respectively. However, there really exists the linear map d that is neither H_t-linear nor H_s-linear.

Lemma 4.0.3. *Let H be a weak Hopf algebra. If $1\varepsilon(a) = \varepsilon_t(a)$ for all $a \in H$, then H is a Hopf algebra.*

Proof. For any $a \in H$, if $1\varepsilon(a) = \varepsilon_t(a)$, we have

$$\varepsilon(ab) = \varepsilon(a1_1)\varepsilon(1_2b) = \varepsilon(a\varepsilon_t(1_1))\varepsilon(1_2b) = \varepsilon(a1\varepsilon(1_1))\varepsilon(1_2b) = \varepsilon(a)\varepsilon(b).$$

This means that ε is a algebra map and so H is an ordinary Hopf algebra. ∎

Example 4.0.4. Let (H^2, D) be a first order differential calculus over H as in Subsection 2.3. Suppose that the map D is H_t-linear. We can get

$$D(\varepsilon_t(a)) = 1 \otimes \varepsilon_t(a) - \varepsilon_t(a) \otimes 1 = 0.$$

This means that $1\varepsilon(a) = \varepsilon_t(a)$. By Lemma 4.0.3, H is an ordinary Hopf algebra. Consequently, if H is a weak Hopf algebra (not a Hopf algebra), then the map D is not H_t-linear; Similarly, D is not H_s-linear.

In the sequel, we will see that Woronowicz's bicovariant differential calculi over weak Hopf algebras must be H_t-bilinear and H_s-bilinear. This is very different from bicovariant differential calculi over quantum groups.

4.1. Left-covariant first order differential calculi

In this subsection, we will discuss left-covariant first order differential calculus.

Definition 4.1.1. Let (Γ, d) be a first order differential calculus over H. (Γ, d) is called A-type if the linear map d is H_t-linear.

Definition 4.1.2. [23]. Let (Γ, d) be a first order differential calculus over H. (Γ, d) is left-covariant if $ad(b) = 0$ implies that $a_1 b_1 \otimes a_2 d(b_2) = 0$ for all $a, b \in \Gamma$.

Note that $1d(1) = 0$. Then $1_1 1'_1 \otimes 1_2 d(1'_2) = 0$. By Lemma 4.0.1, we have

Proposition 4.1.3. *Let (Γ, d) be a left-covariant first order differential calculus over H. Then (Γ, d) is A-type.*

Example 4.1.4. Let $P_1(H^2) = \{\ aS(1_1) \otimes 1_2 b \mid a \otimes b \in H^2\}$ and $D_1(a) = S(1_1) \otimes 1_2 a - aS(1_1) \otimes 1_2$ for all $a \in H$. Then $(P_1(H^2), D_1)$ is a left-covariant first order differential calculus over H and so A-type. Moreover, $(1 \otimes D_1) \circ \Delta = \rho_L \circ D_1$.

Proof. It is straightforward to check that $P_1(H^2)$ is a submodule of H^2. By [23] $(P_1(H^2), D_1)$ is a first order differential calculus over H. It needs to show that $aD_1(b) = 0$ implies that $a_1 b_1 \otimes a_2 D_1(b_2) = 0$ for all $a, b \in H$.

Assume that $aD_1(b) = 0$. We obtain $aS(1_1) \otimes 1_2 b = abS(1_1) \otimes 1_2$. So

$$\begin{aligned}
a_1 b_1 \otimes a_2 D_1(b_2) &= a_1 b_1 \otimes a_2 S(1_1) \otimes 1_2 b_2 - a_1 b_1 \otimes a_2 b_2 S(1_1) \otimes 1_2 \\
&= \rho_L(aS(1_1) \otimes 1_2 b) - a_1 b_1 1'_1 \otimes a_2 b_2 1'_2 S(1_1) \otimes 1_2 \\
&= \rho_L(aS(1_1) \otimes 1_2 b) - a_1 b_1 1'_1 \otimes a_2 b_2 S(1_1) \otimes 1_2 1'_2 \\
&= \rho_L(aS(1_1) \otimes 1_2 b) - \rho_L(abS(1_1) \otimes 1_2 1) \\
&= \rho_L(aS(1_1) \otimes 1_2 b - abS(1_1) \otimes 1_2 1) \\
&= 0.
\end{aligned}$$

■

Lemma 4.1.5. *Let H be a weak Hopf algebra. Then a first order differential calculus (Γ, d) is A-type if and only if there exists a sub-bimodule $N \subset P_1(H^2)$ such that $\Gamma = P_1(H^2)/N$ and $d = \pi \circ D_1$, where π is the canonical epimorphism $P_1(H^2) \longrightarrow \Gamma$.*

Proof. Here we take a method in [23]. If N is a submodule of $P_1(H^2)$, N is also a submodule of H^2. By [23] (Γ, d) is a first order differential calculus. Note that the map π is H-linear. For all $a, b \in H$,

$$\begin{aligned}
\pi \circ D_1(ab) &= \pi[(D_1(a)b + aD_1(b))] \\
&= [\pi \circ (D_1(a))]b + a[\pi \circ D_1(b)].
\end{aligned}$$

The map $\pi \circ D_1$ is surjective. For all $a, b \in H$,

$$\pi \circ D_1(\varepsilon_t(a)b) = \pi[D_1(\varepsilon_t(a)b)] = \pi[\varepsilon_t(a)D_1(b)] = \varepsilon_t(a)[\pi \circ D_1(b)].$$

So (Γ, d) is A-type.

Conversely, assume that (Γ, d) is an A-type first order differential calculus. Namely, d is H_t-linear. For any $aS(1_1) \otimes 1_2 b \in P_1(H \otimes H)$ and $c \in H$, define a map

$$\pi' : P_1(H \otimes H) \longrightarrow \Gamma,\ aS(1_1) \otimes 1_2 b \longmapsto aS(1_1)d(1_2 b).$$

The H-linearity of the map π' follows from the following:

$$\begin{aligned} & c(aS(1_1)d(1_2 b)) = caS(1_1)d(1_2 b), \\ & aS(1_1)d(1_2 bc) = aS(1_1)d(1_2 bc) \\ = \ & aS(1_1)d(1_2)bc + aS(1_1)1_2 d(b)c + aS(1_1)1_2 b)d(c) \\ = \ & aS(1_1)1_2 d(b)c + aS(1_1)1_2 b)d(c) \\ = \ & aS(1_1)1_2 d(b)c + aS(1_1)1_2 bd(c) \\ = \ & aS(1_1)1_2 d(b)c = aS(1_1)d(1_2 b)c. \end{aligned}$$

If $m \in \Gamma$, then there exist a and b in H such that $m = ad(b)$. Note that $aS(1_1) \otimes 1_2 b - abS(1_1) \otimes 1_2 \in P_1(H \otimes H)$. We have

$$\begin{aligned} & \pi'(aS(1_1) \otimes 1_2 b - abS(1_1) \otimes 1_2) \\ = \ & \pi'(aS(1_1) \otimes 1_2 b) - \pi'(abS(1_1) \otimes 1_2) \\ = \ & aS(1_1)d(1_2 b) - abS(1_1)d(1_2) \\ = \ & aS(1_1)1_2 d(b) - 0 = ad(b) = m. \end{aligned}$$

So π' is surjective. Let $ker\pi'$ be the kernel of π'. As an H-bimodule, we have $P_1(H \otimes H)/ker\pi' \cong \Gamma$. For any $a \in H$,

$$\begin{aligned} \pi' \circ D_1(a) &= \pi'(S(1_1) \otimes 1_2 a - aS(1_1) \otimes 1_2) \\ &= S(1_1)d(1_2 a) - aS(1_1)d(1_2) = d(a). \end{aligned}$$

■

Lemma 4.1.6. *Let H be a weak Hopf algebra. Then $P_1(H^2)$ is a left weak Hopf sub-bimodule of $P_1(H \otimes H)$.*

Proof. For any $aS(1_1) \otimes 1_2 b \in P_1(H^2)$, we have

$$\begin{aligned} & (1 \otimes m) \circ \rho_L(aS(1_1) \otimes 1_2 b) \\ = \ & (1 \otimes m)((aS(1_1))_1 (1_2 b)_1 \otimes (aS(1_1))_2 \otimes (1_2 b)_2) \\ = \ & (aS(1_1))_1 (1_2 b)_1 \otimes (aS(1_1))_2 (1_2 b)_2 \\ = \ & (aS(1_1))_1 (1_2 b)_1 \otimes (aS(1_1))_2 (1_2 b)_2 \\ = \ & \Delta(aS(1_1)1_2 b) = 0. \end{aligned}$$

This means $\rho_L(P_1(H^2)) \subset H \otimes P_1(H^2)$. The rest follows Proposition 3.1.6. ■

Theorem 4.1.7. *Let H be a weak Hopf algebra. Then the first order differential calculus (Γ, d) is left-covariant if and only if there exists a left weak Hopf sub-bimodule $N \subset P_1(H^2)$ such that $\Gamma = P_1(H^2)/N$ and $d = \pi \circ D_1$, where π is the canonical epimorphism $P_1(H^2) \longrightarrow \Gamma$.*

Proof. Here the notations is the same as in Lemma 4.1.5. First assume that N is a left weak Hopf sub-bimodule of $P_1(H^2)$. Then the map π is H-linear and left H-colinear. By Example 4.1.4, $(1 \otimes D_1) \circ \Delta = \rho_L \circ D_1$. Since Lemma 4.1.5 holds, it remains to prove that $a\pi[D_1(b)] = 0$ implies $a_1 b_1 \otimes a_2 \pi[D_1(b_2)] = 0$ for all $a, b \in H$. If $a\pi[D_1(b)] = 0$, we can get

$$\begin{aligned} a_1 b_1 \otimes a_2 \pi[D_1(b_2)] &= (a_1 \otimes a_2)(b_1 \otimes \pi(D_1(b_2))) \\ &= (a_1 \otimes a_2)[(1 \otimes \pi)(b_1 \otimes D_1(b_2))] \\ &= (a_1 \otimes a_2)[(1 \otimes \pi)(\rho_L \circ D_1(b))] \\ &= (a_1 \otimes a_2)[(1 \otimes \pi) \circ \rho_L(D_1(b))] \\ &= \Delta(a)[\rho_L \circ \pi((D_1(b))] \\ &= \rho_L[a(\pi \circ D_1(b))] = 0. \end{aligned}$$

Conversely, suppose that (Γ, d) is left-covariant. For all $aS(1_1) \otimes 1_2 b \in ker\pi'$, we have $a_1 b_1 \otimes a_2 d(b_2) = 0$. By Lemma 4.1.5, $a_1 b_1 \otimes a_2 \pi'(S(1_1) \otimes 1_2 b_2) = a_1 b_1 \otimes a_2 \pi'(b_2 S(1_1) \otimes 1_2)$. Since $cd(1) = 0$ for any $c \in H$, then $cS(1_1) \otimes 1_2 \in ker\pi'$. So

$$(1 \otimes \pi') \circ \rho_L(aS(1_1) \otimes 1_2 b) = a_1 b_1 \otimes a_2 \pi'(b_2 S(1_1) \otimes 1_2) = 0.$$

This means that $\rho_L(ker\pi') \subset H \otimes ker\pi'$. The other axioms on a left weak Hopf bimodule are easily checked. ∎

Proposition 4.1.8. *Let (Γ, d) be a left-covariant first order differential calculus over H. Then there uniquely exists a linear map $\rho_L : \Gamma \longrightarrow H \otimes \Gamma$ such that (Γ, ρ_L) is a left weak Hopf bimodule. Moreover, $(1 \otimes d) \circ \Delta = \rho_L \circ d$.*

Proof. Straightforward. ∎

Lemma 4.1.9. *Let H be a weak Hopf algebra. Then*

$$\{\ S(c_1) \otimes c_2 \mid \forall\ c \in ker\varepsilon_s\ \} = {}^{coH}P_1(H^2).$$

Proof. For any $c \in ker\varepsilon_s$, we have $S(c_1)c_2 = \varepsilon_s(c) = 0$ and

$$S(c_1) \otimes c_2 = S(c_1)S(1_1) \otimes 1_2 c_2 \in P_1(H^2) \subset P_1(H \otimes H).$$

It follows from Lemma 4.1.6 and Proposition 3.1.6 that $S(c_1) \otimes c_2 \in {}^{coH}P_1(H^2)$.

Conversely, for any $S(c_1) \otimes c_2 \in {}^{coH}P_1(H^2)$, we have $0 = S(c_1)c_2 = \varepsilon_s(c)$. ∎

Lemma 4.1.10. *Let R be a right ideal of H such that R is contained in $ker\varepsilon_s$. Let $N = R_2(H \otimes R) = \{ \ aS(b_1) \otimes b_2 | \ a \otimes b \in H \otimes R \ \}$. Then N is a left weak Hopf sub-bimodule of $P_1(H^2)$. Moreover, ${}^{coH}N = \{ \ S(b_1) \otimes b_2 | \ b \in R \ \}$.*

Proof. We first check that N is a subspace of $P_1(H^2)$. For any $a \otimes b \in H \otimes R$, we have $a \otimes b \in H \otimes ker\varepsilon_s$ and so $a\varepsilon_s(b) = 0$. Since $aS(b_1) \otimes b_2 = aS(b_1)S(1_1) \otimes 1_2 b_2$ and $aS(b_1)b_2 = a\varepsilon_s(b) = 0$, then $N \subseteq P_1(H^2)$.

Next we show that N is a sub-bimodule of $P_1(H^2)$. For any $c \in H$ and $aS(b_1) \otimes b_2 \in N$, we have

$$c(aS(b_1) \otimes b_2) = caS(b_1) \otimes b_2, \quad (aS(b_1) \otimes b_2)c = aS(b_1) \otimes b_2 c.$$

Note that $a \otimes b \in H \otimes R$. we get $ca \otimes b \in H \otimes R$ and $caS(b_1) \otimes b_2 \in N$. Since R is a right ideal, $ac_1 \otimes bc_2$ is an element in $H \otimes R$. Now compute as follows:

$$ac_1 S(b_1 c_2) \otimes b_2 c_3 = a\varepsilon_t(c_1)S(b_1) \otimes b_2 c_2 = aS(1_1)S(b_1) \otimes b_2 1_2 c = aS(b_1) \otimes b_2 c.$$

So $(aS(b_1) \otimes b_2)c = aS(b_1) \otimes b_2 c \in N$. Thus N is a sub-bimodule.

Now we verify that N is a sub-comodule of $P_1(H^2)$. For any $a \otimes b \in H \otimes R$, we have $a_1 \otimes a_2 \otimes b \in H \otimes H \otimes R$ and

$$\begin{aligned} \rho_L(aS(b_1) \otimes b_2) &= a_1 S(b_1)_1 b_2 \otimes a_2 S(b_1)_2 \otimes b_3 \\ &= a_1 S(b_2) b_3 \otimes a_2 S(b_1) \otimes b_4 \\ &= a_1 \varepsilon_s(b_2) \otimes a_2 S(b_1) \otimes b_3 \\ &= a_1 S(1_2) \otimes a_2 S(b_1 1_1) \otimes b_2 \\ &= a_1 \otimes a_2 S(b_1) \otimes b_2. \end{aligned}$$

So $\rho_L(N) \in H \otimes N$. The other axioms on left weak Hopf modules are straightforward.

Finally we prove that ${}^{coH}N = \{ \ S(b_1) \otimes b_2 | \ b \in R \ \}$. Clearly, $\{ \ S(b_1) \otimes b_2 | \ b \in R \ \} \subset N$. Using Lemma 3.1.6 we get $\{ \ S(b_1) \otimes b_2 | \ b \in R \ \} \subset {}^{coH}N$. Conversely, by Lemma 3.1.6, for any $S(c_1) \otimes c_2 \in {}^{coH}N$, we need to check that c lies in R. Note that there exists $a \in H$ and $b \in R$ such that $S(c_1) \otimes c_2 = aS(b_1) \otimes b_2$. We have

$$c = \varepsilon(aS(b_1))b_2 = \varepsilon(\varepsilon_s(b_1)S^{-1}(a))b_2 = \varepsilon(1_1 S^{-1}(a))b1_2 = b\varepsilon_t(S^{-1}(a)).$$

Then $c \in R$ since R is a right ideal in H. ■

Lemma 4.1.11. *Let N be a left weak Hopf sub-bimodule of $P_1(H^2)$. Then there exists a right ideal R in H, $R \subset ker\varepsilon_s$ such that*

$$N = R_2(H \otimes R) = \{ \ aS(b_1) \otimes b_2 | \ a \otimes b \in H \otimes R \ \}.$$

Proof. Note that ${}^{coH}N \subset {}^{coH}P_1(H^2)$. Define a subspace

$$R := \{ \ c \in ker\varepsilon_s \ | S(c_1) \otimes c_2 \in {}^{coH}N \ \}.$$

We first show that R is a right ideal in H. Clearly, R is a subspace of $ker\varepsilon_s$. Since ${}^{coH}N$ is a H-sub-comodule, we have for any $c \in R$ and $b \in H$,

$$\begin{aligned}
\rho_L(S((cb)_1) \otimes (cb)_2) &= \rho_L(S(b_1)S(c_1) \otimes c_2 b_2)) \\
&= S(b_2)S(c_1)_1 c_2 b_3 \otimes S(b_1)S(c_1)_2 \otimes c_3 b_4 \\
&= S(b_2)b_3 \otimes S(b_1)S(c_1) \otimes c_2 b_4 \\
&= S(1_2) \otimes S(b_1 1_1)S(c_1) \otimes c_2 b_3 \\
&= 1_1 \otimes 1_2 S(b_1)S(c_1) \otimes c_2 b_3 \\
&= 1_1 \otimes 1_2 S((cb)_1) \otimes (cb)_2,
\end{aligned}$$

which means $S((cb)_1) \otimes (cb)_2 \in {}^{coH}N$. So $cb \in R$.

Next we check that $N \subset R_2(H \otimes R)$. By [2] if M is a left-left weak Hopf module, there exists a projection $p : M \longrightarrow {}^{coH}M,\ p(m) = S(m_{(-1)})m_{(0)}$. Additionally, for any $m \in M$, we have

$$m = \varepsilon(1_1 m_{(-1)})1_2 m_{(0)} = \varepsilon(1_1 m_{(-1)})1_2 m_{(0)} = m_{(-1)}S(m_{(0)_{(-1)}})m_{(0)_{(0)}}.$$

So there exists $a \in H$ and $m' \in {}^{coH}M$ such that $m = am'$. Thus $N \subset R_2(H \otimes R)$.

Now we verify that $R_2(H \otimes R) \subset N$. Since N is a left weak Hopf sub-bimodule of $P_1(H^2)$ and ${}^{coH}N$ is a subcomodule of N, we have $S(c_1) \otimes c_2 \in N$. So $aS(c_1) \otimes c_2 \in N$ for all $c \in R$ and $a \in H$. Therefore, $R_2(H \otimes R)$ is contained in N. ■

Proposition 4.1.12. *Let H be a weak Hopf algebra. Then N is a left weak Hopf sub-bimodule of $P_1(H^2)$ if and only if there exists a right ideal R in H, $R \subset ker\varepsilon_s$ such that $N = R_2(H \otimes R) = \{aS(b_1) \otimes b_2 |\ a \otimes b \in H \otimes R\}$.*

Proof. Following Lemma 4.1.10 and 4.1.11.

Similar to Theorem 1.5 in [23], all left-covariant differential calculi over weak Hopf algebras are characterized as follows:

Theorem 4.1.13. *Let H be a weak Hopf algebra. Then the first order differential calculus (Γ, d) is left-covariant if and only if there exists a right ideal R in H, $R \subset ker\varepsilon_s$ such that $\Gamma = P_1(H^2)/N$ and $d = \pi \circ D_1$, where $N = R_2(H \otimes R)$ and π is the canonical epimorphism $P_1(H^2) \longrightarrow \Gamma$.*

Proof. Following Theorem 4.1.7 and Proposition 4.1.12. ■

4.2. Right-covariant first order differential calculi

Here we will consider right-covariant first order differential calculi by a similar way. Some details of some proofs will also by given for the sake of completeness.

Definition 4.2.1. Let (Γ, d) be a first order differential calculus over H. (Γ, d) is called B-type if the linear map d is H_s-linear.

Definition 4.2.2. [23]. Let (Γ, d) be a first order differential calculus over H. (Γ, d) is called right-covariant if $ad(b) = 0$ implies that $a_1 d(b_1) \otimes a_2 b_2 = 0$.

Proposition 4.2.3. *Let (Γ, d) be a left-covariant first order differential calculus over H. Then (Γ, d) is B-type.*

Proof. The proof follows from Lemma 4.0.2. ∎

Example 4.2.4. Let $P_3(H^2) = \{\ a1_1 \otimes S(1_2)b \mid a \otimes b \in H^2\}$ and $D_2(a) = 1_1 \otimes S(1_2)a - a1_1 \otimes S(1_2)$, for any $a \in H$. Then $(P_3(H^2), D_2)$ is a right-covariant first order differential calculus over H and so B-type. Moreover, $(D_2 \otimes 1) \circ \Delta = \rho_R \circ D_2$.

Proof. Similar to Example 4.1.4. ∎

Lemma 4.2.5. *Let H be a weak Hopf algebra. Then a first order differential calculus (Γ, d) is B-type if and only if there exists a sub-bimodule $N \subset P_3(H^2)$ such that $\Gamma = P_3(H^2)/N$ and $d = \pi \circ D_2$, where π is the canonical epimorphism $P_3(H^2) \longrightarrow \Gamma$.*

Proof. If N is a submodule of $P_3(H^2)$, then by [23] (Γ, d) is a first order differential calculus. Note that the map π is H-linear. We have for all $a, b \in H$,

$$\pi \circ D_2(ab) \;\; = \;\; \pi[(D_2(a)b + aD_2(b))] = [\pi \circ (D_2(a))]b + a[\pi \circ D_2(b)].$$

The map $\pi \circ D_2$ is surjective. For all $a, b \in H$,

$$\pi \circ D_1(\varepsilon_s(a)b) = \pi[D_2(\varepsilon_s(a)b)] = \pi[\varepsilon_s(a)D_1(b)] = \varepsilon_s(a)[\pi \circ D_2(b)].$$

So (Γ, d) is B-type.

Conversely, assume that (Γ, d) is a B-type first order differential calculus. Namely, d is H_s-linear. For any $a1_1 \otimes S(1_2)b \in P_3(H \otimes H)$ and $c \in H$, define a map

$$\pi' : P_1(H \otimes H) \longrightarrow \Gamma,\ a1_1 \otimes S(1_2)b \longmapsto a1_1 d(S(1_2)b).$$

Similar to Lemma 4.1.5, the map π' is left and right H-linear. If $m \in \Gamma$, then there exist a and b in H such that $m = ad(b)$. Note that $aS(1_1) \otimes 1_2 b - abS(1_1) \otimes 1_2 \in P_1(H \otimes H)$. We have

$$\pi'(aS(1_1) \otimes 1_2 b - abS(1_1) \otimes 1_2) = ad(b) = m.$$

So π' is surjective. Let $ker\pi'$ be the kernel of π'. Then $P_1(H \otimes H)/ker\pi' \cong \Gamma$ as an H-bimodule. Thus

$$\pi' \circ D_1(a) = \pi'(S(1_1) \otimes 1_2 a - aS(1_1) \otimes 1_2) = d(a). \qquad ∎$$

Lemma 4.2.6. *Let H be a weak Hopf algebra. Then $P_3(H^2)$ is a left weak Hopf sub-bimodule of $P_3(H \otimes H)$.*

Proof. For any $a1_1 \otimes S(1_2)b \in P_3(H^2)$, we have

$$\begin{aligned} & (m \otimes 1) \circ \rho_R(a1_1 \otimes S(1_2)b) \\ = & (m \otimes 1)((a1_1)_1 \otimes (S(1_2)b)_1 \otimes (a1_1)_2(S(1_2)b)_2) \\ = & \Delta(a1_1 S(1_2)b) = 0. \end{aligned}$$

Thus $\rho_R(P_3(H^2)) \subset H \otimes P_3(H^2)$. ∎

Theorem 4.2.7. *Let H be a weak Hopf algebra. Then a first order differential calculus (Γ, d) is right-covariant if and only if there exists a right weak Hopf sub-bimodule $N \subset P_1(H^2)$ such that $\Gamma = P_3(H^2)/N$ and $d = \pi \circ D_2$, where π is the canonical epimorphism $P_3(H^2) \longrightarrow \Gamma$.*

Proof. Here the notations are the same as in Lemma 4.2.5. First assume that N is a right weak Hopf sub-bimodule of $P_3(H^2)$. Then the map π is H-linear and right H-colinear. By Example 4.2.4 and Lemma 4.2.5, we only need to prove that $a\pi[D_2(b)] = 0$ implies $a_1\pi[D_2(b_1)] \otimes a_2 b_2 = 0$ for $a, b \in H$. If $a\pi[D_2(b)] = 0$, then

$$\begin{aligned} a_1\pi[D_2(b_1)] \otimes a_2 b_2 &= \Delta(a)[(\pi \otimes 1)(D_2(b_1) \otimes b_2)] \\ &= (a_1 \otimes a_2)[(\pi \otimes 1)(\rho_R \circ D_2(b))] \\ &= \Delta(a)[\rho_R \circ \pi((D_2(b))] \\ &= \rho_R[a(\pi \circ D_2(b))] = 0. \end{aligned}$$

Assume that (Γ, d) is right-covariant. We have $a_1 d(b_1) \otimes a_2 b_2 = 0$ for any $a1_1 \otimes 1S(_2)b \in ker\pi'$. By Lemma 4.1.5,

$$a_1\pi'(1_1 \otimes S(1_2)b_1) \otimes a_2 b_2 = a_1\pi'(b_1 1_1 \otimes S(1_2)) \otimes a_2 b_2.$$

Note that $cd(1) = 0$ for any $c \in H$. We have $c1_1 \otimes S(1_2) \in ker\pi'$. So we get

$$(\pi' \otimes 1) \circ \rho_R(a1_1 \otimes S(1_2)b) = a_1\pi'(b_1 1_1 \otimes S(1_2)) \otimes a_2 b_2 = 0.$$

So $\rho_R(ker\pi') \subset ker\pi' \otimes H$. The rest is easy. ∎

Proposition 4.2.8. *Let (Γ, d) be a right-covariant first order differential calculus over H. Then there uniquely exists a linear map $\rho_R : \Gamma \longrightarrow \Gamma \otimes H$ such that (Γ, ρ_L) is a right weak Hopf bimodule. Moreover, $(d \otimes 1) \circ \Delta = \rho_R \circ d$.*

Proof. Straightforward.

Lemma 4.2.9. *Let H be a weak Hopf algebra. Then*

$$\{ \ S^{-1}(c_2) \otimes c_1 \mid \forall \ c \in ker\varepsilon_s \ \} = P_3(H^2)^{coH}.$$

Proof. For any $c \in ker\varepsilon_s$, we have $S^{-1}(c_2)c_1 = S^{-1}(\varepsilon_s(c)) = 0$ and

$$S^{-1}(c_2) \otimes c_1 = S^{-1}(c_2)S^{-1}(1_2) \otimes 1_1c_1 = S^{-1}(c_2)1_1 \otimes S(1_2)c_1.$$

Lemma 4.2.6 and Proposition 3.2.6 imply that $S^{-1}(c_2) \otimes c_1 \in P_3(H^2)^{coH}$.

For any $S^{-1}(c_2) \otimes c_1 \in P_3(H^2)^{coH}$, it is easy to see that $\varepsilon_s(c) = 0$. ∎

Lemma 4.2.10. *Let R be a right ideal of H such that R is contained in $ker\varepsilon_s$. Let $N = S_2(R \otimes H) = \{\ bS^{-1}(a_2) \otimes a_1|\ a \otimes b \in R \otimes H\ \}$. Then N is a right weak Hopf sub-bimodule of $P_3(H^2)$. Moreover, $N^{coH} = \{S^{-1}(a_2) \otimes a_1|\ a \in R\}$.*

Proof. We first check that N is a subspace of $P_3(H^2)$. For any $a \otimes b \in R \otimes H$, $a \otimes S(b) \in R \otimes H$. So $\varepsilon_s(a)S(b) = 0$. Since $bS^{-1}(a_2) \otimes a_1 = bS^{-1}(a_2)1_1 \otimes S(1_2)a_1$ and $bS^{-1}(a_2)a_1 = S^{-1}(S(a_1)a_2S(b)) = S^{-1}(\varepsilon_s(a)S(b)) = 0$, we get $N \subseteq P_1(H^2)$.

Next we prove that N is a sub-bimodule of $P_1(H^2)$. In fact

$$c(bS^{-1}(a_2) \otimes a_1) = cbS^{-1}(a_2) \otimes a_1, \quad (bS^{-1}(a_2) \otimes a_1)c = bS^{-1}(a_2) \otimes a_1c,$$

where $c \in H$ and $bS^{-1}(a_2) \otimes a_1 \in N$. Note that $a \otimes b \in R \otimes H$. We have that $a \otimes cb \in H \otimes R$ and $cbS^{-1}(a_2) \otimes a_1 \in N$. We easily get that $ac_1 \otimes bc_2$ lies in $R \otimes H$ since R is a right ideal. Then $bS^{-1}(a_2) \otimes a_1c$ is an element in N since

$$bc_3S^{-1}(a_2c_2) \otimes a_1c_1 = bS^{-1}(a_2\varepsilon_t(c_2)) \otimes a_1c_1 = bS^{-1}(a_2) \otimes a_1c.$$

Now we verify that N is a sub-comodule of $P_1(H^2)$. For any $a \otimes b \in R \otimes H$, we have $a \otimes b_1 \otimes b_2 \in R \otimes H \otimes H$ and

$$\begin{aligned}
\rho_R(bS^{-1}(a_2) \otimes a_1) &= b_1S^{-1}(a_3)_1 \otimes a_1 \otimes b_2S^{-1}(a_3)_2a_2 \\
&= b_1S^{-1}(a_4) \otimes a_1 \otimes b_2S^{-1}(a_3)a_2 \\
&= b_1S^{-1}(a_3) \otimes a_1 \otimes b_2S^{-1}(\varepsilon_s(a_2)) \\
&= b_1S^{-1}(a_21_2) \otimes a_1 \otimes b_2S^{-1}(1_1) \\
&= b_1S^{-1}(a_2) \otimes a_1 \otimes b_2.
\end{aligned}$$

So $\rho_R(N) \subset N \otimes H$. The rest is similar to Lemma 4.1.10. ∎

Lemma 4.2.11. *Let N be a right weak Hopf sub-bimodule of $P_1(H^2)$. Then there exists a right ideal R in H, $R \subset ker\varepsilon_s$ such that*

$$N = S_2(R \otimes H) = \{\ bS^{-1}(a_2) \otimes a_1|\ a \otimes b \in R \otimes H\ \}.$$

Proof. Note that $N^{coH} \subset P_3(H^2)^{coH}$. Define a space

$$R = \{\ c \in ker\varepsilon_s\ |S^{-1}(c_2) \otimes c_1 \in N^{coH}\ \}.$$

We first check that R is a right ideal in H. Clearly, R is a subspace of $ker\varepsilon_s$. Note that N^{coH} is a H-sub-comodule. We have for any $c \in R$ and $b \in H$,

$$\begin{aligned}
&\rho_R(S^{-1}((cb)_2) \otimes (cb)_1) \\
=\ &\rho_R(S^{-1}(b_2)S^{-1}(c_2) \otimes c_1 b_1)) \\
=\ &S^{-1}(b_4)S^{-1}(c_4) \otimes c_1 b_1 \otimes S^{-1}(b_3)S^{-1}(c_3)c_2 b_2 \\
=\ &S^{-1}(b_4)S^{-1}(c_3) \otimes c_1 b_1 \otimes S^{-1}(\varepsilon_s(c_2)b_3)b_2 \\
=\ &S^{-1}(b_4)S^{-1}(c_3) \otimes c_1 1_1 b_1 \otimes S^{-1}(S(1_2)b_3)b_2 \\
=\ &S^{-1}(b_4)S^{-1}(c_2) \otimes c_1 b_1 \otimes S^{-1}(S(b_2)b_3) \\
=\ &S^{-1}(b_2)S^{-1}(c_2) \otimes c_1 b_1 1_1 \otimes 1_2 \\
=\ &S^{-1}(cb)_2) \otimes (cb)_1 1_1 \otimes 1_2,
\end{aligned}$$

So $S^{-1}((cb)_2) \otimes (cb)_1 \in N^{coH}$. Thus $cb \in R$.

Now we show that $N \subset R_2(H \otimes R)$. By [2] if M is a right-right weak Hopf module, there exists a projection $p : N \longrightarrow N^{coH},\ p(n) = n_{(0)}S(n_{(1)})$. Additionally, for any $n \in N$, we have

$$n = \varepsilon(n_{(1)}1_2)n_{(0)}1_1 = \varepsilon(1_1 n_{(-1)})1_2 n_{(0)} = n_{(0)_{(0)}}S(n_{(0)_{(1)}})n_{(1)},$$

so there exists $a \in H$ and $n' \in N^{coH}$ such that $n = an'$. So $N \subset S_2(R \otimes H)$ is clear.

Finally, we verify that $S_2(R \otimes H) \subset N$. Since N is a left weak Hopf sub-bimodule of $P_3(H^2)$ and N^{coH} is a subcomodule of N, we have $S^{-1}(c_2) \otimes c_1 \in N$ for any $c \in R$, and $a \in H$. So $aS^{-1}(c_2) \otimes c_1 \in N$. Therefore, $S_2(R \otimes H)$ is contained in N. ■

Proposition 4.2.12. *Let H be a weak Hopf algebra. Then N is a left weak Hopf sub-bimodule of $P_3(H^2)$ if and only if there exists a right ideal R in H, $R \subset ker\varepsilon_s$ such that $N = S_2(R \otimes H) = \{\ bS^{-1}(a_2) \otimes a_1 |\ a \otimes b \in R \otimes H\ \}$.*

Proof. Following Lemma 4.2.10 and 4.2.11. ■

Using Theorem 4.2.7 and Proposition 4.2.12, Theorem 1.6 of Woronowicz (1989) has been generalized to the case of weak Hopf algebra.

Theorem 4.2.13. *Let H be a weak Hopf algebra. Then a first order differential calculus (Γ, d) is right-covariant if and only if there exists a right ideal R in H, $R \subset ker\varepsilon_s$ such that $\Gamma = P_3(H^2)/N$ and $d = \pi \circ D_2$, where $N = S_2(R \otimes H)$ and π is the canonical epimorphism $P_3(H^2) \longrightarrow \Gamma$.*

4.3. Bicovariant differential calculi

In this subsection, we will investigate bicovariant differential calculus by Woronowicz's fundamental method.

Definition 4.3.1. Let (Γ, d) be a first order differential calculus over H. (Γ, d) is called C-type if it is H_t and H_s-linear.

Definition 4.3.2. [23]. Let (Γ, d) be a first order differential calculus over H. (Γ, d) is called bicovariant if it is left-covariant and right-covariant.

Proposition 4.3.3. Let (Γ, d) be a bicovariant first order differential calculus over H. Then (Γ, d) is C-type.

Proof. Following Proposition 4.1.3 and 4.2.3. ∎

Example 4.3.4. Let $P(H^2) = \{\ a1_1S(1'_1)\otimes 1'_2S(1_2)b \mid a\otimes b \in H^2\}$ and $D_3(a) = 1_1S(1'_1)\otimes 1'_2S(1_2)a - a1_1S(1'_1)\otimes 1'_2S(1_2)$ for all $a \in H$. Then $(P(H^2), D_3)$ is a bicovariant first order differential calculus over H and so C-type.

Proof. Following Lemma 3.3.2, Example 4.1.4 and Example 4.2.4. ∎

Lemma 4.3.5. *Let H be a weak Hopf algebra. Then a first order differential calculus (Γ, d) is C-type if and only if there exists a sub-bimodule $N \subset P(H^2)$ such that $\Gamma = P(H^2)/N$ and $d = \pi \circ D_3$, where π is the canonical epimorphism $P(H^2) \longrightarrow \Gamma$.*

Proof. If N is a submodule of $P_3(H^2)$, then by [23] (Γ, d) is a first order differential calculus. Note that the map π is H-linear. For all $a, b \in H$,

$$\begin{aligned}\pi \circ D_2(ab) &= \pi[(D_2(a)b + aD_2(b))]\\ &= [\pi \circ (D_2(a))]b + a[\pi \circ D_2(b)].\end{aligned}$$

The map $\pi \circ D_2$ is surjective. For all $a, b \in H$,

$$\pi \circ D_1(\varepsilon_s(a)b) = \pi[D_2(\varepsilon_s(a)b)] = \pi[\varepsilon_s(a)D_1(b)] = \varepsilon_s(a)[\pi \circ D_2(b)].$$

So (Γ, d) is B-type.

Assume that (Γ, d) is a B-type first order differential calculus. Then d is H_s-linear. For any $a1_1 \otimes S(1_2)b \in P_3(H \otimes H)$ and $c \in H$, define the map

$$\pi' : P_1(H \otimes H) \longrightarrow \Gamma,\ a1_1 \otimes S(1_2)b \longmapsto a1_1d(S(1_2)b).$$

Similar to Lemma 4.1.5, the map π' is left and right H-linear. If $m \in \Gamma$, then there exist a and b in H such that $m = ad(b)$. It is easy to see that $aS(1_1)\otimes 1_2b - abS(1_1)\otimes 1_2 \in P_1(H \otimes H)$ and $\pi'(aS(1_1) \otimes 1_2b - abS(1_1) \otimes 1_2) = ad(b) = m$. So π' is surjective. Let $ker\pi'$ be the kernel of π'. Then $P_1(H \otimes H)/ker\pi' \cong \Gamma$ as H-bimodules. For any $a \in H$, we obtain

$$\pi' \circ D_1(a) = \pi'(S(1_1) \otimes 1_2a - aS(1_1) \otimes 1_2) = d(a).$$ ∎

Lemma 4.3.6. *Let H be a weak Hopf algebra. Then $P_3(H^2)$ is a left weak Hopf sub-bimodule of $P_3(H\otimes H)$.*

Proof. For any $a1_1\otimes S(1_2)b\in P_3(H^2)$, we have

$$\begin{aligned}&(m\otimes 1)\circ\rho_R(a1_1\otimes S(1_2)b)\\ =\;&(m\otimes 1)((a1_1)_1\otimes(S(1_2)b)_1\otimes(a1_1)_2(S(1_2)b)_2)\\ =\;&\Delta(a1_1S(1_2)b)=0.\end{aligned}$$

Thus $\rho_R(P_3(H^2))\subset H\otimes P_3(H^2)$. ■

Theorem 4.3.7. *Let H be a weak Hopf algebra. Then a first order differential calculus (Γ,d) is right-covariant if and only if there exists a right weak Hopf sub-bimodule $N\subset P_1(H^2)$ such that $\Gamma=P_3(H^2)/N$ and $d=\pi\circ D_2$, where π is the canonical epimorphism $P_3(H^2)\longrightarrow\Gamma$.*

Proof. Here the notations are the same as in Lemma 4.2.5. First assume that N is a right weak Hopf sub-bimodule of $P_3(H^2)$. Then the map π is H-linear and right H-colinear. Moreover, , $(D_2\otimes 1)\circ\Delta=\rho_L\circ D_2$. Using Example 4.2.4 and Lemma 4.2.5 we only need to prove that $a\pi[D_2(b)]=0$ implies $a_1\pi[D_2(b_1)]\otimes a_2b_2=0$ for $a,b\in H$. If $a\pi[D_2(b)]=0$, then

$$\begin{aligned}a_1\pi[D_2(b_1)]\otimes a_2b_2&=\Delta(a)[(\pi\otimes 1)(D_2(b_1)\otimes b_2)]\\&=(a_1\otimes a_2)[(\pi\otimes 1)(\rho_R\circ D_2(b))]\\&=\Delta(a)[\rho_R\circ\pi((D_2(b))]\\&=\rho_R[a(\pi\circ D_2(b))]=0.\end{aligned}$$

Assume that (Γ,d) is right-covariant. For $a1_1\otimes 1S(_2)b\in ker\pi'$, we have $a_1d(b_1)\otimes a_2b_2=0$. By Lemma 4.1.5, $a_1\pi'(1_1\otimes S(1_2)b_1)\otimes a_2b_2=a_1\pi'(b_11_1\otimes S(1_2))\otimes a_2b_2$. Note that $cd(1)=0$ for any $c\in H$. We have $c1_1\otimes S(1_2)\in ker\pi'$. So

$$(\pi'\otimes 1)\circ\rho_R(a1_1\otimes S(1_2)b)=a_1\pi'(b_11_1\otimes S(1_2))\otimes a_2b_2=0.$$

So $\rho_R(ker\pi')\subset ker\pi'\otimes H$. The rest is easy. ■

Proposition 4.3.8. *Let (Γ,d) be a right-covariant first order differential calculus over H. Then there uniquely exists a linear map $\rho_R:\ \Gamma\longrightarrow\Gamma\otimes H$ such that (Γ,ρ_L) is a right weak Hopf bimodule. Moreover, $(d\otimes 1)\circ\Delta=\rho_R\circ d$.*

Proof. Straightforward. ■

Lemma 4.3.9. *Let H be a weak Hopf algebra. Let $e=S^2(1_1)1_2$ and $v=1_1S^2(1_2)$. Then e and v are two idempotents in H with $\varepsilon_s(e)=1$ and $\varepsilon_t(v)=1$.*

Proof. Note that $\varepsilon_t(a)\varepsilon_s(b) = \varepsilon_s(b)\varepsilon_t(a)$. We have

$$\varepsilon_s(e) = \varepsilon_s(1_2 S^2(1_1)) = \varepsilon_s(S(1_2)S^2(1_1)) = \varepsilon_s(S(1_1)1_2) = 1,$$
$$e^2 = S^2(1_1)1_2 S^2(1'_1)1'_2 = S^2(1_1)S^2(1'_1)1_2 1'_2 = S^2(1_1)1_2 = e.$$

Similarly, $v^2 = v$. ■

By Lemma 4.3.9 there exists a Peirce left decomposition $H = eH \oplus (1-e)H$, where eH is a right ideal in H. Consider a coadjoint map

$$\lambda : H \longrightarrow H \otimes H,\ b \longmapsto b_2 \otimes S(b_1)b_3.$$

Lemma 4.3.10. *Let H be a weak Hopf algebra and R a subspace of H. Then*

(1) *The following statements hold:*

$$(i)\ \lambda(eb) = \lambda(b),\ \forall\ b \in H,$$
$$(ii)\ \lambda(R) \subset R \otimes H \Longleftrightarrow \lambda(eR) \subset R \otimes H,$$
$$(iii)\ \lambda(R) \subset eR \otimes H \Longleftrightarrow \lambda(eR) \subset eR \otimes H;$$

(2) *(R, λ) is a right H-comodule $\Longleftrightarrow$ $R = eR \subset eH$ and $\lambda(R) \subset eR \otimes H$;*

(3) *If R is a right ideal in H, so is eR;*

(4) *If R is contained in $ker\varepsilon_s$, so is eR.*

Proof. (1). For any $b \in H$, we first have

$$\begin{aligned}\lambda(S^2(1_1)1_2 b) &= (S^2(1_1)1_2 b)_2 \otimes S((S^2(1_1)1_2 b)_1)(S^2(1_1)1_2 b)_3\\ &= b_2 \otimes S(1_2 b_1)S^2(1_1)b_3 = b_2 \otimes S(b_1)S(1_2)S^2(1_1)b_3\\ &= b_2 \otimes S(b_1)S(1_2)S^2(1_1)b_3 = b_2 \otimes S(b_1)b_3 = \lambda(b).\end{aligned}$$

Part (ii) and (iii) follow from (i).

(2). For any $a \in H$, we have

$$\begin{aligned}(1 \otimes \varepsilon) \circ \lambda(b) &= b_2\varepsilon(S(b_1)b_3) = b_2\varepsilon(S(b_1)\varepsilon_t(b_3)) = 1_1 b_2 \varepsilon(S(b_1)1_2)\\ &= S(1_2)b_2\varepsilon(S(b_1)S(1_1)) = S(\varepsilon(1_1 b_1)1_2)b_2 = eb.\end{aligned}$$

If (R, λ) is a right H-comodule, then $\lambda(R) \subset R \otimes H$ and for any $a \in R$,

$$a = (1 \otimes \varepsilon) \circ \lambda(a) = ea \in eR.$$

If $R \subset eH$ and $\lambda(R) \subset R \otimes H$, then $b = eb$ for all $b \in R$. So $(1 \otimes \varepsilon) \circ \lambda(b) = b$. The coassociativity is easy.

(3). Since R is a right ideal in H, then $RH \subset R$. So $eRH \subset eR$.

(4). If R is contained in $ker\varepsilon_s$, then by Lemma 4.3.9,

$$\varepsilon_s(ec) = \varepsilon_s(\varepsilon_s(e)c) = \varepsilon_s(1c) = 0$$

for all $c \in R$. Thus $eR \subset ker\varepsilon_s$. ■

Lemma 4.3.11. *Let H be a weak Hopf algebra and R a subspace of H. Define the following sets:*

$$N_2(R) = P_3(R_2(H \otimes R)) = \{\ aS(b_1)1_1 \otimes S(1_2)b_2 |\ a \otimes b \in H \otimes R\ \},$$
$$N_2'(R) = R_2(H \otimes eR) = \{\ aS(b_1) \otimes b_2 |\ a \otimes b \in H \otimes eR\ \},$$
$$N_3(R) = P_1(S_2(R \otimes H)) = \{\ bS^{-1}(a_2)S(1_1) \otimes 1_2a_1 |\ a \otimes b \in R \otimes H\ \},$$
$$N_3'(R) = S_2(eR \otimes H) = \{\ bS^{-1}(a_2) \otimes a_1 |\ a \otimes b \in eR \otimes H\ \}.$$

Then (1) $N_2(R) = N_2'(R)$ and $N_3(R) = N_3'(R)$;

(2) If $\lambda(R) \in R \otimes H$, then $N_2'(R) = N_3'(R)$;

(3) If R and R' are two right ideals in H such that $N_2'(R) = N_3'(R')$, then $eR = eR'$.

Proof. (1). For any $b \in R$ and $a \in H$, we have

$$\begin{aligned} aS((eb)_1) \otimes (eb)_2 &= aS(1_2b_1) \otimes S^2(1_1)b_2 \\ &= aS(b_1)S(1_2) \otimes S^2(1_1)b_2 = aS(b_1)1_1 \otimes S(1_2)b_2. \end{aligned}$$

So $R_2(a \otimes eb) = P_3(R_2(a \otimes b))$. For any $c \otimes d \in N_2(R)$, there exists $c' \otimes d' \in H \otimes R$, such that $c \otimes d = P_3(R_2(c' \otimes d'))$. By $R_2(a \otimes eb) = P_3(R_2(a \otimes b))$,

$$c \otimes d = P_3(R_2(c' \otimes d')) = R_2(c' \otimes ed') \in N_2'(R).$$

For any $f \otimes g \in N_2'(R)$, there exists $f' \otimes g' \in H \otimes R$, such that $f \otimes g = R_2(f' \otimes eg')$. By $R_2(a \otimes eb) = P_3(R_2(a \otimes b))$, we get

$$f \otimes g = R_2(f' \otimes eg') = P_3(R_2(f' \otimes g')) \in N_2(R).$$

For any $a' \in R$ and $b' \in H$,

$$b'S^{-1}((ea')_2) \otimes (ea')_1 = bS^{-1}(S^2(1_1)a_2') \otimes 1_2a_1 = bS^{-1}(a_2')S(1_1) \otimes 1_2a_1.$$

So $S_2(ea' \otimes b') = P_1(S_2(a' \otimes b'))$. Similarly, $N_3(R) = N_3'(R)$.

(2) Assume that $\lambda(R) \in R \otimes H$. For any $c \otimes d \in N_2'(R)$, there exists $a \in H$ and $b \in eR$, such that $c \otimes d = aS(b_1) \otimes b_2$. Since $\lambda(R) \in R \otimes H$, we have $\lambda(b) = b_2 \otimes S(b_1)b_3 \in R \otimes R$ and so $b_2 \otimes aS(b_1)b_3 \in R \otimes H$. By (1) we get

$$P_1(S_2(b_2 \otimes aS(b_1)b_3)) \in N_3(R) = N_3'(R).$$

At the same time, we have

$$\begin{aligned} c\otimes d &= aS(b_1)\otimes b_2 = aS((eb)_1)\otimes (eb)_2 \\ &= aS(b_1)1_1\otimes S(1_2)b_2 \\ &= P_1(aS(b_1)S^{-1}(1_2)\otimes 1_1b_2) \\ &= P_1(aS(b_1)S^{-1}(\varepsilon_t(b_3))\otimes b_2) \\ &= P_1(aS(b_1)b_3S^{-1}(b_{2_2})\otimes b_{2_1}) \\ &= P_1(S_2(b_2\otimes aS(b_1)b_3)), \end{aligned}$$

So we conclude that $N'_2(R)\subset N'_3(R)$.

For any $c'\otimes d'\in N'_3(R)$, there exists $a'\in eR$ and $b'\in H$, such that $c'\otimes d' = b'S^{-1}(a'_2)\otimes a'_1$. Since $\lambda(R)\in R\otimes H$, we have $\lambda(a')=a'_2\otimes S(a'_1)a'_3\in R\otimes H$ and so $b'S^{-1}(S(a'_1)a'_3)\otimes a'_2\in H\otimes R$. By (1), $P_3(R_2(b'S^{-1}(S(a'_1)a'_3)\otimes a'_2))\in N_2(R)=N'_2(R)$. Similarly, we do the following computation:

$$\begin{aligned} c'\otimes d' &= b'S^{-1}((ea')_2)\otimes (ea')_1 \\ &= b'S^{-1}(S^2(1_1)a'_2)\otimes 1_2a'_1 \\ &= b'S^{-1}(a'_2)S(1_1)\otimes 1_2a'_1 \\ &= P_3(b'S^{-1}(a'_2)S(1_1)\otimes 1_2a'_1) \\ &= P_3(b'S^{-1}(a'_4)a'_1S(a'_2)\otimes a'_3) \\ &= P_3(R_2(b'S^{-1}(a'_3)a'_1\otimes a'_2)) \\ &= P_3(R_2(b'S^{-1}(S(a'_1)a'_3)\otimes a'_2)), \end{aligned}$$

So $N'_3(R)\subset N'_2(R)$ also holds.

(3) Let R and R' be two right ideals in H such that $N'_2(R)=N'_3(R')$. By Lemma 4.3.10 (3), eR and eR' are also two right ideals in H. By $N'_2(R)=N'_3(R')$, for $a\in eR$, then $S(b_1)\otimes b_2\in N'_2(R)$ and $S(b_1)\otimes b_2\in N'_3(R)$. So there exist $a'\in eR'$ and $b'\in H$ such that

$$S(b_1)\otimes b_2 = b'S^{-1}(a'_2)\otimes a'_1.$$

Applying $\varepsilon\otimes 1$ to the two sides, we can get

$$\begin{aligned} b &= \varepsilon(S(b_1))b_2 = \varepsilon(b'S^{-1}(a'_2))a'_1 = \varepsilon(a'_2S(b'))a'_1 = \varepsilon(\varepsilon_s(a'_2)S(b'))a'_1 \\ &= \varepsilon(S(1_2)S(b'))a'1_1 = \varepsilon(b'1_2)a'1_1 = a'\varepsilon_s(b')\in eR'. \end{aligned}$$

Conversely, for any $a'\in eR'$, there exist $a\in H$ and $b\in eR$, such that

$$aS(b_1)\otimes b_2 = S^{-1}(a'_2)\otimes a'_1.$$

Similarly, we compute as follows:

$$\begin{aligned} a' &= \varepsilon(S^{-1}(a'_2))a'_1 = \varepsilon(aS(b_1))b_2 = \varepsilon(b_1S^{-1}(a))b_2 \\ &= \varepsilon(\varepsilon_s(b_1)S^{-1}(a))b_2 = \varepsilon(1_1S^{-1}(a))b1_2 = b\varepsilon_t(S^{-1}(a)) \in eR. \end{aligned}$$ ∎

Lemma 4.3.12. *Let H be a weak Hopf algebra. Let*

$$N_4 = \{\ S(b_1) \otimes b_2 |\ b \in eH\ \},\ N_5 = \lambda(eH) = \{\ b_2 \otimes S(b_1)b_3 |\ b \in eH\ \}.$$

Then S_1 is a bijective map from N_4 to N_5.

Proof. It is clear that $S_1(S(b_1) \otimes b_2) = b_2 \otimes S(b_1)b_3$ for any $b \in eH$. We show that S_2 is the inverse of S_1. Note that

$$\begin{aligned} S_2(b_2 \otimes S(b_1)b_3) &= S(b_1)b_4S^{-1}(b_3) \otimes b_2 = S(b_1)S^{-1}(\varepsilon_t(b_3)) \otimes b_2 \\ &= S(b_1)S^{-1}(1_2) \otimes 1_1b_2 = S(1_2b_1) \otimes S^2(1_1)b_2 \\ &= S((eb)_1) \otimes (eb)_2 = S(b_1) \otimes b_2. \end{aligned}$$

So $S_2 \circ S_1 = Id_{N_4}$. Similarly, $S_1 \circ S_2 = Id_{N_5}$. ∎

Lemma 4.3.13. *Let R be a right ideal R in H, $R \subset ker\varepsilon_s$ such that (R, λ) is a right H-comodule. Let $N = S_2(R \otimes H) = \{\ bS^{-1}(a_2) \otimes a_1 |\ a \otimes b \in R \otimes H\ \}$. Then N is a weak Hopf sub-bimodule of $P(H^2)$.*

Proof. Since (R, λ) is a right H-comodule, then by Lemma 4.3.10 (2) $R = eR \subset eH$ and $\lambda(R) \subset eR \otimes H$. By Lemma 4.3.11 (3),

$$\{\ aS(b_1) \otimes b_2 |\ a \otimes b \in H \otimes R\ \} = \{\ bS^{-1}(a_2) \otimes a_1 |\ a \otimes b \in R \otimes H\ \} = N.$$

So $N \subset P(H^2)$. By Lemma 4.1.10 (4.2.10), (N, ρ_L) $((N, \rho_R))$ is a left (right) weak Hopf sub-bimodule $P_1(H^2)$ $((P_3(H^2))$. Using Lemma 3.3.3, (N, ρ_L) $((N, \rho_R))$ is a left (right) weak Hopf sub-bimodule of $P(H^2)$. It follows from Lemma 3.3.2 that $(1 \otimes \rho_R) \circ \rho_L = (\rho_L \otimes 1) \circ \rho_R$. ∎

Lemma 4.3.14. *Let N be a weak Hopf sub-bimodule of $P(H^2)$. Then there exists a right ideal R in H, $R \subset ker\varepsilon_s$ such that (R, λ) is a right H-comodule and $N = S_2(R \otimes H) = \{bS^{-1}(a_2) \otimes a_1 |\ a \otimes b \in R \otimes H\}$.*

Proof. Since N is a weak Hopf sub-bimodule of $P(H^2) \subset P_1(H^2)$, by Lemma 4.1.11, there exists a right ideal R'' in H, $R'' \subset ker\varepsilon_s$ such that

$$N = R_2(H \otimes R'') = \{aS(b_1) \otimes b_2 |\ a \otimes b \in H \otimes R''\}.$$

Similarly, there exists a right ideal R' in H, $R' \subset ker\varepsilon_s$ such that

$$N = S_2(R' \otimes H) = \{bS^{-1}(a_2) \otimes a_1 |\ a \otimes b \in R' \otimes H\}.$$

Using Lemma 4.3.11 (3), $eR'' = eR'$ and

$$N = S_2(eR' \otimes H) = \{bS^{-1}(a_2) \otimes a_1 |\ a \otimes b \in eR' \otimes H\}.$$

Following Lemma 4.3.10 , eR' is a right ideal in H, $eR' \subset ker\varepsilon_s$ and $e^2R' = eR' \subset eH$. For any $b \in eR'$, $S(b_1) \otimes b_2 \in N$. Then exists $a' \in eR'$ and $b' \in H$, such that

$$S(b_1) \otimes b_2 = b'S^{-1}(a'_2) \otimes a'_1.$$

On one hand, using Lemma 4.3.12 $\lambda(b) = S_1(S(b_1) \otimes b_2)$. On the other hand,

$$S_1(b'S^{-1}(a'_2) \otimes a'_1) = a'_1 \otimes b'S^{-1}(a'_3)a'_2 = a'1_1 \otimes b'1_2 \in eR' \otimes H.$$

So $\lambda(b) = S_1(S(b_1) \otimes b_2) = a'1_1 \otimes b'1_2 \in eR' \otimes H$. Then $\lambda(eR') \subset eR' \otimes H$. It follows from Lemma 4.3.10 that (eR', λ) is a right H-comodule.

Now let $R := eR'$. Consequently, R is what we need. ■

Proposition 4.3.15. *Let H be a weak Hopf algebra. Then N is a left weak Hopf sub-bimodule of $P(H^2)$ if and only if there exists a right ideal R in H, $R \subset ker\varepsilon_s$ such that (R, λ) is a right H-comodule and*

$$N = S_2(R \otimes H) = \{bS^{-1}(a_2) \otimes a_1 |\ a \otimes b \in R \otimes H\}.$$

Proof. The proof follows from Lemma 4.3.13 and 4.3.14. ■

Applying Theorem 4.3.7 and Proposition 4.3.15, we obtain our main result, which is a generalization of well-known Woronowicz's theorem about bicovariant differential calculi on quantum groups (see [23, Thm 1.8]).

Theorem 4.3.16. *Let H be a weak Hopf algebra with bijective antipode. Then a first order differential calculus (Γ, d) is bicovariant if and only if there exists a right ideal R in H, $R \subset ker\varepsilon_s$ such that (R, λ) is a right H-comodule, $\Gamma = P_3(H^2)/N$ and $d = \pi \circ D_2$, where $N = S_2(R \otimes H)$ and π is the canonical epimorphism $P_3(H^2) \longrightarrow \Gamma$.*

In other word, we have a 1-1 correspondence between bicovariant differential calculi and some special ideals of H:

Corollary 4.3.17. *Let H be a weak Hopf algebra with bijective antipode. Let ε_s be the source map of H. Then there exists a 1-1 correspondence between bicovariant differential calculi and some right ideals of H contained in $ker\varepsilon_s$ such that these ideals are right H-comodules with a coadjoint map.*

Remark 4.3.18. Theorem 4.3.16 means that well-known Woronowicz's theorem about bicovariant differential calculi is still valid in the case of compact face algebras and dynamical quantum groups obtained by dynamical twists of quantum groups. So we can take some methods used in the case of quantum groups to carry out a similar investigation of bicovariant differential calculi on these dynamical quantum groups. The study of this direction is our ongoing program.

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Haixing Zhu
College of Economics and Management
Nanjing Forestry University
Nanjing 210037
P. R. China
E-mail: zhuhaixing@163.com

Shuanhong Wang
Department of Mathematics
Southeast University
Nanjing 210096
P. R. China

Juzhen Chen
College of Science
Nanjing Forestry University
Nanjing 210037
P. R. China