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CHEN'S INEQUALITIES FOR SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD OF QUASI-CONSTANT CURVATURE WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. In this paper, we obtain Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature endowed with a semi-symmetric metric connection. Also, some results of A. Mihai and C. Özgur's paper have been modified.

1. Introduction

According to B.-Y. Chen [5], one of the most important problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Related with famous Nash embedding theorem [22], B.-Y. Chen introduced a new type of Riemannian invariants, known as $\delta-$*invariants* [4,6,13]. The author's original motivation was to provide answers to a question raised by S.S. Chern concerning the existence of minimal isometric immersions into Euclidean space [26]. Therefore, B.-Y. Chen obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold into Euclidean space and established inequalities for submanifolds in real space forms in terms of the sectional curvature, the scalar curvature and the squared mean curvature [7]. Later, he established general inequalities relating $\delta(n_1, \cdots, n_k)$ and the squared mean curvature for submanifolds in real space forms [8]. Similar inequalities also hold for Lagrangian submanifolds of complex space forms. In [9], B.-Y. Chen proved that, for any $\delta(n_1, \cdots, n_k)$, the equality case holds if and only if the Lagrangian submanifold is minimal. This interesting phenomenon inspired people to look for a more sharp inequality. In 2007, T. Oprea improved the inequality on $\delta(2)$ for Lagrangian submanifolds in complex space forms [27]. Recently, B.-Y. Chen and F. Dillen established

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general inequalities for submanifolds in complex space forms and provided some examples showing these new improved inequalities are best possible [14]. However, it was pointed out in [15] that the proof of the general inequality given [14] is incorrect when $\sum_{i=1}^{k} \frac{1}{2+n_i} > \frac{1}{3}$. In [16], B.-Y. Chen, F. Dillen, J. Van der Veken and L. Vrancken corrected the proof of the general inequality in the case $n_1 + \cdots + n_k < n$ and showed that the inequality can be improved in the case $n_1 + \cdots + n_k = n$.

Such invariants and inequalities have many nice applications to several areas in mathematics [10].

Afterwards, many papers studied Chen's inequalities for different submanifolds in various ambient spaces, like complex space forms [20], generalized complex space forms[1], $(\kappa, \mu)-$contact space forms [23], Riemannian manifold of quasi-constant curvature[18], Euclidean space [19] and locally conformal almost cosymplectic manifolds [29].

Recently, A. Mihai and C. Özgür proved Chen's inequalities for submanifolds of real space forms, complex space forms and Sasakian space forms with semi-symmetric metric connections [2,3]. In this paper, we obtain Chen first inequalities and Chen-Ricci inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature endowed with a semi-symmetric metric connection by using algebraic lemmas. We should point out that our approaches are different from B.Y. Chen's. Moreover, we prove a result of A. Mihai and C. Özgür [2, Theorem 4.1.] is incorrect and the Corollary 4.2 from [2] isn't ideal. For the sake of correcting the results, we establish Chen-Ricci inequalities for submanifolds of real space forms with a semi-symmetric metric connection at the end of Section 5.

2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of Riemannian manifolds endowed with a semi-symmetric metric connection are briefly presented.

Let N^{n+p} be an $(n+p)-$dimensional Riemannian manifold with *Riemannian metric* g, the linear connection $\overline{\nabla}$ and the *Riemannian connection* $\hat{\overline{\nabla}}$. For the vector fields $\overline{X}, \overline{Y}$ on N^{n+p} the torsion tensor field $\overline{T}$ of the linear connection $\overline{\nabla}$ is defined by $\overline{T}(\overline{X}, \overline{Y}) = \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} - [\overline{X}, \overline{Y}]$. A liner connection $\overline{\nabla}$ is said to be a *semi-symmetric connection* if the torsion tensor $\overline{T}$ of the connection $\overline{\nabla}$ satisfies $\overline{T}(\overline{X}, \overline{Y}) = \phi(\overline{Y})\overline{X} - \phi(\overline{X})\overline{Y}$, where ϕ is a 1-form on N^{n+p}. Further, if $\overline{\nabla}$ satisfies $\overline{\nabla}g = 0$, then $\overline{\nabla}$ is called a *semi-symmetric metric connection*[25]. In [25], K. Yano obtained a relation between the semi-symmetric metric connection $\overline{\nabla}$ and the Riemannian connection $\hat{\overline{\nabla}}$ which is given by $\overline{\nabla}_{\overline{X}}\overline{Y} = \hat{\overline{\nabla}}_{\overline{X}}\overline{Y} + \phi(\overline{Y})\overline{X} - g(\overline{X}, \overline{Y})P$, where P is a vector field given by $g(P, \overline{X}) = \phi(\overline{X})$ for any vector field $\overline{X}$ on N^{n+p}.

Let M^n be an $n-$dimensional submanifold of an $(n+p)$-dimensional manifold N^{n+p} with the semi-symmetric metric connection $\overline{\nabla}$ and the Riemannian connection

$\hat{\overline{\nabla}}$. On M^n we consider the induced semi-symmetric metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\hat{\nabla}$. We denote by R and $\hat{R}$ the curvature tensors associated to ∇ and $\hat{\nabla}$.

The Gauss formulas with respect to ∇, respectively $\hat{\nabla}$, can be written as the following

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \hat{\overline{\nabla}}_X Y = \hat{\nabla}_X Y + \hat{h}(X,Y),$$

for any vector fields X, Y on M^n, where h is a $(0,2)$ symmetric tensor on M^n and $\hat{h}$ is the second fundamental form associated to Riemaniann connection $\hat{\nabla}$ [30].

We will consider a Riemanniann manifold N^{n+p} of *quasi-constant curvature* [17] endowed with a semi-symmetric metric connection $\overline{\nabla}$ and the Riemannian connection $\hat{\overline{\nabla}}$.

From [17], the curvature tensor $\hat{\overline{R}}$ with respect to the Levi-Civita connection $\hat{\overline{\nabla}}$ on N^{n+p} is expressed by

$$\begin{aligned} \hat{\overline{R}}(X,Y,Z,W) = {} & a[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \\ & + b[g(X,Z)T(Y)T(W) - g(X,W)T(Y)T(Z) \\ & + g(Y,W)T(X)T(Z) - g(Y,Z)T(X)T(W)], \end{aligned} \tag{2.1}$$

where a, b are scalar functions and T is a $1-$form defined by

$$g(X,U) = T(X), \tag{2.2}$$

and U is a unit vector field. If $b = 0$, it can be easily seen that the manifold reduces to a space of constant curvature.

Decomposing the vector field U on M uniquely into its tangent and normal components U^T and $U^\perp$, respectively, we have

$$U = U^T + U^\perp. \tag{2.3}$$

The *curvature tensor* $\overline{R}$ with respect to the semi-symmetric metric connection $\overline{\nabla}$ on N^{n+p} can be written as [30]

$$\begin{aligned} \overline{R}(X,Y,Z,W) = {} & \hat{\overline{R}}(X,Y,Z,W) + \alpha(Y,Z)g(X,W) - \alpha(X,Z)g(Y,W) \\ & + \alpha(X,W)g(Y,Z) - \alpha(Y,W)g(X,Z), \end{aligned} \tag{2.4}$$

for any vector fields X, Y, Z, W on M^n, where α is a $(0,2)$-tensor field defined by

$$\alpha(X,Y) = (\hat{\overline{\nabla}}_X \phi)Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(P)g(X,Y).$$

Denote λ the trace of α.

From (2.1) and (2.4) it follows that the *curvature tensor* $\overline{R}$ can be expressed as

$$\begin{aligned}\overline{R}(X,Y,Z,W) = & a[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \\ & + b[g(X,Z)T(Y)T(W) - g(X,W)T(Y)T(Z) \\ & + g(Y,W)T(X)T(Z) - g(Y,Z)T(X)T(W)] \\ & + \alpha(Y,Z)g(X,W) - \alpha(X,Z)g(Y,W) \\ & + \alpha(X,W)g(Y,Z) - \alpha(Y,W)g(X,Z).\end{aligned} \tag{2.5}$$

The Gauss equation with respect to semi-symmetric metric connection is [30]

$$\begin{aligned}R(X,Y,Z,W) = & \overline{R}(X,Y,Z,W) + g(h(X,Z),h(Y,W)) \\ & - g(h(X,W),h(Y,Z)).\end{aligned} \tag{2.6}$$

In N^{n+p} we can choose a local orthonormal frame

$$e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}, \tag{2.7}$$

such that, restricting to M^n, $e_1, e_2, \cdots, e_n$ are tangent to M^n. We write $h^r_{ij} = g(h(e_i,e_j),e_r)$. The squared length of h is $\| h \|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j))$ and the *mean curvature vector* of M associated to ∇ is $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i)$. Similarly, the *mean curvature vector* of M^n associated to $\hat{\nabla}$ is $\hat{H} = \frac{1}{n}\sum_{i=1}^{n} \hat{h}(e_i,e_i)$.

If $\hat{h}^r_{ij} = k^r g_{ij}$, where k^r are real-valued functions on M, then M is said to be *totally umbilical with respect to Levi-Civita connection*. Similarly, if $h^r_{ij} = k^r g_{ij}$, then M is said to be *totally umbilical with respect to semi-symmetric metric connection* [30].

Let $\pi \subset T_xM$ and $\pi^\perp \subset T_x^\perp M$ be plane sections for any x in M^n and $K(\pi)$ the sectional curvature of M^n associated to the induced semi-symmetric metric connection ∇. The *scalar curvature* τ at x is defined by

$$\tau(x) = \sum_{1\le i<j\le n} K_{ij}. \tag{2.8}$$

Suppose L is an l-dimensional subspace of T_xM, $x \in M$, $l \geq 2$ and $\{e_1, \cdots, e_l\}$ an orthonormal basis of L. We define the *scalar curvature* $\tau(L)$ of the $l-$plane L by

$$\tau(L) = \sum_{1\le \mu<\nu\le l} K(e_\mu \wedge e_\nu). \tag{2.9}$$

For simplicity we put

$$\begin{aligned}\Psi_1(L) &= \sum_{1\le i<j\le l} [\alpha(e_i,e_i) + \alpha(e_j,e_j)], \\ \Psi_2(L) &= \sum_{1\le i<j\le l} [g(U^T,e_i)^2 + g(U^T,e_j)^2].\end{aligned} \tag{2.10}$$

For an integer $k \geq 0$ we denote by $S(n,k)$ the set of $k-$tuples $(n_1, \cdots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. We denote by $S(n)$ the set of unordered $k-$ tuples with $k \geq 0$ for a fixed n. For each $k-$ tuples $(n_1, \cdots, n_k) \in S(n)$, B.-Y. Chen defined a *Riemannian invariant* $\delta(n_1, \cdots, n_k)$ as follows [8]

$$\delta(n_1, \cdots, n_k)(x) = \tau(x) - S(n_1, \cdots, n_k)(x), \tag{2.11}$$

where

$$S(n_1, \cdots, n_k)(x) = \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

and $L_1, \cdots, L_k$ run over all k mutually orthogonal subspaces of T_xM such that $\dim L_j = n_j,\ j \in \{1, \cdots, k\}$. In particular, we have $\delta(2) = \tau(x) - \inf K$, where K is the sectional curvature.

For each $(n_1, \cdots, n_k) \in S(n)$, we put

$$c(n_1, \cdots, n_k) = \frac{n^2\big(n+k-1-\sum_{j=1}^{k} n_j\big)}{2\big(n+k-\sum_{j=1}^{k} n_j\big)}, \quad d(n_1, \cdots, n_k) = \frac{1}{2}[n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)].$$

According to the formula (7) from [30] we have

Lemma 2.1. [30]. *If P is a tangent vector field on M^n, we have $H = \hat{H}, h = \hat{h}$.*

On the other hand, Z. Nakao proved

Lemma 2.2. [30, Theorem 3]. *A submanifold M of a Riemannian manifold N is totally umbilical if and only if it is totally umbilical with respect to the semi-symmetric metric connection.*

We recall the well-known Chen's lemma:

Lemma 2.3. [7]. *Let $a_1, a_2, \cdots, a_n, b$ be $(n+1)(n \geq 2)$ real numbers such that*

$$(\sum_{i=1}^{n} a_i)^2 = (n-1)(\sum_{i=1}^{n} a_i^2 + b),$$

then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Most of the geometers (cf. [1, 2, 3, 7, 18, 23, 29]) established inequalities relating $\delta(2)$ and the squared mean curvature for different submanifolds in various ambient spaces by using the above algebraic lemma except for T. Oprea (cf. [27, 28]). In [28], T. Oprea gave an another proof of Chen's inequalities for submanifolds in a real space form by using optimization techniques applied in the setup of Riemannian geometry. We will use another algebraic lemma to obtain inequalities relating $\delta(2)$ and the squared mean curvature.

Lemma 2.4. *Let* $f(x_1, x_2, \cdots, x_n) \quad (n \geq 3)$*be a function in* R^n *defined by*

$$f(x_1, x_2, \cdots, x_n) = (x_1 + x_2)\sum_{i=3}^{n} x_i + \sum_{3\leq i<j\leq n} x_i x_j.$$

If $x_1 + x_2 + \cdots + x_n = (n-1)\varepsilon$, then we have

$$f(x_1, x_2, \cdots, x_n) \leq \frac{(n-1)(n-2)}{2}\varepsilon^2,$$

with the equality holding if and only if $x_1 + x_2 = x_3 = \cdots = x_n = \varepsilon$.

Proof. By simple calculation, we have

$$\begin{aligned} f(x_1, x_2, \cdots, x_n) &= (x_1 + x_2)\sum_{i=3}^{n} x_i + \sum_{3\leq i<j\leq n} x_i x_j \\ &= \frac{1}{2}\{(x_1 + x_2 + \cdots + x_n)^2 - [(x_1 + x_2)^2 + x_3^2 + \cdots + x_n^2]\} \\ &= \frac{1}{2}\{(n-1)^2\varepsilon^2 - [(x_1 + x_2)^2 + x_3^2 + \cdots + x_n^2]\}. \end{aligned} \tag{2.12}$$

On the other hand, by the Cauchy-Schwartz inequality we have

$$[(x_1 + x_2) + x_3 + \cdots + x_n]^2 \leq (n-1)[(x_1 + x_2)^2 + x_3^2 + \cdots + x_n^2], \tag{2.13}$$

with the equality holding if and only if $x_1 + x_2 = x_3 = \cdots = x_n$.

Noting that $(x_1 + x_2) + \cdots + x_n = (n-1)\varepsilon$, from (2.13) we have

$$(x_1 + x_2)^2 + x_3^2 + \cdots + x_n^2 \geq (n-1)\varepsilon^2. \tag{2.14}$$

Using (2.12) and (2.14) we derive

$$f(x_1, x_2, \cdots, x_n) \leq \frac{1}{2}[(n-1)^2\varepsilon^2 - (n-1)\varepsilon^2] = \frac{(n-1)(n-2)}{2}\varepsilon^2,$$

which represents Lemma 2.4 to prove. ∎

In Section 5, we use a more simple way to obtain the relation between the Ricci curvature and the spared mean curvature. We need the following lemma.

Lemma 2.5. *Let* $f(x_1, x_2, \cdots, x_n)$ *be a function in* R^n *defined by*

$$f(x_1, x_2, \cdots, x_n) = x_1 \sum_{i=2}^{n} x_i.$$

If $x_1 + x_2 + \cdots + x_n = 2\varepsilon$, then we have

$$f(x_1, x_2, \cdots, x_n) \leq \varepsilon^2,$$

with the equality holding if and only if $x_1 = x_2 + x_3 + \cdots + x_n = \varepsilon$.

Proof. From $x_1 + x_2 + \cdots + x_n = 2\varepsilon$, we have

$$\sum_{i=2}^{n} x_i = 2\varepsilon - x_1.$$

It follows that

$$f(x_1, x_2, \cdots, x_n) = x_1(2\varepsilon - x_1) = -(x_1 - \varepsilon)^2 + \varepsilon^2,$$

which represents Lemma 2.5 to prove. ∎

3. Chen First Inequality

For submanifolds of a Riemannian manifold of quasi-constant curvature endowed with a semi-symmetric metric connection we establish the following optimal inequality relating $\delta(2)$ and squared mean curvature, which will call Chen first inequality.

Theorem 3.1. *Let* M^n, $n \geq 3$, *be an* $n-$*dimensional submanifold of an* $(n+p)-$*dimensional Riemannian manifold of quasi-constant curvature* N^{n+p} *endowed with a semi-symmetric metric connection, then we have*

$$\tau(x) - K(\pi) \leq \frac{(n+1)(n-2)}{2}a + b[(n-1) \parallel U^T \parallel^2 - \parallel U_\pi \parallel^2] \\ -(n-2)\lambda - \text{trace}(\alpha\mid_{\pi^\perp}) + \frac{n^2(n-2)}{2(n-1)} \parallel H \parallel^2,$$

where π is a $2-$*plane section of* T_xM^n, $x \in M^n$.

Remark 3.2. For $b = 0$, Theorem 3.1 is due to A. Mihai and C. Özgür [2, Theorem 3.1].

Proof. We consider the point $x \in M^n$, choose a local orthonormal frame (2.7) such that $\{e_1, e_2\}$ being an orthonormal frame in the 2-plane which minimize the sectional curvature at the point x. We remark that

$$U_\pi = pr_\pi U, \quad \alpha(e_1, e_1) + \alpha(e_2, e_2) = \lambda - \text{trace}(\alpha\mid_{\pi^\perp}). \tag{3.1}$$

Using (2.3), (2.5) and (2.6) we have

$$R_{ijij} = a + b[g(U^T, e_i)^2 + g(U^T, e_j)^2] - \alpha(e_i, e_i) \\ -\alpha(e_j, e_j) + \sum_{r=n+1}^{n+p} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2], \tag{3.2}$$

it follows that

$$\begin{aligned}\tau(x) &= \sum_{1\le i<j\le n} R_{ijij} = \frac{n^2-n}{2}a + b(n-1) \parallel U^T \parallel^2 \\ &\quad -(n-1)\lambda + \sum_{r=n+1}^{n+p} \sum_{1\le i<j\le n} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2].\end{aligned} \tag{3.3}$$

Using (3.1) and (3.2) we have

$$\begin{aligned}R_{1212} &= a + b[g(U^T, e_1)^2 + g(U^T, e_2)^2] - \alpha(e_1, e_1) \\ &\quad -\alpha(e_2, e_2) + \sum_{r=n+1}^{n+p} [h^r_{11}h^r_{22} - (h^r_{12})^2] \\ &= a + b \parallel U_\pi \parallel^2 - [\lambda - \text{trace}(\alpha \mid_{\pi^\perp})] + \sum_{r=n+1}^{n+p} [h^r_{11}h^r_{22} - (h^r_{12})^2].\end{aligned} \tag{3.4}$$

From (3.3) and (3.4) one gets

$$\begin{aligned}\tau(x) - K(\pi) &= \frac{(n+1)(n-2)}{2}a + b[(n-1) \parallel U^T \parallel^2 - \parallel U_\pi \parallel^2] \\ &\quad -(n-2)\lambda - \text{trace}(\alpha \mid_{\pi^\perp}) \\ &\quad + \sum_{r=n+1}^{n+p} \left[\sum_{1\le i<j\le n} h^r_{ii}h^r_{jj} - h^r_{11}h^r_{22} - \sum_{1\le i<j\le n} (h^r_{ij})^2 + (h^r_{12})^2 \right] \\ &= \frac{(n+1)(n-2)}{2}a + b[(n-1) \parallel U^T \parallel^2 - \parallel U_\pi \parallel^2] \\ &\quad -(n-2)\lambda - \text{trace}(\alpha \mid_{\pi^\perp}) \\ &\quad + \sum_{r=n+1}^{n+p} \left[(h^r_{11} + h^r_{22}) \sum_{3\le i\le n} h^r_{ii} + \sum_{3\le i<j\le n} h^r_{ii}h^r_{jj} \right. \\ &\quad \left. - \sum_{3\le j\le n} (h^r_{1j})^2 - \sum_{2\le i<j\le n} (h^r_{ij})^2 \right] \\ &\le \frac{(n+1)(n-2)}{2}a + b[(n-1) \parallel U^T \parallel^2 - \parallel U_\pi \parallel^2] \\ &\quad -(n-2)\lambda - \text{trace}(\alpha \mid_{\pi^\perp}) \\ &\quad + \sum_{r=n+1}^{n+p} \left[(h^r_{11} + h^r_{22}) \sum_{3\le i\le n} h^r_{ii} + \sum_{3\le i<j\le n} h^r_{ii}h^r_{jj} \right].\end{aligned} \tag{3.5}$$

Let us consider the quadratic forms $f_r : \mathbb{R}^n \to \mathbb{R}$, defined by

$$f_r(h_{11}^r, h_{22}^r, \cdots, h_{nn}^r) = (h_{11}^r + h_{22}^r) \sum_{3 \le i \le n} h_{ii}^r + \sum_{3 \le i < j \le n} h_{ii}^r h_{jj}^r.$$

We consider the problem $max f_r$, subject to $\Xi : h_{11}^r + h_{22}^r + \cdots + h_{nn}^r = k^r$, where k^r is a real constant. From Lemma 2.4, we see that the solution $(h_{11}^r, h_{22}^r, \cdots, h_{nn}^r)$ of the problem in question must satisfy

$$h_{11}^r + h_{22}^r = h_{ii}^r = \frac{k^r}{n-1}, \quad i = 3, \cdots, n, \tag{3.6}$$

with the following holds

$$f_r \le \frac{n-2}{2(n-1)} (k^r)^2. \tag{3.7}$$

Form (3.5) and (3.7) we have

$$\begin{aligned} \tau(x) - K(\pi) \le & \frac{(n+1)(n-2)}{2} a + b[(n-1) \parallel U^T \parallel^2 - \parallel U_\pi \parallel^2] \\ & -(n-2)\lambda - \text{trace}(\alpha \mid_{\pi^\perp}) + \sum_r \frac{n-2}{2(n-1)} (k^r)^2 \\ = & \frac{(n+1)(n-2)}{2} a + b[(n-1) \parallel U^T \parallel^2 - \parallel U_\pi \parallel^2] \\ & -(n-2)\lambda - \text{trace}(\alpha \mid_{\pi^\perp}) + \frac{n^2(n-2)}{2(n-1)} \parallel H \parallel^2, \end{aligned}$$

which represents the inequality to prove. ■

Corollary 3.3. *If P is a tangent vector field on M^n, then $H = \hat{H}$, here we used Lemma 2.1. In this case the inequality proved in Theorem 3.1 becomes*

$$\begin{aligned} \tau(x) - K(\pi) \le & \frac{(n+1)(n-2)}{2} a + b[(n-1) \parallel U^T \parallel^2 - \parallel U_\pi \parallel^2] \\ & -(n-2)\lambda - \text{trace}(\alpha \mid_{\pi^\perp}) + \frac{n^2(n-2)}{2(n-1)} \parallel \hat{H} \parallel^2 . \end{aligned} \tag{3.8}$$

Corollary 3.4. *If P is a tangent vector field on M^n, then $h = \hat{h}$. In these conditons the equality case of* (3.8) *holds at a point $x \in M$ if and only if, with respect to a suitable orthonormal basis $\{e_A\}$ at x, the shape operators $A_r = A_{e_r}$ take the following forms:*

$$A_{n+1} = \begin{pmatrix} h_{11}^{n+1} & 0 & 0 & \cdots & 0 \\ 0 & h_{22}^{n+1} & 0 & \cdots & 0 \\ 0 & 0 & h_{11}^{n+1}+h_{22}^{n+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & h_{11}^{n+1}+h_{22}^{n+1} \end{pmatrix}$$

and

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, r = n+2, \cdots, n+p.$$

Proof. If the equality case of (3.8) holds at a point $x \in M$, then the equality cases of (3.5) and (3.7) hold, it follows that

$$\sum_{3 \le i \le n} (h_{1i}^r)^2 = 0, \sum_{2 \le i < j \le n} (h_{ij}^r)^2 = 0, \ \forall r,$$

$$h_{11}^r + h_{22}^r = h_{ii}^r, 3 \le i \le n, \quad \forall r.$$

So choose a suitable orthonormal basis, the shape operators take the desired forms. ■

Corollary 3.5. *Under the same assumptions as in Theorem* 3.1, *if U is tangent to M^n, we have*

$$\tau(x) \ -K(\pi) \le \frac{(n+1)(n-2)}{2}a + b[n-1- \parallel U_\pi \parallel^2] \\ -(n-2)\lambda - \text{trace}(\alpha \mid_{\pi^\perp}) + \frac{n^2(n-2)}{2(n-1)} \parallel H \parallel^2,$$

If U is normal to M^n, we have

$$\tau(x) - K(\pi) \le \frac{(n+1)(n-2)}{2}a - (n-2)\lambda - \text{trace}(\alpha \mid_{\pi^\perp}) + \frac{n^2(n-2)}{2(n-1)} \parallel H \parallel^2 .$$

4. Chen's General Inequality

Next we prove a generalization of Theorem 3.1 in terms of Chen's invariant $\delta(n_1, \cdots, n_k)$.

Theorem 4.1. *If M^n $(n \geq 3)$ is a submanifold of a Riemannian manifold of quasi-constant curvature N^{n+p} endowed with a semi-symmetric metric connection, then we have*

$$\delta(n_1, \cdots, n_k) \leq c(n_1, \cdots, n_k) \parallel H \parallel^2 + d(n_1, \cdots, n_k)a - (n-1)\lambda + \sum_{j=1}^{k} \Psi_1(L_j) + b[(n-1) \parallel U^T \parallel^2 - \sum_{j=1}^{k} \Psi_2(L_j)], \tag{4.1}$$

for any $k-$tuples $(n_1, \cdots, n_k) \in S(n)$. If P is a tangent vector field on M^n, the equality case of (4.1) *holds at $x \in M^n$ if and only if there exist an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_xM and an orthonormal basis $\{e_{n+1}, \cdots, e_{n+p}\}$ of $T_x^{\perp}M$ such that the shape operators of M^n in N^{n+p} at x have the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, A_{e_r} = \begin{pmatrix} A_1^r & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_k^r & 0 \\ 0 & \cdots & 0 & \varsigma_r I \end{pmatrix}, r = n+2, \cdots, n+p,$$

where $a_1, \cdots, a_n$ satisfy

$$a_1 + \cdots + a_{n_1} = \cdots = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k} = a_{n_1+\cdots+n_k+1} = \cdots = a_n$$

and each A_j^r is a symmetric $n_j \times n_j$ submatrix satisfying trace$(A_1^r) = \cdots =$ trace$(A_k^r) = \varsigma_r$. I is an identity matrix.

Remark 4.2. *For $\delta(2)$, inequality* (4.1) *is due to Theorem 3.1.*

Proof. Choose an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ for T_xM^n and $\{e_{n+1}, e_{n+2}, \cdots, e_{n+p}\}$ for the normal space $T_x^{\perp}M^n$ such that the mean curvature vector H is in the direction of the normal vector to e_{n+1}. For convenience, we set

$$\begin{aligned} & a_i = h_{ii}^{n+1}, \quad i = 1, 2, \cdots, n, \\ & b_1 = a_1, \quad b_2 = a_2 + \cdots + a_{n_1}, \quad b_3 = a_{n_1+1} + \cdots + a_{n_1+n_2}, \cdots, \\ & b_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+n_2+\cdots+n_{k-1}+n_k}, \\ & b_{k+2} = a_{n_1+\cdots+n_k+1}, \cdots, b_{\gamma+1} = a_n, \\ & \Delta_1 = \{1, \cdots, n_1\}, \cdots, \\ & \Delta_k = \{(n_1 + \cdots + n_{k-1}) + 1, \cdots, n_1 + \cdots + n_k\}, \\ & \Delta_{k+1} = (\Delta_1 \times \Delta_1) \cup \cdots \cup (\Delta_k \times \Delta_k). \end{aligned}$$

Let $L_1, \cdots, L_k$ be mutually orthogonal subspaces of T_xM with dim$L_j = n_j$, defined by

$$L_j = Span\{e_{n_1+\cdots+n_{j-1}+1}, \cdots, e_{n_1+\cdots+n_j}\}, \quad j = 1, \cdots, k.$$

From (2.5), (2.6), (2.8), (2.9) and (2.10) we have

$$\tau(L_j) = \frac{n_j(n_j-1)}{2}a + b\Psi_2(L_j) - \Psi_1(L_j) + \sum_{r=n+1}^{n+p}\sum_{\mu_j<\nu_j}[h^r_{\mu_j\mu_j}h^r_{\nu_j\nu_j} - (h^r_{\mu_j\nu_j})^2], \tag{4.2}$$

$$2\tau = n(n-1)a + 2b(n-1)\parallel U^T\parallel^2 - 2(n-1)\lambda + n^2\parallel H\parallel^2 - \parallel h\parallel^2 . \tag{4.3}$$

We can rewrite (4.3) as

$$n^2\parallel H\parallel^2 = (\parallel h\parallel^2 + \eta)\gamma,$$

or equivalently,

$$(\sum_{i=1}^{n} h^{n+1}_{ii})^2 = \gamma[\sum_{i=1}^{n}(h^{n+1}_{ii})^2 + \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h^r_{ij})^2 + \eta], \tag{4.4}$$

where

$$\eta = 2\tau - 2c(n_1,\cdots,n_k)\parallel H\parallel^2 - n(n-1)a - 2(n-1)b\parallel U^T\parallel^2 + 2(n-1)\lambda, \tag{4.5}$$

$$\gamma = n + k - \sum_{j=1}^{k} n_j.$$

From (4.4) we deduce

$$(\textstyle\sum_{i=1}^{\gamma+1} b_i)^2 = \gamma[\eta + \displaystyle\sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h^r_{ij})^2 - 2\sum_{\mu_1<\nu_1} a_{\mu_1}a_{\nu_1} - \cdots - 2\sum_{\mu_k<\nu_k} a_{\mu_k}a_{\nu_k}],$$

where $\mu_j,\nu_j \in \Delta_j$, for all $j = 1,\cdots,k$. Applying Lemma 2.3, we derive

$$\sum_{j=1}^{k}\sum_{\mu_j<\nu_j} a_{\mu_j}a_{\nu_j} \geq \frac{1}{2}[\eta + \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h^r_{ij})^2],$$

it follows that

$$\sum_{j=1}^{k}\sum_{r=n+1}^{n+p}\sum_{\mu_j<\nu_j}[h^r_{\mu_j\mu_j}h^r_{\nu_j\nu_j} - (h^r_{\mu_j\nu_j})^2] \geq \frac{\eta}{2} + \frac{1}{2}\sum_{r=n+1}^{n+p}\sum_{(\mu,\nu)\notin\Delta_{k+1}}(h^r_{\mu\nu})^2 + \sum_{r=n+2}^{n+p}\sum_{\mu_j\in\Delta_j}(h^r_{\mu_j\mu_j})^2 \geq \frac{\eta}{2}. \tag{4.6}$$

From (4.2) and (4.6) we have

$$\sum_{j=1}^{k}\tau(L_j) \geq \sum_{j=1}^{k}[\frac{n_j(n_j-1)}{2}a + b\Psi_2(L_j) - \Psi_1(L_j)] + \frac{1}{2}\eta. \tag{4.7}$$

Using (2.11), (4.5) and (4.7), we derive the desired inequality.

The equality case of (4.1) at a point $x \in M$ holds if and only if we have the equality in all the previous inequality and also in the Lemma 2.3, thus, the shape operators take the desired forms. ∎

From Theorem 4.1, we have

Corollary 4.3. *If M^n $(n \geq 3)$ is a submanifold of an $(n+p)-$dimensional real space form $N^{n+p}(c)$ of constant curvature c endowed with a semi-symmetric metric connection, then we have*

$$\begin{aligned}\delta(n_1,\cdots,n_k) \leq\ & c(n_1,\cdots,n_k) \parallel H \parallel^2 + d(n_1,\cdots,n_k)c \\ & -(n-1)\lambda + \sum_{j=1}^{k}\Psi_1(L_j),\end{aligned} \tag{4.8}$$

for any $k-$tuples $(n_1,\cdots,n_k) \in S(n)$. If P is a tangent vector field on M^n, the equality case of (4.8) holds at $x \in M^n$ if and only if there exist an orthonormal basis $\{e_1,\cdots,e_n\}$ of T_xM and an orthonormal basis $\{e_{n+1},\cdots,e_{n+p}\}$ of $T_x^{\perp}M$ such that the shape operators of M^n in N^{n+p} at x have the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},\ A_{e_r} = \begin{pmatrix} A_1^r & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_k^r & 0 \\ 0 & \cdots & 0 & \varsigma_r I \end{pmatrix},\ r=n+2,\cdots,n+p,$$

where $a_1,\cdots,a_n$ satisfy

$$a_1+\cdots+a_{n_1} = \cdots = a_{n_1+\cdots+n_{k-1}+1}+\cdots+a_{n_1+\cdots+n_k} = a_{n_1+\cdots+n_k+1} = \cdots = a_n$$

and each A_j^r is a symmetric $n_j \times n_j$ submatrix satisfying trace$(A_1^r) = \cdots =$ trace$(A_k^r) = \varsigma_r$. I is an identity matrix.

Remark 4.4. For $\delta(2)$, inequality (4.8) is due to A. Mihai and C. Özgür [2, Theorem 3.1].

5. Chen-Ricci Inequality

In [11], B.-Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for any $n-$dimensional Riemannian submanifold of a real space form $R^m(c)$ of constant sectional curvature c as follows

Theorem 5.1. (See [11, Theorem 4]). *Let M be an $n-$dimensional submanifold of a real space form $R^m(c)$. Then the following statements are true.*

(i) *For each unit vector $X \in T_pM$, we have*

$$\|H\|^2 \geq \frac{4}{n^2}[Ric(X) - (n-1)c]. \tag{5.1}$$

(ii) *If $H(p) = 0$, then a unit vector $X \in T_pM$ satisfies the equality case of* (5.1) *if and only if X belongs to the relative null space $N(p)$ given by*

$$N(p) = \{X \in T_pM \mid h(X,Y) = 0,\ \forall Y \in T_pM\}.$$

(iii) *The equality case of* (5.1) *holds for all unit vectors $X \in T_pM$ if and only if either p is a geodesic point or $n = 2$ and p is an umbilical point.*

Afterwards, many papers studied similar Chen-Ricci inequalities for different submanifolds in various ambient manifolds[12,21,24]. Besides, after putting an extra condition on the ambient manifold, like semi-symmetric metric connections in the case of real space forms [2], one proves the results similar to that of Theorem 5.1.

In this section, we establish Chen-Ricci inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature endowed with a semi-symmetric metric connection.

Theorem 5.2. *Let M^n, $n \geq 2$, be an $n-$dimensional submanifold of an $(n+p)-$dimensional Riemannian manifold of quasi-constant curvature N^{n+p} endowed with a semi-symmetric metric connection $\overline{\nabla}$. Then:*

(i) *For each unit vector X in T_xM we have*

$$\begin{aligned} Ric(X) \leq{}& (n-1)a + b[(n-2)g(U^T, X)^2 + \| U^T \|^2] \\ & -(n-2)\alpha(X,X) - \lambda + \frac{n^2}{4} \| H \|^2 . \end{aligned} \tag{5.2}$$

(ii) *If $H(x) = 0$, then a unit tangent vector X at x satisfies the equality case of* (5.2) *if and only if $X \in N(x) = \{X \in T_xM \mid h(X,Y) = 0, \forall Y \in T_xM\}$.*

(iii) *The equality of* (5.2) *holds for all unit tangent vector at x if and only if either*
(1) *$n \neq 2$, $h^r_{ij} = 0,\ i,j = 1,2,\cdots,n,\ r = n+1,\cdots,n+p$ or*
(2) *$n = 2$, $h^r_{11} = h^r_{22},\ \ h^r_{12} = 0,\ r = 3,\cdots,2+p$,*
where h is a $(0,2)$ symmetric tensor on M^n.

Proof. (i) Let$X \in T_xM$ be a unit tangent vector at x. We choose the local field of orthonormal frames (2.7) at x such that $e_1 = X$. From the equation (3.2) we have

$$\begin{aligned} Ric(X) ={}& \sum_{i=2}^{n} R_{1i1i} = (n-1)a + (n-1)bg(U^T, e_1)^2 \\ & + b\sum_{i=2}^{n} g(U^T, e_i)^2 - (n-1)\alpha(X,X) - \sum_{i=2}^{n} \alpha(e_i, e_i) \end{aligned} \tag{5.3}$$

$$
\begin{aligned}
&+\sum_{r=n+1}^{n+p}\sum_{i=2}^{n}[h_{11}^r h_{ii}^r-(h_{1i}^r)^2]\\
&\leq (n-1)a+(n-2)bg(U^T,X)^2+b\parallel U^T\parallel^2\\
&\quad -(n-2)\alpha(X,X)-\lambda+\sum_{r=n+1}^{n+p}\sum_{i=2}^{n}h_{11}^r h_{ii}^r.
\end{aligned}
$$

Let us consider the quadratic forms $f_r:\mathbb{R}^n\to\mathbb{R}$, defined by

$$
f_r(h_{11}^r,h_{22}^r,\cdots,h_{nn}^r)=\sum_{i=2}^{n}h_{11}^r h_{ii}^r.
$$

We consider the problem $max f_r$, subject to $\Xi: h_{11}^r+h_{22}^r+\cdots+h_{nn}^r=k^r$, where k^r is a real constant. From Lemma 2.5, we can see that the solution $(h_{11}^r,h_{22}^r,\cdots,h_{nn}^r)$ of the problem in question must satisfy

$$
h_{11}^r=\sum_{i=2}^{n}h_{ii}^r=\frac{k^r}{2}, \tag{5.4}
$$

with the following holds

$$
f_r\leq\frac{(k^r)^2}{4}. \tag{5.5}
$$

From (5.3) and (5.5) we have

$$
\begin{aligned}
Ric(X)&\leq (n-1)a+(n-2)bg(U^T,X)^2+b\parallel U^T\parallel^2\\
&\quad -(n-2)\alpha(X,X)-\lambda+\sum_{r=n+1}^{n+p}\frac{(k^r)^2}{4}\\
&=(n-1)a+(n-2)bg(U^T,X)^2+b\parallel U^T\parallel^2\\
&\quad -(n-2)\alpha(X,X)-\lambda+\frac{n^2}{4}\parallel H\parallel^2 .
\end{aligned}
$$

(ii) For each unit vector X at x, if the equality case of inequality (5.2) holds, from (5.3), (5.4) and (5.5) we have

$$
h_{1i}^r=0,\quad i\neq 1,\ \forall\ r, \tag{5.6}
$$

$$
h_{11}^r+h_{22}^r+\cdots+h_{nn}^r-2h_{11}^r=0,\quad \forall r. \tag{5.7}
$$

Noting that $H(x)=0$, we have $h_{11}^r=0$, then $h_{1j}^r=0,\ \forall j,r$, i.e. $X\in N(x)$.

(iii) For all unit vector X at x, if the equality case of inequality (5.2) holds, noting that X is arbitrary, by computing $Ric(e_j), j = 2, 3, \cdots, n$ and combining (5.6) and (5.7) we have

$$h^r_{ij} = 0, \quad i \neq j, \quad \forall r,$$

$$h^r_{11} + h^r_{22} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \quad \forall i, r.$$

We can distinguish two cases:

(1) $n \neq 2$, $h^r_{ij} = 0, \ i, j = 1, 2, \cdots, n, \ r = n+1, \cdots, n+p$ or
(2) $n = 2$, $h^r_{11} = h^r_{22}, \quad h^r_{12} = 0, \ r = 3, \cdots, 2+p$.

The converse is trivial. ■

Theorem 5.3. *If the equality case of inequality* (5.2) *holds for all unit tangent vector X of M^n, then M^n is a totally umbilical submanifold. Moreover, we have*

(i) *The equality case of inequality* (5.2) *holds for all unit tangent vector X of M^2 if and only if M^2 is a totally umbilical submanifold.*

(ii) *If P is a tangent vector field on M^n and $n \geq 3$, M^n is a totally geodesic submanifold.*

Proof. For $n = 2$, the equality case of inequality (5.2) holds for all unit tangent vector X of M^2 if and only if M^2 is a totally umbilical submanifold with respect to the semi-symmetric metric connection. Then from Lemma 2.2, M^2 is a totally umbilical submanifold with respect to the Levi-Civita connection. For $n \geq 3$, from Theorem 5.2 the the equality case of inequality (5.2) holds for all unit tangent vector X of M^n if and only if $h^r_{ij} = 0, \forall i, j, r$. According to the formula (7) from [30], we have $\hat{h}^r_{ij} = h^r_{ij} + k^r g_{ij}$, where k^r are real-valued functions on M. Thus, we have $\hat{h}^r_{ij} = k^r g_{ij}$, which implies M^n is a totally umbilical submanifold.

If P is a tangent vector field on M^n, from Lemma 2.1 we have $\hat{h} = h$. For $n \geq 3$, from Theorem 5.2 the the equality case of inequality (5.2) holds for all unit tangent vector X of M^n if and only if $h^r_{ij} = 0, \forall i, j, r$. Thus we have $\hat{h}^r_{ij} = 0, \forall i, j, r$, which implies M^n is a totally geodesic submanifold. ■

In [2], A. Mihai and C. Özgür proved:

Theorem 5.4. (See [2, Theorem 4.1]) *Let M^n be an $n-$dimensional submanifold of an $(n+p)-$dimensional real space form $N^{n+p}(c)$ endowed with a semi-symmetric metric connection. Then*

(i) *For each unit vector X in T_xM we have*

$$Ric(X) \leq (n-1)c + \frac{n^2 \parallel H \parallel^2}{4} + (n-2)\alpha(X, X) - (2n-3)\lambda. \tag{5.8}$$

(ii) *If* $H(x) = 0$*, then a unit tangent vector* X *at* x *satisfies the equality case of* (5.8) *if and only if* $X \in N(x) = \{X \in T_xM \mid h(X, Y) = 0, \forall Y \in T_xM\}$.

Further on, they obtained

Corollary 5.5. (see [2, Corollary 4.2]). *If* P *is tangent to* M^n*, then the equality case of inequality* (5.8) *holds for all unit tangent vectors at* x *if and only if either* x *is a totally geodesic point, or* $n = 2$ *and* x *is a totally umbilical point.*

Remark 5.6. Without the condition that P is tangent to M, we can also classify submanifolds in real space forms endowed with semi-symmetric metric connection satisfying the equality case of (5.8).

Remark 5.7. For $n \neq 2$, if the equality case of (5.9) holds for all unit tangent vectors X at x, from Corollary 5.5, we know that $h_{ij}^r = 0, \ \forall i, j, r$. Further, using the equation of Gauss we have

$$Ric(X) = \sum_{i=2}^{n} R_{1i1i} = (n-1)c - (n-2)\alpha(X, X) - \lambda,$$

here is a contradiction with the equality case of (5.8).

Remark 5.8. In the proof of Theorem 4.1 in [2], they wrote

$$\begin{aligned} n^2 \parallel H \parallel^2 &\geq \frac{1}{2} n^2 \parallel H \parallel^2 + 2\big(\tau - \sum_{2 \leq i < j \leq n}\big) K_{ij} + 2 \sum_{r=n+1}^{n+p} \sum_{j=2}^{n} (h_{1j}^r)^2 \\ &= -2(n-1)c + 2(2n-3)\lambda - 2(n-2)\alpha(e_1, e_1), \end{aligned}$$

but according to the formula (4.2) and (4.3) in [2], one gets

$$\begin{aligned} n^2 \parallel H \parallel^2 &\geq \frac{1}{2} n^2 \parallel H \parallel^2 + 2\big(\tau - \sum_{2 \leq i < j \leq n}\big) K_{ij} + 2 \sum_{r=n+1}^{n+p} \sum_{j=2}^{n} (h_{1j}^r)^2 \\ &= -2(n-1)c + 2\lambda + 2(n-2)\alpha(e_1, e_1). \end{aligned}$$

This is the reason they made a mistake.

Under these circumstances it becomes necessary to give a theorem, which could present a sharp inequality between the Ricci-curvature and the squared mean curvature with respect to the semi-symmetric metric connection. From Theorem 5.2 and Theorem 5.3 we have

Corollary 5.9. *Let* M^n *be an* $n-$*dimensional submanifold of an* $(n+p)-$*dimensional real space form* $N^{n+p}(c)$ *of constant curvature* c *endowed with a semi-symmetric metric connection. Then:*

(i) *For each unit vector X in T_xM we have*

$$Ric(X) \leq (n-1)c - (n-2)\alpha(X,X) - \lambda + \tfrac{n^2}{4} \parallel H \parallel^2 . \tag{5.9}$$

(ii) *If $H(x) = 0$, then a unit tangent vector X at x satisfies the equality case of* (5.9) *if and only if $X \in N(x) = \{X \in T_xM \mid h(X,Y) = 0, \forall Y \in T_xM\}$.*

(iii) *If the equality case of inequality* (5.9) *holds for all unit tangent vector X of M^n, then M^n is a totally umbilical submanifold. Moreover, we have*

(1) *The equality case of inequality* (5.9) *holds for all unit tangent vector X of M^2 if and only if M^2 is a totally umbilical submanifold.*

(2) *If P is a tangent vector field on M^n and $n \geq 3$, M^n is a totally geodesic submanifold.*

6. $k-$Ricci Curvature

Let L be a $k-$plane section of T_xM^n, $x \in M$, and X a unit vector in L. We choose an orthonormal frame $e_1, \cdots, e_k$ of L such that $e_1 = X$. In [11], B.-Y. Chen defined the $k-$Ricci curvature of L at X by

$$Ric_L(X) = K_{12} + K_{13} + \cdots + K_{1k}. \tag{6.1}$$

The scalar curvature of a $k-$plane section L is given by

$$\tau(L) = \sum_{1 \leq i < j \leq n} K_{ij}. \tag{6.2}$$

For an integer $k,\ 2 \leq k \leq n$, the Riemannian invariant Θ_k on M^n defined by

$$\Theta_k(x) = \frac{1}{k-1}\inf\{Ric_L(X) \mid L, X\},\ x \in M, \tag{6.3}$$

where L runs over all $k-$plane sections in T_xM and X runs over all unit vectors in L.

From (2.8), (6.1) and (6.2), it follows that for any $k-$plane section $L_{i_1 \cdots i_k}$ spanned by $\{e_{i_1}, \cdots, e_{i_k}\}$, one has

$$\tau(L_{i_1 \cdots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \cdots, i_k\}} Ric_{L_{i_1 \cdots i_k}}(e_i) \tag{6.4}$$

and

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \tau(L_{i_1 \cdots i_k}). \tag{6.5}$$

From (6.3), (6.4) and (6.5) we obtain

$$\tau(x) \geq \frac{n(n-1)}{2}\Theta_k(x). \tag{6.6}$$

In this section, we prove a relationship between the $k-$Ricci curvature of M^n (intrinsic invariant) and the mean curvature $\parallel H \parallel$ (extrinsic invariant), as another answer of the basic problem in submanifold theory which we have mentioned in the introduction. In this section, we assume that the vector field P is tangent to M^n.

Theorem 6.1. *Let M^n, $n \geq 3$, be an $n-$dimensional submanifold of an $(n+p)-$dimensional Riemannian manifold of quasi-constant curvature N^{n+p} endowed with a semi-symmetric metric connection $\overline{\nabla}$, then for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\parallel H \parallel^2 (x) \geq \Theta_k(x) - a - \frac{2b}{n} \parallel U^T \parallel^2 + \frac{2}{n}\lambda.$$

Remark 6.2. For $b = 0$, Theorem 6.1 is due to A. Mihai and C. Özgür [2, Theorem 5.2].

Proof. We choose the orthonormal frame (2.7) at x such that the e_{n+1} is in the direction of the mean curvature vector $H(x)$ and $\{e_1, \cdots, e_n\}$ diagonalize the shape operator A_{n+1}. Then the shape operators take the following forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \tag{6.7}$$

$$\text{trace}A_r = 0, \ r = n+2, \cdots, n+p.$$

From (4.3) and (6.7) we have

$$\begin{aligned} n^2 \parallel H \parallel^2 &= 2\tau + \parallel h \parallel^2 - (n^2-n)a - 2b(n-1) \parallel U^T \parallel^2 + 2(n-1)\lambda \\ &= 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - (n^2-n)a \\ &\quad -2b(n-1) \parallel U^T \parallel^2 + 2(n-1)\lambda. \end{aligned} \tag{6.8}$$

Using the Cauchy-Schwartz inequality we have

$$(n \parallel H \parallel)^2 = (\sum_{i=1}^{n} a_i)^2 \leq n \sum_{i=1}^{n} a_i^2,$$

it follows that

$$\sum_{i=1}^{n} a_i^2 \geq n \parallel H \parallel^2 . \tag{6.9}$$

From (6.8) and (6.9) we have

$$n^2 \parallel H \parallel^2 \geq 2\tau + n \parallel H \parallel^2 - (n^2 - n)a - 2b(n-1) \parallel U^T \parallel^2 + 2(n-1)\lambda,$$

which implies

$$\parallel H \parallel^2 \geq \frac{2\tau}{n(n-1)} - a - \frac{2b}{n} \parallel U^T \parallel^2 + \frac{2}{n}\lambda. \tag{6.10}$$

Using (6.6) and (6.10) we have

$$\parallel H \parallel^2 (x) \geq \Theta_k(x) - a - \frac{2b}{n} \parallel U^T \parallel^2 + \frac{2}{n}\lambda.$$ ■

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