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SIGN-CHANGING SOLUTIONS FOR A CLASS OF DAMPED VIBRATION PROBLEMS WITH IMPULSIVE EFFECTS

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Abstract. In this paper, some sufficient conditions are obtained for the existence and multiplicity of sign-changing solutions for the damped vibration problem with impulsive effects

$$\begin{cases} -u''(t)+g(t)u'(t)=f(t,u(t)), & \text{a.e. } t\in[0,T];\\ u(0)=u(T)=0, & \\ \Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)=I_j(u(t_j)), & j=1,2,\ldots,p, \end{cases}$$

where $t_0=0<t_1<t_2<\ldots<t_p<t_{p+1}=T, g\in L^1(0,T;\mathbb{R}), I_j:\mathbb{R}\to\mathbb{R}, j=1,2,\ldots,p$ are continuous, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with subcritical growth condition:

(A) $|f(t,u)|\le C(1+|u|^{s-1}), \forall t\in[0,T], u\in\mathbb{R}, s\in[2,+\infty)$.

The sign-changing solutions are sought by means of some sign-changing critical point theorems and two examples are presented to illustrate the feasibility and effectiveness of our results.

1. INTRODUCTION

Consider the damped vibration problem with impulse

$$\begin{cases} -u''(t)+g(t)u'(t)=f(t,u(t)), & \text{a.e. } t\in[0,T];\\ u(0)=u(T)=0, & \\ \Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)=I_j(u(t_j)), & j=1,2,\ldots,p, \end{cases} \tag{1.1}$$

where $t_0=0<t_1<t_2<\ldots<t_p<t_{p+1}=T, g\in L^1(0,T;\mathbb{R}), I_j:\mathbb{R}\to\mathbb{R}, j=1,2,\ldots,p$ are continuous, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with subcritical growth condition:

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(A) $|f(t,u)| \leq C(1+|u|^{s-1}), \forall t \in [0,T], u \in \mathbb{R}, s \in [2,+\infty)$.

Impulsive effects exist widely in many evolution processes in which their states are changed abruptly at certain moments of time. The theory of impulsive differential systems has been developed by numerous mathematicians (see [1-4]). Applications of impulsive differential equations with or without delays occur in biology, medicine, mechanics, engineering, chaos theory and so on (see [5-7]).

For a second order differential equation $u'' = f(t,u,u')$, one usually considers impulses in the position u and the velocity u'. However, in the motion of spacecraft one has to consider instantaneous impulses depending on the position that result in jump discontinuities in velocity, but with no changes in position (see [8,9]). The impulse only on the velocity occurs also in impulsive mechanics (see [10]).

In recent years, impulsive and periodic boundary value problems have been studied extensively in the literature. There have been many approaches to study periodic solutions of differential equations, such as method of lower and upper solutions, fixed-point theory, coincidence degree theory and so on. In [11], authors used the method of lower and upper solutions with monotone iterative technique to study impulsive differential equations. In [12], authors used the Krasnoselskii<¯s fixed point theorem in a cone to impulsive differential equations and obtained the existence of positive solutions. However, the study of solutions for impulsive differential equations using variational methods has received considerably less attention (see, for example [9,13-15]).

Especially, when $g(t) \equiv 0$, authors in [13-15] used variational methods to study the existence and multiplicity of solutions for problems (1.1). But, to the best of authors' knowledge, when $g(t) \not\equiv 0$, the existence and multiplicity of sign-changing solutions for problem (1.1) have not been studied yet. Our purpose of this paper is to study the sign-changing solutions of problem (1.1) in an appropriate space of functions and the existence and multiplicity of sign-changing solutions for problem (1.1) by some sign-changing critical point theorems.

2. Preliminaries

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we make a variational structure. From this variational structure, we can reduce the problem of finding solutions of problem (1.1) to the one of seeking the critical points of a corresponding functional.

In the Sobolev space $H_0^1(0,T)$, consider the inner product

$$\langle u,v\rangle_{H_0^1(0,T)} = \int_0^T u'(t)v'(t)\,\mathrm{d}t,$$

inducing the norm

$$\|u\|_{H_0^1(0,T)} = \left(\int_0^T (u'(t))^2\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Let $G(t) = \int_0^t g(s)\mathrm{d}s$. Since $g \in L^1(0,T;\mathbb{R})$, $G : [0,T] \to \mathbb{R}$ is absolutely continuous. Therefore,

$$\alpha = \min_{t\in[0,T]} e^{G(t)} > 0,$$

$$\beta = \max_{t\in[0,T]} e^{G(t)} > 0.$$

We also consider the inner product

$$\langle u, v\rangle = \int_0^T e^{G(t)} u'(t)v'(t)\,\mathrm{d}t, \quad \forall u, v \in H_0^1(0,T)$$

and the norm

$$\|u\| = \left(\int_0^T e^{G(t)}\big(u'(t)\big)^2\,\mathrm{d}t\right)^{\frac{1}{2}}, \quad \forall u \in H_0^1(0,T).$$

Then the norm $\|\cdot\|$ and the norm $\|\cdot\|_{H_0^1(0,T)}$ are equivalent and there exists $C_1 > 0$ such that

$$\|u\|_\infty \le C_1\|u\|, \quad \forall u \in H_0^1(0,T). \tag{2.1}$$

Let $\lambda_k (k = 1, 2, \dots)$ denote the eigenvalues and $\varphi_k (k = 1, 2, \dots)$ the corresponding eigenfunctions of the problem

$$\begin{cases} -u''(t) = \lambda u(t), & t \in [0,T]; \\ u(0) = u(T) = 0, \end{cases}$$

where each eigenvalue λ_k is repeated as often as multiplicity recall that $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots, \lambda_k \longrightarrow \infty$. Then φ_1 is positive (or negative) and the eigenfunction associated to $\lambda_i (i \ge 2)$ is sign-changing. Let X_k denote the eigenspace associated to λ_k, then $H_0^1(0,T) = \overline{\bigoplus_{i\in\mathbb{N}} X_i}$. We denote by $\|\cdot\|_p$ the norm in $L^p(0,T)$.

For $u \in H^2(0,T)$, we have that u and u' are both absolutely continuous, and $u'' \in L^2(0,T)$. Hence, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ for any $t \in [0,T]$.

If $u \in H_0^1(0,T)$, then u is absolutely continuous and $u' \in L^2(0,T)$. In this case, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ may not hold for some $t \in (0,T)$. It allows to consider impulsive effects in the derivative.

Take $v \in H_0^1(0,T)$ and multiply two sides of the equality

$$u''(t) + g(t)u'(t) + f(t, u(t)) = 0$$

by $e^{G(t)}v$ and integrate from 0 to T, we have

$$
\begin{aligned}
&\int_0^T e^{G(t)} u''(t)v(t)\,\mathrm{d}t + \int_0^T e^{G(t)} g(t)u'(t)v(t)\,\mathrm{d}t \\
&+ \int_0^T e^{G(t)} f(t,u(t))v(t)\,\mathrm{d}t = 0.
\end{aligned} \tag{2.2}
$$

Moreover,

$$
\begin{aligned}
&\int_0^T \left(e^{G(t)}u''(t)v(t) + e^{G(t)}g(t)u'(t)v(t)\right)\mathrm{d}t \\
={}& \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} \left(e^{G(t)}u''(t)v(t) + e^{G(t)}g(t)u'(t)v(t)\right)\mathrm{d}t \\
={}& \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} v(t)\,\mathrm{d}e^{G(t)}u'(t) \\
={}& \sum_{j=0}^{p} \left(e^{G(t_{j+1}^-)}u'(t_{j+1}^-)v(t_{j+1}^-) - e^{G(t_j^+)}u'(t_j^+)v(t_j^+) - \int_{t_j}^{t_j+1} e^{G(t)}u'(t)v'(t)\,\mathrm{d}t\right) \\
={}& -\sum_{j=1}^{p} e^{G(t_j)}\Delta u'(t_j)v(t_j) - u'(0)v(0) + e^{G(T)}u'(T)v(T) - \int_0^T e^{G(t)}u'(t)v'(t)\,\mathrm{d}t \\
={}& -\sum_{j=1}^{p} e^{G(t_j)}I_j(u(t_j))v(t_j) - \int_0^T e^{G(t)}u'(t)v'(t)\,\mathrm{d}t.
\end{aligned}
$$

Combining (2.2), we have

$$
\int_0^T e^{G(t)}u'(t)v'(t)\,\mathrm{d}t + \sum_{j=1}^{p} e^{G(t_j)}I_j(u(t_j))v(t_j) - \int_0^T e^{G(t)}f(t,u(t))v(t)\,\mathrm{d}t = 0.
$$

Considering the above, we introduce the following concept solution for problem (1.1).

Definition 2.1. We say that a function $u \in H_0^1(0,T)$ is a weak solution of problem (1.1) if the identity

$$
\int_0^T e^{G(t)}u'(t)v'(t)\,\mathrm{d}t + \sum_{j=1}^{p} e^{G(t_j)}I_j(u(t_j))v(t_j) = \int_0^T e^{G(t)}f(t,u(t))v(t)\,\mathrm{d}t
$$

holds for any $v \in H_0^1(0,T)$.

Consider the functional $\varphi : H_0^1(0,T) \to \mathbb{R}$ defined by

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p} e^{G(t_j)} \int_0^{u(t_j)} I_j(t)\,\mathrm{d}t - \int_0^T e^{G(t)} F(t,u(t))\,\mathrm{d}t \\ &= \frac{1}{2}\|u\|^2 - J_1(u) - J_2(u) \\ &= \frac{1}{2}\|u\|^2 - J(u),\end{aligned}$$

where $F(t,s) = \int_0^s f(t,\tau)\mathrm{d}\tau$ and

$$\begin{aligned}J_1(u) &= -\sum_{j=1}^{p} e^{G(t_j)} \int_0^{u(t_j)} I_j(t)\,\mathrm{d}t,\ J_2(u) \\ &= \int_0^T e^{G(t)} F(t,u(t))\,\mathrm{d}t,\ J(u) = J_1(u) + J_2(u).\end{aligned}$$

Using the subcritical growth condition (A) and the continuity of $I_j, j = 1, 2, \ldots, p$, one has that $\varphi \in C^1(H_0^1(0,T), \mathbb{R})$. For any $v \in H_0^1(0,T)$, we have

$$\varphi'(u)v = \int_0^T e^{G(t)} u'(t)v'(t)\,\mathrm{d}t + \sum_{j=1}^{p} e^{G(t_j)} I_j(u(t_j))v(t_j) - \int_0^T e^{G(t)} f(t,u(t))v(t)\,\mathrm{d}t.$$

Thus, the solutions of problem (1.1) are corresponding to the critical points of φ.

Lemma 2.1. *J_1' is continuous on $H_0^1(0,T)$.*

Proof. Let $\{u_k\} \subseteq H_T^1$ and $u_k \to u$. By (2.1), $\|u_k - u\|_\infty \to 0$. Thus, we have

$$\begin{aligned}\|J_1'(u_k) - J_1'(u)\| &= \sup_{v \in H_0^1(0,T), \|v\| \le 1} \left|\langle J_1'(u_k) - J_1'(u), v\rangle\right| \\ &= \sup_{v \in H_0^1(0,T), \|v\| \le 1} \left|\sum_{j=1}^{p} e^{G(t_j)} \left[I_j(u_k(t_j)) - I_j(u(t_j))\right] v(t_j)\right| \\ &\le \sup_{v \in H_0^1(0,T), \|v\| \le 1} \sum_{j=1}^{p} \beta \left|I_j(u_k(t_j)) - I_j(u(t_j))\right| \|v\|_\infty \\ &\le C_1 \beta \sup_{v \in H_0^1(0,T), \|v\| \le 1} \sum_{j=1}^{p} \left|I_j(u_k(t_j)) - I_j(u(t_j))\right| \|v\| \\ &= C_1 \beta \sum_{j=1}^{p} \left|I_j(u_k(t_j)) - I_j(u(t_j))\right|.\end{aligned}$$

The continuity of I_j and this imply that $J_1'(u_k) \to J_1'(u)$ in $H_0^1(0,T)$. The proof is complete. ■

Lemma 2.2. *J' is continuous on $H_0^1(0,T)$.*

Proof. By condition (A) and [16, Lemma 2.3], J_2' is continuous on $H_0^1(0,T)$. By Lemma 2.1 and $J(u) = J_1(u) + J_2(u)$, we have J' is continuous on $H_0^1(0,T)$. The proof is complete. ∎

To prove our main results, we need the following definitions and theorems.

Definition 2.2. ([17], p. 81). Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$. I is said to satisfy (PS) condition on X if any sequence $\{x_n\} \subseteq X$ for which $I(x_n)$ is bounded and $I'(x_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in X.

Definition 2.3. ([18]). Let E is Hilbert space, $I \in C^1(E,R)$. We say that I satisfies (w-PS) condition on E if $\{u_n\} \subseteq E$ and $I(u_n)$ is bounded, $I'(u_n) \to 0$, we have either $\{u_n\}$ is bounded and has a convergent subsequence or $\|I'(u_n)\|\|u_n\| \to \infty$.

Remark 2.1. It is clear that the (PS) condition implies the (w-PS) condition.

Theorem 2.1. ([19], Theorem 3.2). *Assume that H is Hilbert space, f satisfies (PS) condition on H and $f'(u)$ has the expression $f'(u) = u - Au$. D_1 and D_2 are open convex subsets of H, $D_1 \bigcap D_2 \neq \emptyset$, $A(\partial D_1) \subset D_1, A(\partial D_2) \subset D_2$. If there exists a path $h : [0,1] \to H$ such that*

$$h(0) \in D_1 \backslash D_2, h(1) \in D_2 \backslash D_1$$

and

$$\inf_{u \in \overline{D_1} \bigcap \overline{D_2}} f(u) > \sup_{t \in [0,1]} f(h(t)).$$

Then f has at least four critical points: $u_1 \in D_1 \bigcap D_2, u_2 \in D_1 \backslash \overline{D_2}, u_3 \in D_2 \backslash \overline{D_1}, u_4 \in H \backslash (\overline{D_1} \bigcup \overline{D_2})$.

Theorem 2.2. ([18], Theorem 2.1). *Let E be a Hilbert space with inner product $\langle , \rangle$ and norm $\| \cdot \|$. Assume that E has an orthogonal decomposition $E = N \bigoplus M$ with $dimN < \infty$. Let $G \in C^1(E,R)$ and the gradient G' be of the form*

$$G'(u) = u - J'(u),$$

where $J' : E \to E$ is a continuous operator. Let P denote a closed convex positive cone of E; $D_0^{(i)}$ be an open convex subset of $E, i = 1,2$, $S = E \backslash W$, $W = D_0^{(1)} \bigcup D_0^{(2)}$. Assume

(H_1) *$J'(D_0^{(i)}) \subset D_0^{(i)}$, $i = 1,2$.*

(H_2) *If $D_0^{(1)} \bigcap D_0^{(2)} = \emptyset$, then either $D_0^{(1)} = \emptyset$ or $D_0^{(2)} = \emptyset$.*

(H_3) *There exist a* $\delta > 0$ *and* $z_0 \in N$ *with* $\|z_0\| = 1$ *such that*

$$B := \{u \in M : \|u\| \geq \delta\} \bigcup \{kz_0 + v : v \in M, k \geq 0, \|kz_0 + v\| = \delta\} \subset S.$$

Let G *map bounded sets to bounded sets and satisfy (w-PS) condition and*

$$b_0 = \inf_M G \neq -\infty, a_0 = \sup_N G \neq +\infty.$$

Then G *has a critical point in* S *with critical value* $\geq \inf_B G$.

3. Main Results

Our main results of this paper are as follows.

Theorem 3.1. *Suppose that* (A) *and the following conditions are satisfied.*

(f_1) *there exists* $\eta > 2$ *such that*

$$0 \leq \eta F(t,u) \leq f(t,u)u, \ \ \forall t \in [0,T], \forall u \in \mathbb{R}.$$

Moreover $f(t,u) = o(|u|)$ *as* $u \to 0$ *uniformly in* $t \in [0,T]$*;*

(f_2) $\eta \int_0^u I_j(\tau)\,\mathrm{d}\tau \geq I_j(u)u \ \ \forall u \in \mathbb{R}, j = 1,2,\dots,p$*;*

(f_3) $I_j(u) \geq 0, \ \ \forall u \in \mathbb{R}, j = 1,2,\dots,p$*;*

(f_4) *there exists* $a_j > 0$ *such that*

$$\left| \int_0^u I_j(\tau)\,\mathrm{d}\tau \right| \leq a_j, \ \ \forall u \in \mathbb{R}, j = 1,2,\dots,p.$$

Then problem (1.1) *has four solutions: one zero solution, one positive solution, one negative solution and one sign-changing solution.*

Theorem 3.2. *Assume that* $(\mathrm{A}), (f_1), (f_2), (f_3), (f_4)$ *and the following condition are satisfied.*

(f_5) $\lambda_l u^2 - W_1(t) \leq 2e^{G(t)}F(t,u) \leq \lambda_{l+1}u^2 + W_2(t)$, *a.e* $t \in [0,T], u \in \mathbb{R}$, *where* $W_1, W_2 \in L^1(0,T), l \geq 2$.

Then problem (1.1) *has at least one sign-changing solution.*

4. Proofs of Theorems

For $\mu_{0>0}$, let

$$\begin{aligned} P \ &= \{u \in H_0^1(0,T) : u(t) \geq 0 \ \ a.e. \ \ t \in [0,T]\}, \\ D_0(\mu_0) \ &= \{u \in H_0^1(0,T) : dist(u,P) < \mu_0\}, \\ -D_0(\mu_0) \ &= \{u \in H_0^1(0,T) : dist(u,-P) < \mu_0\}. \end{aligned}$$

Lemma 4.1. *Assume that* $(\mathrm{A}), (f_1)$ *and* (f_2) *hold, then* φ *satisfies (PS) condition.*

Proof. Assume that $\{u_n\} \subseteq H_0^1(0,T)$ is a (PS) sequence, that is, $|\varphi(u_n)| \leq C_2, \varphi'(u_n) \to 0$. By (f_1) and (f_2), we have

$$\begin{aligned}
&\eta C_2 + \|\varphi'(u_n)\|\|u_n\| \\
&\geq \eta\varphi(u_n) - \langle \varphi'(u_n), u_n\rangle \\
&= \frac{\eta}{2}\|u_n\|^2 + \eta\sum_{j=1}^{p} e^{G(t_j)}\int_0^{u_n(t_j)} I_j(t)\,\mathrm{d}t - \eta\int_0^T e^{G(t)}F(t,u_n(t))\,\mathrm{d}t \\
&\quad -\langle u_n, u_n\rangle - \sum_{j=1}^{p} e^{G(t_j)} I_j(u_n(t_j))u_n(t_j) + \int_0^T e^{G(t)} f(t,u_n(t))u_n(t)\,\mathrm{d}t \\
&= \frac{\eta-2}{2}\|u_n\|^2 - \int_0^T e^{G(t)}\big(\eta F(t,u_n(t)) - f(t,u_n(t))u_n(t)\big)\,\mathrm{d}t \\
&\quad + \sum_{j=1}^{p} e^{G(t_j)}\left(\eta\int_0^{u_n(t_j)} I_j(t)\,\mathrm{d}t - I_j(u_n(t_j))u_n(t_j)\right) \\
&\geq \frac{\eta-2}{2}\|u_n\|^2.
\end{aligned}$$

Thus $\{u_n\}$ is bounded in $H_0^1(0,T)$. Hence there exists a subsequence of $\{u_n\}$ (for simplicity denoted again by $\{u_n\}$) such that

$$u_n \rightharpoonup u \quad \text{in } H_0^1(0,T), \tag{4.1}$$

$$u_n \to u \quad \text{uniformly in } C([0,T]). \tag{4.2}$$

On the other hand, we have

$$\begin{aligned}
&\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle \\
&= \int_0^T |u_n'(t) - u'(t)|^2 \mathrm{d}t \\
&\quad + \sum_{j=1}^{p} e^{G(t_j)}\big(I_j(u_n(t_j)) - I_j(u(t_j))\big)\big(u_n(t_j) - u(t_j)\big) \\
&\quad - \int_0^T e^{G(t)}\big(f(t,u_n(t) - f(t,u(t)))\big)\big(u_n(t) - u(t)\big)\mathrm{d}t.
\end{aligned} \tag{4.3}$$

From (A), (4.1), (4.2) and (4.3), it follows that $u_n \to u$ in $H_0^1(0,T)$. Thus, φ satisfies the (PS) condition. The proof is complete. ∎

Lemma 4.2. *Assume that* $(\mathrm{A}), (f_1)$ *and* (f_3) *hold, then there exists* $\epsilon_0 > 0$ *such that*

(i) $J'(\partial D_0(\epsilon_0)) \subset D_0(\epsilon_0)$*, and if* $u \in D_0(\epsilon_0)$ *is the solution of problem (1.1), then* $u \in P$*;*

(ii) $J'(\partial(-D_0(\epsilon_0))) \subset -D_0(\epsilon_0)$*, and if* $u \in -D_0(\epsilon_0)$ *is the solution of problem* (1.1)*, then* $u \in -P$*.*

Proof. Let $u^{\pm} = \max\{\pm u, 0\}$. for all $w \in -P$, we have $w(t) \leq 0$, so $-w(t) \geq 0$. Hence, $\forall w \in -P, r \in [2, +\infty)$,

$$\|u^+\|_r^r \leq \int_{u(t)\geq 0} |u^+ - w|^r \mathrm{d}t + \int_{u(t)<0} |-u^- - w|^r \mathrm{d}t = \int_0^T |u-w|^r \mathrm{d}t. \tag{4.4}$$

Therefore, (4.4) implies that $\|u^+\|_r \leq \inf\limits_{w\in(-P)} \|u-w\|_r$. Moreover, by Sobolev embedding Theorem, when $r \in [2, +\infty)$, the embedding $H_0^1(0,T) \hookrightarrow L^r(\Omega)$ is continuous. So there exists $C_r > 0$ such that for all $u \in H_0^1(0,T)$, if $r \in [2, +\infty)$,

$$\|u^+\|_r \leq \inf_{w\in(-P)} \|u-w\|_r \leq C_r \inf_{w\in(-P)} \|u-w\| = C_r \mathrm{dist}(u, -P). \tag{4.5}$$

By (A) and (f_1): $\forall \epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$f(t,u)u \leq \epsilon u^2 + C_\epsilon |u|^s, \quad \forall t \in [0,T], \quad \forall u \in \mathbb{R}. \tag{4.6}$$

Assume $v = J'(u)$. Then by (4.5), (4.6) and (f_3), for ϵ small enough,

$$\begin{aligned}
\mathrm{dist}(v,-P)\|v^+\| &\leq \|v^+\|^2 \\
&= \langle v, v^+\rangle \\
&\leq -\sum_{j=1}^p e^{G(t_j)} I_j(u(t_j))v^+(t_j) + \int_0^T e^{G(t)} f(x,u^+)v^+ \mathrm{d}t \\
&\leq \int_0^T e^{G(t)} f(x,u^+)v^+ \mathrm{d}t \\
&\leq \int_0^T \beta(\epsilon|u^+| + C_\epsilon |u^+|^{s-1})||v^+| \mathrm{d}t \\
&\leq \Big(\frac{1}{2}\mathrm{dist}(u,-P) + C_3 \mathrm{dist}(u,-P)^{s-1}\Big)\|v^+\|,
\end{aligned}$$

where

$$\beta = \max_{t\in[0,T]} e^{G(t)} > 0.$$

That is,

$$\mathrm{dist}(J'(u), -P) \leq \frac{1}{2}\mathrm{dist}(u,-P) + C_3(\mathrm{dist}(u,-P)^{s-1}). \tag{4.7}$$

So there exists $\epsilon_0 > 0$ such that $\mathrm{dist}(J'(u), -P) \leq \frac{3}{4}\epsilon_0$ for every $u \in \partial(-D_0(\epsilon_0))$. Thus $J'(\partial(-D_0(\epsilon_0))) \subset -D_0(\epsilon_0)$. If $u \in D_0(\epsilon_0)$ is the solution of problem (1.1), then $\varphi'(u) = u - J'(u) = 0$, that is, $J'(u) = u$. By (4.7), $u \in -P$, (ii) holds. (i) can be proved analogously. The proof is complete. ■

Lemma 4.3. *Assume that* (f_1) *and* (f_4) *hold, then*

$$\inf_{\overline{D_0(\epsilon_0)} \bigcap \overline{-D_0(\epsilon_0)}} \varphi(u) = d_0 > -\infty.$$

Proof. By $(f_1), (f_4)$, (4.6) and Holder inequality, we have that there exists $C_4 > 0$ such that

$$\begin{aligned}
\varphi(u) &\geq \sum_{j=1}^{p} e^{G(t_j)} \int_0^{u(t_j)} I_j(t)\,\mathrm{d}t - \int_0^T e^{G(t)} F(t, u(t))\,\mathrm{d}t \\
&\geq -\beta pa - \frac{\beta}{\eta} \int_0^T f(t, u(t))u(t)\,\mathrm{d}t \\
&\geq -\beta pa - \frac{\beta C_4}{\eta}(\|u\|_2^2 + \|u\|_s^s),
\end{aligned} \tag{4.8}$$

where $a = \max\{a_1, a_2, \ldots, a_p\}$. According to (4.5), for any $u \in \overline{D_0(\epsilon_0)} \bigcap \overline{-D_0(\epsilon_0)}$, one has

$$\|u^+\|_s \leq C_s \mathrm{dist}(u, -P) \leq C_s\epsilon_0, \|u^-\|_s \leq C_s \mathrm{dist}(u, P) \leq C_s\epsilon_0. \tag{4.9}$$

Thus, by (4.8) and (4.9), we get

$$\inf_{\overline{D_0(\epsilon_0)} \bigcap \overline{-D_0(\epsilon_0)}} \varphi(u) = d_0 > -\infty.$$ ■

Now, we prove Theorem 3.1. *Proof.* By (f_1), there are two positive constants C_5 and C_6 such that $\forall u \in R, \forall t \in [0, T]$ (see [13, p. 2863]),

$$F(t, u) \geq C_5|u|^\eta - C_6.$$

For any finitely dimensional subspace $\widetilde{V}$ of $H_0^1(0, T)$, by (f_4), there exists a constant $C_7 > 0$ such that for all u in $\widetilde{V}$ it holds

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p} e^{G(t_j)} \int_0^{u(t_j)} I_j(t)\mathrm{d}t - \int_0^T e^{G(t)} F(t, u)\mathrm{d}t \\
&\leq \frac{1}{2}\|u\|^2 + \beta pa - \alpha C_5\|u\|_\eta^\eta + \alpha C_6 T \\
&\leq \frac{1}{2}\|u\|^2 + \beta pa - \alpha C_7\|u\|^\eta + \alpha C_6 T.
\end{aligned} \tag{4.10}$$

Since $\eta > 2$, by (4.10), there are two positive numbers C_8 and C_9 such that

$$\varphi(u) \leq -C_8\|u\|^2 + C_9, \quad \forall u \in V_0. \tag{4.11}$$

$\varphi_k(k = 1, 2, \ldots)$ is the eigenfunctions of the problem

$$\begin{cases} -u''(t) = \lambda u(t), & t \in [0, T]; \\ u(0) = u(T) = 0, \end{cases}$$

corresponding the eigenvalues $\lambda_k(k = 1, 2, \ldots)$ and X_k is the eigenspace associated to λ_k, $H_0^1(0, T) = \overline{\bigoplus_{i\in\mathbb{N}} X_i}$, thus φ_1 is positive (or negative) and the eigenfunction associated to $\lambda_i(i \geq 2)$ is sign-changing. Since $\varphi_2 \in H_0^1(0, T)$ is sign-changing, that is $\varphi_2^+ \neq 0, \varphi_2^- \neq 0$. Let $V_0 = \operatorname{span}\{\varphi_2^+, \varphi_2^-\}$, then V_0 is the finitely dimensional subspace of $H_0^1(0, T)$. Define a path $h : [0, 1] \mapsto H_0^1(0, T)$,

$$h(t) = t\frac{\sqrt{R_0}}{\|\varphi_2^+\|}\varphi_2^+ + (1 - t)\frac{\sqrt{R_0}}{\|\varphi_2^-\|}(-\varphi_2^-),$$

where $R_0 = \max\{\frac{d_0 - C_9 - 1}{-C_8}, 1\}$, then by (4.11)

$$\begin{aligned} \varphi(h(t)) &= \varphi\left(t\frac{\sqrt{R_0}}{\|\varphi_2^+\|}\varphi_2^+ + (1 - t)\frac{\sqrt{R_0}}{\|\varphi_2^-\|}(-\varphi_2^-)\right) \\ &\leq -C_8R_0 + C_9 \\ &\leq d_0 - 1. \end{aligned} \tag{4.12}$$

Thus, by Lemma 4.3 and (4.12),

$$\inf_{u\in\overline{D_0(\epsilon_0)}\bigcap\overline{-D_0(\epsilon_0)}} \varphi(u) > \sup_{t\in[0,1]} \varphi(h(t)).$$

Obviously, $h(0) \in -D_0(\epsilon_0), h(1) \in D_0(\epsilon_0)$, thus $h(0) \in -D_0(\epsilon_0)\backslash D_0(\epsilon_0)$. If not, $h(0) \in -D_0(\epsilon_0) \cap D_0(\epsilon_0)$, by Lemma 4.3, $G(h(0)) \geq d_0$. This is a contradiction. Analogously, $h(1) \in D_0(\epsilon_0)\backslash - D_0(\epsilon_0)$. Moreover, $0 \in -D_0(\epsilon_0) \cap D_0(\epsilon_0)$, by Lemma 4.1, Lemma 4.2 and Theorem 2.1, problem (1.1) has four solutions: $u_1 \in \overline{D_0(\epsilon_0)\bigcap(-D_0(\epsilon_0))}, u_2 \in D_0(\epsilon_0)\backslash\overline{-D_0(\epsilon_0)}, u_3 \in (-D_0(\epsilon_0))\backslash\overline{D_0(\epsilon_0)}, u_4 \in H_0^1(0, T)\backslash(\overline{D_0(\epsilon_0)\bigcup -D_0(\epsilon_0)})$. That is, u_1 is a zero solution, u_2 is a positive solution, u_3 is a negative solution and u_4 is a sign-changing solution. The proof is complete. ∎

To prove Theorem 3.2, we first let $N = X_1 \bigoplus X_2 \bigoplus \cdots \bigoplus X_l$ $(l \geq 2)$, $M = \overline{\bigoplus_{i=l+1}^{\infty} X_i}$, then $H_0^1(0, T) = N \bigoplus M$. We take $z_0 \in X_l$, $\|z_0\| = 1$ and define

$$B = \{u \in M : \|u\| \geq \delta\} \bigcup\{u = kz_0 + v : v \in M, k \geq 0, \|u\| = \delta\}.$$

Then each element of B is sign-changing.

Lemma 4.4. $\mathrm{dist}(B, -P \bigcup P) = d_1 > 0.$

Proof. B and $-P \bigcup P$ are two closed subsets of $H_0^1(0,T)$. Note that $B \bigcap (-P \bigcup P) = \emptyset$ and $H_0^1(0,T)$ is a normal space, the conclusion is readily to be shown. The proof is complete. ■

Lemma 4.5. *Assume that* $(\mathrm{A}), (f_1)$ *and* (f_3) *hold, then there exists* $\mu_0 \in (0, d_1)$ *such that* $J'(\pm D_0(\mu_0)) \subset \pm D_0(\mu_0)$.

Proof. Assume $v = J'(u)$, by (4.7), there is $\mu_0 < d_1$ (cf. Lemma 4.4) such that $\mathrm{dist}(J'(u), -P) \le \frac{3}{4}\mu_0$ for every $u \in -D_0(\mu_0)$. In a similar way, $\mathrm{dist}(J'(u), P) \le \frac{3}{4}\mu_0$ for every $u \in D_0(\mu_0)$. The conclusion follows. The proof is complete. ■

Now, we prove Theorem 3.2.

Proof. Assume

$$D_0^{(1)} = D_0(\mu_0), D_0^{(2)} = -D_0(\mu_0),$$
$$W = D_0^{(1)} \bigcup D_0^{(2)}, S = V \backslash W.$$

By Lemma 4.4, $B \subset S$, that is, the condition (H_3) of Theorem 2.2 holds. Lemma 4.5 says that condition (H_1) of Theorem 2.2 is also satisfied. Since $0 \in D_0^{(1)} \bigcap D_0^{(2)}$, then (H_2) holds automatically. By Lemma 4.1 and Remark 2.1, φ satisfies w-PS condition. Moreover, note that $\|v\|^2 \le \mu_l \|v\|_2^2$ for all $v \in N$ and $\mu_{l+1}\|w\|_2^2 \le \|w\|^2$ for all $w \in M$. Combine (f_4) and (f_5), we have that for any $v \in N, w \in M$,

$$\begin{aligned}
\varphi(v) &= \frac{1}{2}\|v\|^2 + \sum_{j=1}^{p} e^{G(t_j)} \int_0^{v(t_j)} I_j(t)\,\mathrm{d}t - \int_0^T e^{G(t)} F(t,v)\,\mathrm{d}t \\
&\le \frac{1}{2}\|v\|^2 + \beta p a - \frac{\mu_l}{2}\|v\|_2^2 + \frac{\int_0^T W_1(t)\,\mathrm{d}t}{2} \\
&\le \beta p a + \frac{\int_0^T W_1(t)\,\mathrm{d}t}{2}
\end{aligned}$$

and

$$\begin{aligned}
\varphi(w) &= \frac{1}{2}\|w\|^2 + \sum_{j=1}^{p} e^{G(t_j)} \int_0^{w(t_j)} I_j(t)\,\mathrm{d}t - \int_0^T e^{G(t)} F(t,w)\,\mathrm{d}t \\
&\ge \frac{1}{2}\|w\|^2 - \beta p a - \frac{\mu_{l+1}}{2}\|w\|_2^2 - \frac{\int_0^T W_2(t)\,\mathrm{d}t}{2} \\
&\ge -\beta p a - \frac{\int_0^T W_2(t)\,\mathrm{d}t}{2}.
\end{aligned}$$

Therefore, we have
$$\sup_N G = a_0 < \infty, \quad \inf_M = b_0 > -\infty.$$
Since (A) holds, φ maps bounded sets to bounded sets. By Theorem 2.2, φ has a critical point in S. Therefore, problem (1.1) has a sign-changing solution. The proof is complete. ■

5. Examples

Example 5.1. Let $T = \frac{\pi}{2}, t_1 = \frac{1}{3}, t_2 = \frac{1}{2}$. Consider the damped vibration problem with impulse

$$(5.1) \qquad \begin{cases} -u''(t) + g(t)u'(t) = f(t, u(t)), & \text{a.e. } t \in [0, \frac{\pi}{2}]; \\ u(0) = u(\frac{\pi}{2}) = 0, \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I_j(u(t_j)), & j = 1, 2, \end{cases}$$

where

$$I_1(u) = I_2(u) = \begin{cases} -(u-5)^2 + 25, & u \in [0, 10]; \\ 0, & u \in (-\infty, 0) \bigcup (10, +\infty) \end{cases}$$

and $f(t, u) = \frac{1}{t+1}u^2, g(t) = -1 - t$. All conditions of Theorem 3.1 hold because of $s = 3, \eta = 3, a_1 = a_2 = 250$. According to Theorem 3.1, problem (5.1) has four solutions: one zero solution, one positive solution, one negative solution and one sign-changing solution.

Example 5.2. Let $T = 1, t_1 = \frac{1}{3}$. Consider the damped vibration problem with impulse

$$\begin{cases} -u''(t) + g(t)u'(t) = f(t, u(t)), & \text{a.e. } t \in [0, 1]; \\ u(0) = u(1) = 0, \\ \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = I_1(u(t_1)), \end{cases}$$

where

$$(5.2) \qquad I_1(u) = \begin{cases} 0, & u \in (-\infty, 0) \bigcup (100, +\infty), \\ -u + 100, & u \in [50, 100], \\ u, & u \in [0, 50), \end{cases}$$

$\lambda_2 = 4\pi^2$ and $\lambda_3 = 9\pi^2$ are the eigenvalue the problem

$$\begin{cases} -u''(t) = \lambda u(t), & t \in [0, 1]; \\ u(0) = u(1) = 0. \end{cases}$$

and $f(t,u) = \frac{\lambda_2+\lambda_3}{2}u = \frac{13\pi^2}{2}u, g(t) = \sin t + \cos t$. All conditions of Theorem 3.2 hold because of $s = 2, \eta = 2, a_1 = 2500, W_1(t) = W_2(t) = 0$. According to Theorem 3.2, problem (5.2) has at least one sign-changing solution.

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REFERENCES

1. Z. Luo and J. J. Nieto, New results for the periodic boundary value problem for impulsive integro-differential equations, *Nonlinear Analysis*, **70** (2009), 2248-2260.
2. Y. K. Chang, J. J. Nieto and W. S. Li, On impulsive hyperbolic differential inclusions with nonlocal initial conditions, *J. Optim. Theory Appl.*, **140** (2009), 431-442.
3. J. J. Nieto, Variational formulation of a damped Dirichlet impulsive problem, *Appl. Math. Letters*, **23** (2010), 940-942.
4. J. Xiao and J. J. Nieto, Variational approach to some damped Dirichlet nonlinear impulsive differential equations, *Journal of the Franklin Institute*, **348** (2011), 369-377.
5. B. Dai and D. Zhang, The existence and multiplicity of solutions for second-order impulsive differential equations on the half-line, *Results in Mathematics*, **63** (2013), 135-149.
6. H. R. Sun, Y. N. Li, J. J. Nieto and Q. Tang, *Existence of Solutions for Sturm-liouville Boundary Value Problem of Impulsive Differential Equations*, Abstract and Applied Analysis 2012, art. no. 707163.
7. J. Xiao, J. J. Nieto and Z. Luo, Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods, *Communications in Nonlinear Science and Numerical Simulation*, **17** (2012), 426-432.
8. T. E. Carter, Necessary and sufficient conditions for optimal impulsive rendezvous with linear equations of motion, *Dynam. Control*, **10** (2000), 219-227.
9. J. J. Nieto and D. O'Regan, Variational approach to impulsive differential equations, *Nonlinear Anal. Real World Appl.*, **10** (2009), 680-690.
10. S. Pasquero, On the simultaneous presence of unilateral and kinetic constraints in time-dependent impulsive mechanics, *J. Math. Phys.*, **47** (2006), 082903.19 pages.
11. E. K. Lee and Y. H. Lee, Multiple positive solutions of singular two point boundary value problems for second order impulsive differential equation, *Appl. Math. Comput.*, **158** (2004), 745-759.
12. X. N. Lin and D. Q. Jiang, Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, *J. Math. Anal. Appl.*, **321** (2006), 501-514.

13. J. W. Zhou and Y. K. Li, Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects, *Nonlinear Anal.*, **71** (2009), 2856-2865.

14. H. Zhang and Z. X. Li, Variational approach to impulsive differential equations with periodic boundary conditions, *Nonlinear Anal. Real World Appl.*, **11** (2010), 67-78.

15. M. Galewski, On variational impulsive boundary value problems, *Central European Journal of Mathematics*, **10** (2012), 1969-1980.

16. X. Li, X. Wu and K. Wu, On a class of damped vibration problems with super-quadratic potentials, *Nonlinear Anal.*, **72** (2010), 135-142.

17. J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, Berlin, 1989.

18. W. Zou, Sign-changing saddle point, *J. Funct. Anal.*, **219** (2005), 433-468.

19. Z. Liu and J. Sun, Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations, *J. Differential Equations*, **172** (2001), 257-299.

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