TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 6, pp. 1981-1990, December 2014
DOI: 10.11650/tjm.18.2014.4030
This paper is available online at http://journal.taiwanmathsoc.org.tw

DISTANCE SETS WITH DIAMETER GRAPH BEING CYCLE

Xianglin Wei*, Guogang Li, Yue Cong and Feixing Gao

Abstract. A point set X in the plane is called a k-distance set if there are exactly k different distances between two distinct points in X. Let $D = D(X)$ be the diameter of a finite set X, and let $X_D = \{x \in X : d(x,y) = D$ for some $y \in X\}$, the diameter graph $DG(X_D)$ of X_D is the graph with X_D as its vertices and where two vertices $x, y \in X_D$ are adjacent if $d(x,y) = D$. We prove the set X having at most five distances with $DG(X_D) = C_7$ has the unique $X_D = R_7$, and the set X having at most six distances with $DG(X_D) = C_9$ has the unique $X_D = R_9$, and give a conjecture for k-distance set with $DG(X_D) = C_{2k-3}$.

1. INTRODUCTION

A point set X in the Euclidean plane is called a k-distance set if it determines exactly k different distances. For two planar point sets, we say that they are isomorphic if there exists a similar transformation from one to the other. Let $d(x,y)$ denote the distance between two planar points x and y. Let R_n denote the vertex set of a regular convex n-gon, $R_n - i$ denote a set of $n-i$ vertices of R_n. Let $g(k)$ be the largest possible cardinality of k-distance set. A k-distance set X is said to be maximum if X has $g(k)$ points. Erdős-Fishburn [1] determined $g(k)$ for $k \leq 5$ and classified maximum k-distance sets for $k \leq 4$, and conjectured $g(6) = 13$. Shinohara [4] classified 3-distance sets with at least five points. Shinohara [5] proved the uniqueness of the 12-point 5-distance set and classified 8-point 4-distance sets.

Let $D = D(X)$ be the diameter of a finite set X, and let $X_D = \{x \in X : d(x,y) = D$ for some $y \in X\}$ and $m = m(X) = |X_D|$. The diameter graph $DG(X_D)$ of X_D is the graph with X_D as its vertices and where two vertices $x, y \in X_D$ are adjacent if $d(x,y) = D$. Clearly $DG(X_D)$ has no isolated vertex. We denote a cycle with n

Received November 13, 2013, accepted April 14, 2014.
Communicated by Gerard Jennhwa Chang.
2010 *Mathematics Subject Classification*: 52C20, 52A10, 52C15.
Key words and phrases: k-Distance set, Diameter graph, Different distance, Cycle.
This research was supported by National Natural Science Foundation of China and Natural Science Foundation of Hebei Province (No. A2014208095).
*Corresponding author.

vertices by C_n. When indexing a set of t points, we identify indices modulo t. Let $X_D = \{1, 2, 3, \ldots, m\}$, here the points $1, 2, 3, \ldots, m$ are consecutive and always in counter-clockwise order, we say segment $[i, i+1]$ be an edge of X_D for every $i \in X_D$.

2. Related Lemmas

Lemma 1. [2, 3]. *Suppose S is the vertex set of a convex n-gon, $n \geq 3$, that determines exactly t different distances. Then $t \geq \lfloor n/2 \rfloor$. Moreover:*

(i) *if n is odd and $t = (n-1)/2$, S is R_n;*

(ii) *if n is even, $t = n/2$, and $n \geq 8$, S is R_n or $R_{n+1} - 1$;*

(iii) *if $(n, t) = (7, 4)$, S is $R_8 - 1$ or $R_9 - 2$;*

(iv) *if $(n, t) = (9, 5)$, S is $R_{10} - 1$ or $R_{11} - 2$.*

Lemma 2. [1]. *Let D be the diameter of an n-point planar set X with $n \geq 3$ and $m = |X_D|$. Then*

(a) *if $m \geq 3$, the points in X_D are the vertices of a convex m-gon;*

(b) *D can be eliminated as an interpoint distance by removing at most $\lceil \frac{m}{2} \rceil$ points from X, where $\lceil \frac{m}{2} \rceil$ is the smallest integer at least $m/2$.*

Lemma 3. [6]. *For a planar point set X with $m = |X_D|$, let $X_D = \{1, 2, \ldots, m\}$, m points are consecutive with counter-clockwise order. If for a subset $S \subset X_D$, $S = \{k, k+1, k+2, \ldots, k+l-1\}$, the segment $[k, k+l-1]$ is the max-length segment of S and $d(k, k+i) < d(k, k+l-1)$ for any $i = 1, 2, 3, \ldots, l-2$, then $d(k, k+1) < d(k, k+2) < d(k, k+3) < \ldots < d(k, k+l-1) \leq D$.*

In the following some proofs are omitted because of the restriction of the length of the paper.

3. The Unique Set R_7 with $DG(X_D) = C_7$

In the following proof we try to conclude a contradiction if $X_D \neq R_7$. For brevity, we do not always say "a contradiction".

Theorem 4. *Let X be a 5-distance set. If $DG(X_D) = C_7$, then $X_D = R_7$.*

Proof. Let X be a 5-distance set, and 5 distances are $D = d_1 > d_2 > d_3 > d_4 > d_5$. By lemma 2, we know X_D is a convex set. Let $X_D = \{1, 2, 3, \ldots, 7\}$, points $1, 2, 3, 4, 5, 6, 7$ are consecutive and always in counter-clockwise order. Since $g(3) = 7$ [1], X_D has at least 3 distinct distances. If X_D is a 3-distance set, then by lemma 1 (i), $X_D = R_7$. If X_D is a 4-distance set, then by lemma 1 (iii), $X_D = R_8 - 1$ or $X_D = R_9 - 2$, but $DG(R_8 - 1) \neq C_7$, $DG(R_9 - 2) \neq C_7$. So in the following we need to consider the case that X_D is 5-distance. By Lemma 3, we can see that

$d(x, x+1) \leq d_3$ for any $x \in X_D$. If all the seven edges of X_D have the same length, then clearly all points of X_D lie on a circle, and hence $X_D = R_7$, which is not a 5-distance. So we can conclude that not all edges of X_D have the same length. Now we depart three Parts to prove.

Part I. Every edge of X_D has d_4-length or d_5-length. If there are six edges of X_D having the same length, then clearly all points of X_D lie on a circle, which leads to a contradiction. So in the following we may assume that at most five edges of X_D have the same length.

Case 1. There are two edges of X_D which have d_5-length (If there are two edges of X_D which have d_4-length, the proof is similar). Without loss of generality, we may assume $d(1,7) = d_5$, and consider three types by symmetry. At first assume $d(6,7) = d_5$. Then $d(2,4) > d(1,6)$ since $\angle 234 = \angle 176 > \frac{\pi}{2}$, which contradicts the fact $d(2,4) = d(1,6)$ since $\triangle 214 \cong \triangle 126$. Secondly assume $d(5,6) = d_5$. Then $d(1,3) > d(5,7)$ since $\angle 123 > \angle 567 > \frac{\pi}{2}$, which contradicts the fact $d(1,3) = d(5,7)$ since $\triangle 173 \cong \triangle 715$. Thirdly assume $d(4,5) = d_5$. Then $d_3 = d(4,6) < d(2,7) = d_2$ since $\angle 217 > \angle 456 > \frac{\pi}{2}$. But $\angle 237 < \angle 341 < \frac{\pi}{2}$, that is to say $d_2 = d(2,7) < d(1,3) \leq d_2$.

Case 2. There are three edges of X_D which have d_5-length (If there are three edges of X_D which have d_4-length, the proof is similar). Without loss of generality, we may assume $d(1,7) = d_5$, and consider four types by symmetry. At first assume $d(6,7) = d(5,6) = d_5$. Then $d(1,3) > d(5,7)$ since $\angle 123 = \angle 567 > \frac{\pi}{2}$, which contradicts the fact $d(1,3) = d(5,7)$ since $\triangle 143 \cong \triangle 745$. Secondly assume $d(6,7) = d(4,5) = d_5$. Then $d_3 = d(4,6) < d(2,7) = d_2$ since $\angle 217 > \angle 456 > \frac{\pi}{2}$. But $\angle 267 < \angle 715 < \frac{\pi}{2}$, that is to say, $d_2 = d(2,7) < d(5,7) \leq d_2$. Thirdly assume $d(6,7) = d(3,4) = d_5$. Then $d_3 = d(2,7) < d(3,5) = d_2$ since $\angle 345 > \angle 217 > \frac{\pi}{2}$. But $\angle 325 < \angle 547 < \frac{\pi}{2}$, that is to say, $d_2 = d(3,5) < d(5,7) \leq d_2$. At last assume $d(5,6) = d(3,4) = d_5$. Then $d_3 = d(4,6) < d(3,5) = d_2$ since $\angle 345 > \angle 456 > \frac{\pi}{2}$. But $\angle 365 < \angle 341 < \frac{\pi}{2}$, that is to say, $d_2 = d(3,5) < d(1,3) \leq d_2$.

Part II. There is only one edge of X_D which has d_3-length.

Without loss of generality, we may assume $d(1,2) = d_3$. By Lemma 3, $d(1,3) = d(2,7) = d_2$.

Case 1. $d(2,3) = d(1,7) = x$. Then $d(3,4) = d(6,7)$ since $\angle 314 = \angle 317 - \angle 417 = \angle 723 - \angle 623 = \angle 726$, $d(4,5) = d(5,6)$ since $\angle 526 = \angle 521 - \angle 621 = \angle 512 - \angle 412 = \angle 514$. In fact $d(2,4) = d(1,6)$, $d(3,5) = d(5,7)$, $12||37||46$, $5 \in\perp 12 =\perp 37 =\perp 46$, all points of X_D is symmetry about $\perp 12$.

(1) $d(2,3) = d(4,5) = d_4$. If $d(3,4) = d_4$, clearly no segment of X_D has d_5-length. If $d(3,4) = d_5$, then $\angle 341 < \angle 436 < \frac{\pi}{2}$, which leads to $d_2 = d(1,3) < d(4,6) \leq d_2$.

(2) $d(2,3) = d_4$, $d(4,5) = d_5$. If $d(3,4) = d_4$, then $\angle 237 < \angle 325 < \frac{\pi}{2}$, which leads to $d_2 = d(2,7) < d(3,5) \leq d_2$. If $d(3,4) = d_5$, then $\angle 341 < \angle 436 < \frac{\pi}{2}$, which

leads to $d_2 = d(1,3) < d(4,6) \le d_2$.

(3) $d(2,3) = d_5$, $d(4,5) = d_4$. If $d(3,4) = d_4$, then $d_2 \ge d(3,5) > d(4,6) \ge d_3$ since $\angle 345 > \angle 456 > \frac{\pi}{2}$, and $d_2 \ge d(3,5) > d(2,4) = d(1,6) \ge d_3$ since $\angle 345 > \angle 234$. Now $\angle 465 = \angle 165 - \angle 164 = \angle 467 - \angle 164 = \angle 167$, which implies $d_5 = d(1,7) = d(4,5) = d_4$. If $d(3,4) = d_5$, then points 1, 2, 3, 4, 6, 7 lie on a circle, $d(3,5) = d(5,7) = d_3$ since $\angle 325 = \angle 517 < \angle 237 < \frac{\pi}{2}$, and $d(2,7) = d_2$, $d(2,4) = d(1,6) = d_4$ by the same reason. Now $\angle 125 = \angle 124 - \angle 524 = \angle 754 - \angle 452 = \angle 752$, which implies $\triangle 125 \cong \triangle 752$ and $d_1 = d(1,5) = d(2,7) = d_2$.

(4) $d(2,3) = d(4,5) = d_5$. If $d(3,4) = d_5$, clearly all points of X_D lie on the circle, which leads to $d_3 = d(1,2) = d(4,5) = d_5$. If $d(3,4) = d_4$, $\angle 237 < \angle 325 < \frac{\pi}{2}$, which leads to $d_2 = d(2,7) < d(3,5) \le d_2$.

Case 2. $d(2,3) \ne d(1,7)$. Without loss of generality, we may suppose $d(2,3) = d_4$ and $d(1,7) = d_5$. At first assume $d(6,7) = d_4$. If $d(3,4) = d_4$, then $\angle 267 < \angle 143 < \frac{\pi}{2}$, which leads to $d_2 = d(2,7) < d(1,3) = d_2$; if $d(3,4) = d_5$, then $\angle 267 < \angle 476 < \frac{\pi}{2}$, which leads to $d_2 = d(2,7) < d(4,6) \le d_2$. Secondly assume $d(6,7) = d_5$. Now $d(1,3) = d(4,6) = d_2$ since $\triangle 173 \cong \triangle 674$. Clearly $d(3,4) = d_4$, since otherwise $\angle 267 < \angle 476 < \frac{\pi}{2}$, which leads to $d_2 = d(2,7) < d(4,6) = d_2$. Clearly $d(4,5) = d_4$, since otherwise $\angle 173 < \angle 517 < \frac{\pi}{2}$, which leads to $d_2 = d(1,3) < d(5,7) \le d_2$. If $d(5,6) = d_5$, then $\angle 321 > \angle 217$, $d_2 = d(1,3) > d(2,7) = d_2$; if $d(5,6) = d_4$, then $\frac{\pi}{2} < \angle 456 < \angle 345$, which leads to $d_2 = d(4,6) < d(3,5) \le d_2$.

Part III. At least two edges of X_D have d_3-length.

Without loss of generality, we may assume $d(1,2) = d_3$. By Lemma 3, $d(2,7) = d(1,3) = d_2$.

Case 1. $d(1,7) = d_3$ (If $d(2,3) = d_3$, the proof is similar). By Lemma 3, $d(2,7) = d(1,3) = d(1,6) = d_2$. Clearly it is easy to see that $d(2,3) = d(6,7)$ and $d(3,4) = d(5,6)$. In fact $d(2,4) = d(5,7)$, $d(3,5) = d(4,6)$, all points of X_D is symmetry about $\perp_{45}$. Clearly $d(2,3) \ne d_3$, $d(3,4) \ne d_3$, $d(4,5) \ne d_3$, since otherwise all edges of X_D must be d_3-length, which contradicts 5-distance. Since $d(6,7) \le d_4$, $\angle 341 = \angle 347 - \angle 147 < \angle 437 - \angle 637 = \angle 436 < \frac{\pi}{2}$, which implies $d_2 = d(1,3) < d(4,6) \le d_2$.

Case 2. $d(4,5) = d_3$ (If $d(5,6) = d_3$, the proof is similar). By the former case we can see that $d(3,4) \ne d_3$ and $d(5,6) \ne d_3$. If $d(6,7) = d_3$, then clearly all edges of X_D must be d_3-length, which contradicts 5-distance. If $d(2,3) = d(1,7)$, then $\angle 321 \ne \angle 217$ and $d_2 = d(2,7) \ne d(1,3) = d_2$. So $d(2,3) \ne d(1,7)$. We may assume $d(2,3) = d_4$ and $d(1,7) = d_5$ (If $d(2,3) = d_5$ and $d(1,7) = d_4$, the proof is similar). If $d(3,4) = d(5,6)$, clearly $d_5 = d(1,7) = d(1,2) = d_3$; if $d(3,4) = d_5$ and $d(5,6) = d_4$, then $\angle 345 \ne \angle 217$, and hence $d_2 = d(3,5) \ne d(2,7) = d_2$; if $d(3,4) = d_4$ and $d(5,6) = d_5$, then $\angle 123 > \angle 217 > \frac{\pi}{2}$, and hence $d_2 = d(1,3) > d(2,7) = d_2$.

Case 3. $d(3,4) = d_3$ (If $d(6,7) = d_3$, the proof is similar). Then $\angle 173 = \angle 174 -$

$\angle 374 < \angle 714 - \angle 514 = \angle 715 < \frac{\pi}{2}$, which implies $d_2 = d(1,3) < d(5,7) \leq d_2$.

Until now we have proved $d(i, i+1) \leq d_4$ for $i = 2, 3, 4, 5, 6, 7 \in X_D$, that is to say, there is only one edge $[1, 2]$ of X_D whose length is d_3, which has been proved in Part II. ∎

4. The Unique Set with $DG(X_D) = C_9$

In the following proof we try to conclude a contradiction if $X_D \neq R_9$. For brevity, we do not always say "a contradiction".

Theorem 5. *Let X be a 6-distance set. If $DG(X_D) = C_9$, then $X_D = R_9$.*

Proof. Let X be a 6-distance set, and 6 distances are $D = d_1 > d_2 > d_3 > d_4 > d_5 > d_6$. By lemma 2, we know X_D is a convex set. Let $X_D = \{1, 2, 3, \ldots, 9\}$, points $1, 2, 3, 4, 5, 6, 7, 8, 9$ are consecutive and always in counter-clockwise order. Since $g(4) = 9$ [1], X_D has at least 4 distinct distances. If X_D is a 4-distance set, then by lemma 1 (i), $X_D = R_9$. If X_D is a 5-distance set, then by lemma 1 (iv), $X_D = R_{10} - 1$ or $X_D = R_{11} - 2$, but $DG(R_{10} - 1) \neq C_9$, $DG(R_{11} - 2) \neq C_9$. So in the following we need to consider the case that X_D is a 6-distance set. By Lemma 3, we can see that $d(x, x+1) \leq d_4$ for any $x \in X_D$. If all the nine edges of X_D have the same length, then clearly all points of X_D lie on a circle, and hence $X_D = R_9$, which is not 6-distance. So not all edges of X_D have the same length. Now we depart three Parts to prove.

Part I. Every edge of X_D has d_5-length or d_6-length. If there exist eight edges of X_D which have the same length, then clearly all points of X_D lie on the circle, which implies a contradiction. So in the following we may assume at most seven edges of X_D have the same length.

Case 1. There are two edges of X_D which have d_5-length (If there are two edges of X_D which have d_6-length, the proof is similar). Without loss of generality, we may assume $d(1, 2) = d_5$, and consider four types by symmetry. At first assume $d(1, 9) = d_5$. Then points 2, 3, 4, 5, 6, 7, 8, 9 lie on a circle, points 1, 2, 5, 7 lie on a circle, and so deduce points 1, 2, 5, 6 lie on a circle, which implies $d_6 = d(5, 6) = d(1, 2) = d_5$. Secondly assume $d(8, 9) = d_5$. Then points 2, 3, 4, 6, 7, 8 lie on a circle, points 1, 2, 3, 9 lie on a circle, points 1, 2, 8, 9 lie on a circle, and so conclude points 1, 2, 6, 7 lie on the circle, which implies $d_6 = d(6, 7) = d(1, 2) = d_5$. Thirdly assume $d(7, 8) = d_5$. Then points 1, 3, 4, 5, 6, 8, 9 lie on a circle, points 1, 2, 3, 9 lie on a circle, points 2, 3, 6, 7 lie on a circle, and at last we conclude all points of X_D lie on the circle, which implies $d_6 = d(2, 3) = d(7, 8) = d_5$. At last assume $d(6, 7) = d_5$. Then clearly all points of X_D lie on the circle, which implies $d_6 = d(2, 3) = d(6, 7) = d_5$.

Case 2. There are three edges of X_D which have d_5-length (If there are three edges of X_D which have d_6-length, the proof is similar). Without loss of generality, we may assume $d(1,9) = d_5$.

(1) There are at least two d_5-length edges which are consecutive. We should consider four types by symmetry. At first assume $d(1,2) = d(8,9) = d_5$. Then points 2, 3, 6, 7 lie on a circle, points 3, 4, 5, 6 lie on a circle, points 4, 5, 6, 7 lie on a circle, and so conclude points 2, 3, 4, 5 lie on the circle, which implies $d(2,4) = d(3,5)$, but in fact $\angle 234 \neq \angle 345$, that is to say, $d(2,4) \neq d(3,5)$. Secondly assume $d(1,2) = d(7,8) = d_5$. Then points 2, 4, 5, 7 lie on a circle, points 4, 5, 6, 7 lie on a circle, points 2, 3, 6, 7 lie on a circle, and so conclude points 3, 4, 5, 6 lie on the circle, which implies $d(3,5) = d(4,6)$, but in fact $\angle 345 \neq \angle 456$, that is to say, $d(3,5) \neq d(4,6)$. Thirdly assume $d(1,2) = d(6,7) = d_5$, or at last assume $d(1,2) = d(5,6) = d_5$. Then $d_4 = d(5,7) < d(1,8) = d_3$ since $\angle 198 > \angle 567 > \frac{\pi}{2}$, and so $d(1,7) = d_2$ by lemma 3. But $\angle 167 < \angle 376 < \frac{\pi}{2}$, which leads to $d_2 = d(1,7) < d(3,6) \leq d_2$.

(2) There are not two d_5-length edges which are consecutive. We should consider three types by symmetry. At first assume $d(7,8) = d(5,6) = d_5$. Then $d_4 = d(2,9) < d(5,7) = d_3$ since $\angle 567 > \angle 219 > \frac{\pi}{2}$, and so $d(5,8) = d(4,7) = d_2$ by lemma 3. But $\angle 548 \neq \angle 437$, which leads to $d_2 = d(5,8) \neq d(4,7) = d_2$. Secondly assume $d(7,8) = d(4,5) = d_5$. Then $d_4 = d(4,6) < d(7,9) = d_3$ since $\angle 789 > \angle 456 > \frac{\pi}{2}$, and so $d(6,9) = d(1,7) = d_2$ by lemma 3. But $\angle 127 \neq \angle 659$, which leads to $d_2 = d(6,9) \neq d(1,7) = d_2$. At last assume $d(6,7) = d(3,4) = d_5$. Clearly $d_3 \leq d(1,7) < d(2,8) \leq d_2$ since $\frac{\pi}{2} > \angle 278 > \angle 127$, that is to say, $d(2,8) = d_2$ and $d(1,7) = d_3$. Similarly $d(1,8) = d_4$ and $d(7,9) = d_5$ since $\angle 198 > \angle 789 > \frac{\pi}{2}$. Now we conclude 134679 is a regular hexagon, and $X_D = R_{12} - 3$, but $DG(R_{12} - 3) \neq C_9$.

Case 3. There are four edges of X_D which have d_5-length (If there are four edges of X_D which have d_6-length, the proof is similar). Without loss of generality, we may assume $d(1,2) = d_5$.

(1) There are at least three d_5-length edges which are consecutive. We should consider three types by symmetry. At first assume $d(1,9) = d(8,9) = d(7,8) = d_5$. Then points 2, 3, 4, 5, 6, 7 lie on a circle, points 1, 2, 5, 7 lie on a circle, and so conclude points 1, 2, 5, 6 lie on the circle, which implies $d_6 = d(5,6) = d(1,2) = d_5$. Secondly assume $d(1,9) = d(8,9) = d(6,7) = d_5$. Then $d_4 = d(5,7) < d(7,9) = d_3$ since $\angle 789 > \angle 567 > \frac{\pi}{2}$, and so $d(6,9) = d(1,7) = d_2$ by lemma 3. But $\angle 916 \neq \angle 127$, which leads to $d_2 = d(6,9) \neq d(1,7) = d_2$. Thirdly assume $d(1,9) = d(8,9) = d(5,6) = d_5$. Then $d_4 = d(5,7) < d(7,9) = d(1,3) = d_3$ since $\angle 123 = \angle 789 > \angle 567 > \frac{\pi}{2}$, and so $d(6,9) = d(1,4) = d_2$ by lemma 3. But $\angle 194 \neq \angle 619$, which leads to $d_2 = d(6,9) \neq d(1,4) = d_2$.

(2) There are just two d_5-length edges which are consecutive. We should consider six types by symmetry. At first assume $d(1,9) = d(6,7) = d(7,8) = d_5$. Then $d_4 = d(5,7) < d(1,8) = d_3$ since $\angle 198 > \angle 567 > \frac{\pi}{2}$, and so $d(2,8) = d(1,7) = d_2$

by lemma 3. But $\angle 167 \neq \angle 278$, which leads to $d_2 = d(1,7) \neq d(2,8) = d_2$. Secondly assume $d(1,9) = d(5,6) = d(6,7) = d_5$. Then $d_4 = d(4,6) < d(1,8) = d_3$ since $\angle 198 > \angle 456 > \frac{\pi}{2}$, and so $d(1,7) = d_2$ by lemma 3. In this way $\angle 127 < \angle 376 < \frac{\pi}{2}$, which leads to $d_2 = d(1,7) < d(3,6) \leq d_2$. Thirdly assume $d(1,9) = d(5,6) = d(7,8) = d_5$. Then $d_4 = d(4,6) < d(7,9) = d_3$ since $\angle 789 > \angle 456 > \frac{\pi}{2}$, and so $d(6,9) = d(1,7) = d_2$ by lemma 3. But $\angle 127 \neq \angle 619$, which leads to $d_2 = d(1,7) \neq d(6,9) = d_2$. Fourth assume $d(1,9) = d(4,5) = d(7,8) = d_5$. Then $d_4 = d(1,8) < d(7,9) = d(1,3) = d_3$ since $\angle 123 = \angle 789 > \angle 198 > \frac{\pi}{2}$, and so $d(6,9) = d(1,4) = d_2$ by lemma 3. But $\angle 194 \neq \angle 619$, which leads to $d_2 = d(1,4) \neq d(6,9) = d_2$. Fifth assume $d(1,9) = d(3,4) = d(7,8) = d_5$. Then $d_4 = d(7,9) < d(3,5) = d_3$ since $\angle 345 > \angle 789 > \frac{\pi}{2}$, and so $d(2,5) = d(3,6) = d_2$ by lemma 3. But $\angle 265 \neq \angle 326$, which leads to $d_2 = d(2,5) \neq d(3,6) = d_2$. At last assume $d(1,9) = d(4,5) = d(6,7) = d_5$. Then $d_4 = d(5,7) < d(6,8) = d_3$ since $\angle 678 > \angle 567 > \frac{\pi}{2}$, and so $d(5,8) = d_2$ by lemma 3. In this way $\angle 598 < \angle 389 < \frac{\pi}{2}$, which leads to $d_2 = d(5,8) < d(3,9) \leq d_2$.

(3) Any two d_5-length edges are not consecutive. We may assume $d(8,9) = d(6,7) = d(4,5) = d_5$. Then $d_4 = d(1,8) < d(7,9) = d_3$ since $\frac{\pi}{2} < \angle 891 < \angle 789$, and so $d(1,7) = d_2$ by lemma 3. In this way $\angle 167 < \angle 376 < \frac{\pi}{2}$, which leads to $d_2 = d(1,7) < d(3,6) \leq d_2$.

Part II. There exists only one edge of X_D which has d_4-length. Without loss of generality, we may assume $d(1,2) = d_4$. By Lemma 3, $d(1,3) = d(2,9) = d_3$, $d(1,4) = d(2,8) = d(3,9) = d_2$.

Case 1. $d(1,9) = d(2,3) = x$. Then $d(5,6) = d(6,7)$ since $d(2,9) = d(1,3) = d_3$, $d(3,4) = d(8,9)$ since $\angle 829 = \angle 329 - \angle 328 = \angle 913 - \angle 914 = \angle 413$, and $d(4,5) = d(7,8)$ since $\angle 827 = \angle 328 - \angle 327 = \angle 914 - \angle 915 = \angle 415$. Until now we can see $d(3,6) = d(6,9)$, $d(4,6) = d(6,8)$, $d(3,5) = d(7,9)$, $d(2,4) = d(1,8)$, that is to say, all points of X_D is symmetry about $\perp_{12}$.

(1) $d(1,9) = d_5$ and $d(5,6) = d_5$. If $d(3,4) = d_5$ and $d(4,5) = d_6$, then $\angle 349 < \angle 238 < \frac{\pi}{2}$, which leads to $d_2 = d(3,9) < d(2,8) = d_2$. If $d(3,4) = d_6$ and $d(4,5) = d_5$, then $\angle 238 < \angle 487 < \frac{\pi}{2}$, which leads to $d_2 = d(2,8) < d(4,7) \leq d_2$. If $d(3,4) = d(4,5) = d_6$, then $\angle 265 < \angle 326 < \angle 238 < \frac{\pi}{2}$, which leads to $d_3 \leq d(2,5) < d(3,6) < d(2,8) = d_2$. If $d(3,4) = d(4,5) = d_5$, then clearly no segment of X_D has d_6-length, a contradiction.

(2) $d(1,9) = d_5$ and $d(5,6) = d_6$. If $d(3,4) = d(4,5) = d_5$, then $\angle 278 < \angle 619 < \frac{\pi}{2}$, which leads to $d_2 = d(2,8) < d(6,9) \leq d_2$. If $d(3,4) = d(4,5) = d_6$, then $\angle 451 < \angle 349 < \frac{\pi}{2}$, which leads to $d_2 = d(1,4) < d(3,9) = d_2$. If $d(3,4) = d_5$ and $d(4,5) = d_6$, then $\angle 349 < \angle 194 < \frac{\pi}{2}$, which leads to $d_2 = d(3,9) < d(1,4) = d_2$. If $d(3,4) = d_6$ and $d(4,5) = d_5$, then $\angle 278 < \angle 487 < \frac{\pi}{2}$, which leads to $d_2 = d(2,8) < d(4,7) \leq d_2$.

(3) $d(1,9) = d_6$ and $d(5,6) = d_6$. If $d(3,4) = d_5$ and $d(4,5) = d_6$, then $\angle 349 < \angle 437 < \frac{\pi}{2}$, which leads to $d_2 = d(3,9) < d(4,7) \leq d_2$. If $d(3,4) = d_6$ and $d(4,5) = d_5$, then $\angle 238 < \angle 349 < \frac{\pi}{2}$, which leads to $d_2 = d(2,8) < d(3,9) = d_2$. If $d(3,4) = d(4,5) = d_5$, then $\angle 349 < \angle 278 < \frac{\pi}{2}$, which leads to $d_2 = d(3,9) < d(2,8) = d_2$. If $d(3,4) = d(4,5) = d_6$, then all points of X_D lie on the circle, a contradiction.

(4) $d(1,9) = d_6$ and $d(5,6) = d_5$. If $d(3,4) = d_5$ and $d(4,5) = d_6$, then $\angle 349 < \angle 437 < \frac{\pi}{2}$, which leads to $d_2 = d(3,9) < d(4,7) \leq d_2$. If $d(3,4) = d_6$ and $d(4,5) = d_5$, then $\angle 238 < \angle 349 < \frac{\pi}{2}$, which leads to $d_2 = d(2,8) < d(3,9) = d_2$. If $d(3,4) = d(4,5) = d_5$, then $\angle 389 < \angle 278 < \frac{\pi}{2}$, which leads to $d_2 = d(3,9) < d(2,8) = d_2$. If $d(3,4) = d(4,5) = d_6$, then points 1, 2, 3, 4, 5, 7, 8, 9 lie on a circle, from this we can see $d(5,7) = d_3$. If $d(4,6) = d(6,8) = d_3$, then points 2, 4, 6, 9 lie on a circle, and so conclude all points of X_D lie on the circle, which implies $d_5 = d(6,7) = d(2,3) = d_6$. So $d(4,6) = d(6,8) = d_4$. Since $d_5 \leq d(3,5) \leq d_4$, we know points 2, 4, 6, 8 lie on a circle when $d(3,5) = d_4$, points 3, 5, 6, 7 lie on a circle when $d(3,5) = d_5$. Hence we conclude all points of X_D lie on the circle, which implies $d_5 = d(6,7) = d(2,3) = d_6$.

Case 2. $d(2,3) \neq d(1,9)$. Without loss of generality, we may assume $d(2,3) = d_6$ and $d(1,9) = d_5$. When $d(5,6) = d(6,7) = d_5$, $\frac{\pi}{2} > \angle 376 > \angle 619 > \angle 167$, which implies $d_2 \geq d(3,6) > d(6,9) > d(1,7) \geq d_3$. When $d(5,6) = d(6,7) = d_6$, $\frac{\pi}{2} > \angle 326 > \angle 659 > \angle 265$, which implies $d_2 \geq d(3,6) > d(6,9) > d(2,5) = d(1,7) \geq d_3$. When $d(5,6) = d_6$ and $d(6,7) = d_5$, $\angle 219 > \angle 123 > \frac{\pi}{2}$, which implies $d_3 \leq d(1,3) < d(2,9) \leq d_3$. Now we only need to consider $d(5,6) = d_5$ and $d(6,7) = d_6$.

Clearly $d(7,8) = d_6$, since otherwise $\angle 238 < \angle 376 < \frac{\pi}{2}$, which leads to $d_2 = d(2,8) < d(3,6) \leq d_2$. And $d(8,9) = d_5$, since otherwise $\angle 154 < \angle 548 < \frac{\pi}{2}$, which leads to $d_2 = d(1,4) < d(5,8) \leq d_2$. In this way $\angle 349 < \angle 437 < \frac{\pi}{2}$, which leads to $d_2 = d(3,9) < d(4,7) \leq d_2$.

Part III. At least two edges of X_D have d_4-length. Without loss of generality, we may assume $d(1,2) = d_4$. By Lemma 3, $d(2,9) = d(1,3) = d_3$, $d(2,8) = d(3,9) = d(1,4) = d_2$.

Case 1. $d(5,6) = d_4$ (If $d(6,7) = d_4$, the proof is similar).

By Lemma 3, $d(4,6) = d(5,7) = d_3$, $d(3,6) = d(4,7) = d(5,8) = d_2$. Clearly $d(2,3) = d(4,5)$ since $\angle 564 = \angle 561 - \angle 461 = \angle 216 - \angle 316 = \angle 213$, and $d(6,7) = d(1,9)$ since $\angle 657 = \angle 652 - \angle 752 = \angle 125 - \angle 925 = \angle 129$, $d(7,8) = d(8,9)$ since $\angle 748 = \angle 743 - \angle 843 = \angle 934 - \angle 834 = \angle 938$, $d(3,4) = d(4,5)$ since $\angle 394 = \angle 398 - \angle 498 = \angle 589 - \angle 489 = \angle 584$, $d(6,7) = d(7,8)$ since $\angle 637 = \angle 632 - \angle 732 = \angle 823 - \angle 723 = \angle 827$. Until now we conclude $d(2,3) = d(3,4) = d(4,5)$ and $d(6,7) = d(7,8) = d(8,9) = d(1,9)$. If $d(2,3) = d_4$, then all points of

X_D lie on a circle, that is to say, any edge of X_D must be d_4-length. So $d(2,3) \leq d_5$. By the same reason, $d(6,7) \leq d_5$. If $d(2,3) = d(6,7)$, then clearly all points of X_D lie on a circle, which implies $d_4 = d(1,2) = d(6,7) \leq d_5$. So $d(2,3) \neq d(6,7)$. When $d(2,3) = d_5$ and $d(6,7) = d_6$, $\angle 321 > \angle 567 > \frac{\pi}{2}$, which implies $d_3 = d(1,3) > d(5,7) = d_3$. When $d(2,3) = d_6$ and $d(6,7) = d_5$, points 1, 6, 7, 8, 9 lie on a circle, points 2, 4, 7, 8 lie on a circle,points 3, 4, 7, 9 lie on a circle, points 1, 4, 5, 8 lie on a circle. If $d(6,9) = d_2$, then points 1, 4, 6, 9 lie on a circle, combining this with the former results we conclude all points of X_D lie on the circle, which implies $d_5 = d(1,9) = d(5,6) = d_4$; if $d(6,9) = d_3$, then $d(7,9) = d_4$ by Lemma 3, points 5, 6, 7, 9 lie on a circle, combining this with the former results we conclude all points of X_D lie on the circle, which implies $d_5 = d(1,9) = d(5,6) = d_4$.

Case 2. $d(4,5) = d_4$ (If $d(7,8) = d_4$, the proof is similar). Now $\angle 194 = \angle 195 - \angle 495 < \angle 915 - \angle 615 = \angle 916 < \frac{\pi}{2}$, which implies $d_2 = d(1,4) < d(6,9) \leq d_2$.

Case 3. $d(3,4) = d_4$ (If $d(8,9) = d_4$, the proof is similar). Now $\angle 389 = \angle 489 - \angle 483 < \angle 498 - \angle 495 = \angle 598 < \frac{\pi}{2}$, which implies $d_2 = d(3,9) < d(5,8) \leq d_2$.

Case 4. $d(2,3) = d_4$ (If $d(1,9) = d_4$, the proof is similar). By the former case 3 we can conclude $d(1,9) \leq d_5$. By Lemma 3, $d(2,4) = d_3$, $d(2,5) = d_2$. Now $\angle 265 = \angle 165 - \angle 162 < \angle 156 - \angle 159 = \angle 659 < \frac{\pi}{2}$, which implies $d_2 = d(2,5) < d(6,9) \leq d_2$.

That is to say, there is only one edge $[1,2]$ of X_D whose length is d_4, which has been proved in Part II. ■

When X is a 3-distance set with $DG(X_D) = C_3$, clearly $X_D = R_3$. When X is a 4-distance set with $DG(X_D) = C_5$, X_D can be R_5 and the other two configurations, see Lemma 6 in [7].

Conjecture 6. Let X be a k-distance set for $k \geq 7$. If $DG(X_D) = C_{2k-3}$, then $X_D = R_{2k-3}$.

References

1. P. Erdös and P. Fishburn, Maximum planar sets that determine k distances, *Discrete Mathematics*, **160** (1996), 115-125.

2. P. Erdös and P. Fishburn, Convex nonagons with five intervertex distance, *Geometria Dedicata*, **60** (1996), 317-332.

3. P. Fishburn, Convex polygons with few intervertex distance, *Computational Geometry*, **5** (1995), 65-93.

4. M. Shinohara, Classification of three-distance sets in two dimensional Euclidean space, *European Journal of Combinatorics*, **25** (2004), 1039-1058.

5. M. Shinohara, Uniqueness of maximum planar five-distance sets, *Discrete Mathematics*, **308** (2008), 3048-3055.

6. X. Wei, Classification of eleven-point five-distance sets in the plane, *Ars Combinatoria*, **102** (2011), 505-515.

7. W. Lan and X. Wei, Classification of seven-point four-distance sets in the plane, *Mathematical Notes*, **93** (2013), 510-522.

Xianglin Wei, Guogang Li, Yue Cong and Feixing Gao
College of Science
Hebei University of Science and Technology
Shijiazhuang 050018
P. R. China
E-mail: wxlhebtu@126.com