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SEMICLASSICAL SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER-MAXWELL EQUATIONS WITH CRITICAL NONLINEARITY

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Abstract. In this paper, by using variational methods and critical point theory, we study the existence of semiclassical solutions for the following nonlinear Schrödinger-Maxwell equations

$$\begin{cases} -\varepsilon^2 \triangle u + V(x)u + \phi u = K(x)|u|^4 u + f(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle \phi = 4\pi u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$, $V(x) \geq 0$ and $K(x) > 0$ for all $x \in \mathbb{R}^3$, under some more assumptions on V, K and f, we prove that the system has at least one nontrivial solution for sufficient small $\varepsilon > 0$. Our approach is much more straightforward.

1. Introduction and Statement of the Main Results

In this paper, we study the following electrostatic nonlinear Schrödinger-Maxwell equations

$$(\mathrm{SM}_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \triangle u + V(x)u + \phi u = K(x)|u|^4 u + f(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle \phi = 4\pi u^2, & \text{in } \mathbb{R}^3. \end{cases}$$

Such a system is also called Schrödinger-Poisson equations, which arise in an interesting physical context. Indeed a similar system arises in many mathematical physics contexts, such as in quantum electrodynamics, to describe the interaction between a charge particle interacting with the electromagnetic field, and also in semiconductor theory, in nonlinear optics and in plasma physics. For a more physical background of system $(\mathrm{SM}_\varepsilon)$, we refer the readers to [7, 12] and the references therein.

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Since it was first introduced by V. Benci and D. Fortunato in [12] (where $\varepsilon \equiv 1$), system $(\mathrm{SM}_\varepsilon)$ has been widely studied by many authors. The case $\varepsilon \equiv 1$ and $V \equiv 1$ (or being radially symmetric), has been studied under various conditions on f in [4, 7, 17, 26]. When $\varepsilon \equiv 1$, $V(x)$ is not a constant, the existence of infinitely many large solutions for $(\mathrm{SM}_\varepsilon)$ has been considered in [8, 16, 23, 29] via the fountain theorem (cf. [30, 35]). For more results of system $(\mathrm{SM}_\varepsilon)$ in the case $\varepsilon \equiv 1$, we refer the reader to [2, 20] and the references therein. Meanwhile, When $\varepsilon \equiv 1$ and $f(x,s) = f(s)$, based on the well known Pohozaev type identity [15, 21, 22], some results on the nonexistence of nontrivial solutions for system $(\mathrm{SM}_\varepsilon)$ were obtained in [8]. When ε is not a constant, we refer the reader to [25, 31] and the references therein. Here we strongly recommend the readers to [3] which include many aspects of $(\mathrm{SM}_\varepsilon)$. For more results of system (1.1), we refer the reader to [32, 33] and the references therein.

In the paper [31], the authors studied the existence of semiclassical solutions of $(\mathrm{SM}_\varepsilon)$ under the following assumptions on V and f:

(V0) $V \in C(\mathbb{R}^3, \mathbb{R}); V(0) = \min V = 0$; and there is $b > 0$ such that the set $\mathcal{V}^b := \{x \in \mathbb{R}^3 : V(x) < b\}$ has finite Lebesgue measure.

(K) $K \in C(\mathbb{R}^3)$ and $0 < K_1 := \inf K \leq \sup K := K_2 < \infty$;

(f1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and $f(x,u) \to o(|u|)$ uniformly for $x \in \mathbb{R}^3$ as $u \to 0$.

(f2) There are $c_0 > 0$ and $q < 6$ such that $|f(x,u)| \leq c_0(1 + |u|^{q-1})$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

(f3) There are $a_0 > 0, p > 4$ and $\mu > 4$ such that $F(x,u) \geq a_0|u|^p$ and $\mu F(x,u) \leq f(x,u)u$ for all (x,u), here and subsequently, we always denote $F(x,u) = \int_0^u f(x,s)ds$.

Then the authors established the following theorem :

Theorem A. [31]. *Let* (V0), (K) *and* (f1)-(f3) *be satisfied. Then for any* $\sigma > 0$ *there is* $\varepsilon_\sigma > 0$ such that if $\varepsilon \leq \varepsilon_\sigma$, $(\mathrm{SM}_\varepsilon)$ *has at least one least energy solution* u_ε.

We remark that (f3) implies that there exist some $\alpha > 0$ and $M > 0$ such that

$$F(x,u) \geq \alpha|u|^\mu, \quad \text{for} \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, \quad |u| \geq M. \tag{1.1}$$

And assumption (f3) is too strict for many cases, for example, let $f(x,u) = 4u^3 \ln(2 + |\cos|x|| + u^2) + \frac{2u^5}{2+|\cos|x||+u^2}$, then $F(x,u) = u^4 \ln(2 + |\cos|x|| + u^2)$, one can easily show that $F(x,u)$ dose not satisfy (1.1). Actually by L'Hospital Principle we have

$$\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^\mu} \to 0, \quad \text{for any} \quad \mu > 4. \tag{1.2}$$

That is, $f(x,u)$ does not satisfy (1.1). Hence Theorem A can not be applied in this case. However, one will see later that our results in this paper can also work in this case.

In Theorem A, the number ε_σ is rather vague, so a natural question arises: **Can we use a more accurate number to replace** ε_σ ? and we give a positive answer in the present paper.

Let $\lambda = \varepsilon^{-2}$, (SM$_\varepsilon$) becomes

$$(\mathrm{SM}_\lambda) \qquad \begin{cases} -\triangle u + \lambda V(x)u + \lambda\phi u = \lambda K(x)|u|^4 u + \lambda f(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle\phi = 4\pi u^2, & \text{in } \mathbb{R}^3. \end{cases}$$

Now, let's introduce some notations. For any $1 \le r < \infty$, $L^r(R^3)$ is the usual Lebesgue space with the norm

$$\|u\|_r = (\int_{\mathbb{R}^3} |u|^r dx)^{\frac{1}{r}}.$$

$H^1(\mathbb{R}^3)$ is the usual Sobolev space with the norm

$$\|u\|_{H^1} = (\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx)^{\frac{1}{2}}.$$

Define the space (see for instance in [30])

$$\mathcal{D}^{1,2} = \{u \in L^{2^*}(\mathbb{R}^3) \mid \quad \nabla u \in L^2(\mathbb{R}^3)\}$$

with the norm

$$\|u\|_{\mathcal{D}^{1,2}} = (\int_{\mathbb{R}^3} |\nabla u|^2)^{\frac{1}{2}},$$

where $2^* = 6$ is the critical Sobolev exponent of $\mathbb{R}^3$. Then $\mathcal{D}^{1,2} \hookrightarrow L^{2^*}$, let S be the best embedding constant of this embedding, i.e. S satisfies:

$$\|u\|_6 \le S\|u\|_{\mathcal{D}^{1,2}}. \tag{1.3}$$

Applying Lax-Milgram theorem (see [34]), for every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ (see [19]) such that

$$-\triangle\phi_u = 4\pi u^2. \tag{1.4}$$

Moreover, ϕ_u has the following integral expression

$$\phi_u = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy.$$

Thus $\phi_u \ge 0$, from (1.3) and (1.4), for any $u \in H^1(\mathbb{R}^3)$ using Hölder inequality we have

$$\|\phi_u\|^2_{\mathcal{D}^{1,2}} = 4\pi \int_{\mathbb{R}^3} \phi_u u^2 dx \le 4\pi\|\phi_u\|_6 \|u\|^2_{\frac{12}{5}} \le 4\pi S\|\phi_u\|_{\mathcal{D}^{1,2}}\|u\|^2_{\frac{12}{5}}.$$

This implies that

$$\|\phi_u\|_{\mathcal{D}^{1,2}} \leq 4\pi S\|u\|^2_{\frac{12}{5}}, \tag{1.5}$$

and

$$\int_{\mathbb{R}^3} \phi_u u^2 dx \leq 4\pi S^2\|u\|^4_{\frac{12}{5}}. \tag{1.6}$$

Let

$$E = \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^2 \mathrm{d}x < +\infty\right\}$$

and

$$\|u\|_{\lambda\dagger} = \left\{\int_{\mathbb{R}^3} \left[|\nabla u|^2 + \lambda V(x)|u|^2\right] \mathrm{d}x\right\}^{1/2}, \quad \forall\, u \in E.$$

Analogous to the proof of [27, Lemma 1], by using (V0) and the Sobolev inequality, one can demonstrate that there exists a constant $\gamma_0 > 0$ independent of λ such that

$$\|u\|_{H^1} \leq \gamma_0\|u\|_{\lambda\dagger}, \quad \forall\, u \in E, \ \ \lambda \geq 1. \tag{1.7}$$

This shows that $(E, \|\cdot\|_{\lambda\dagger})$ is a Hilbert space for $\lambda > 0$. Furthermore, by virtue of the Sobolev embedding theorem, we have

$$\|u\|_s \leq \gamma_s\|u\|_{H^1} \leq \gamma_s\gamma_0\|u\|_{\lambda\dagger}, \quad \forall\, u \in E, \ \ \lambda \geq 1, \ \ 2 \leq s \leq 2^*, \tag{1.8}$$

Now we define a functional I_ε on $E \times \mathcal{D}^{1,2}$ by

$$\begin{aligned} I_\varepsilon(u,\phi) \;=\; & \frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2|\nabla u|^2 + V(x)u^2)dx - \frac{1}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi|^2 dx \\ & + \frac{1}{2}\int_{\mathbb{R}^3}\phi u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx - \frac{1}{6}\int_{\mathbb{R}^3} K(x)u^6 dx. \end{aligned} \tag{1.9}$$

From the discussion above we know that I_ε is well defined and $I_\varepsilon \in C^1(E \times \mathcal{D}^{1,2})$, it is well known that I_ε's critical points are the solutions of system (SM_ε). Moreover, by (1.4), we have

$$\begin{aligned} I_\varepsilon(u,\phi) \;=\; & \frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2|\nabla u|^2 + V(x)u^2)dx \\ & + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx - \frac{1}{6}\int_{\mathbb{R}^3} K(x)u^6 dx. \end{aligned} \tag{1.10}$$

Let

$$\begin{aligned} \Phi_\lambda(u) \;=\; & \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + \lambda V(x)u^2)dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx \\ & - \lambda\int_{\mathbb{R}^3} F(x,u)dx - \frac{\lambda}{6}\int_{\mathbb{R}^3} K(x)u^6 dx \\ =\; & \frac{1}{2}\|u\|^2_{\lambda\dagger} + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx - \lambda\int_{\mathbb{R}^3} F(x,u)dx - \frac{\lambda}{6}\int_{\mathbb{R}^3} K(x)u^6 dx, \end{aligned} \tag{1.11}$$

And it is well known that if $u \in E$ is a critical point of Φ_λ, (i.e. $\Phi'_\lambda(u) = 0$) then (u, ϕ_u) is a solution of system (SM$_\lambda$). Moreover, for $\forall u, v \in E$, one can easily know

$$\begin{aligned}\langle \Phi'_\lambda(u), v\rangle = &\int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x)uv)dx + \lambda \int_{\mathbb{R}^3} \phi_u uvdx \\ &-\lambda \int_{\mathbb{R}^3} f(x,u)vdx - \lambda \int_{\mathbb{R}^3} K(x)|u|^4 uvdx.\end{aligned}$$

Theorem 1.1 can be restated as

Theorem B. *Let* (V0), (K) *and* (f1)-(f3) *be satisfied. Then for any* $\sigma > 0$ *there is* $\Lambda_\sigma > 0$ *such that if* $\lambda \geq \Lambda_\sigma$, (1.4) *has at least one positive solution* u_λ *of least energy.*

To state the main results of this paper, we assume that

$(f2')$ There are $c_0 > 0$ and $1 < p < 5$ such that $|f(x,u)| \leq c_0(1 + |u|^p)$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

$(f3')_1$ There exists $a_0 > 0$ such that $F(x,u) \geq a_0|u|^4$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

$(f3')_2$ There are $a_0 > 0$ and $4 < q < 6$ such that $F(x,u) \geq a_0|u|^q$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

Assumption (f1) implies that there exists some $R_0 > 0$ such that

$$tf(x,t) \leq \frac{b}{3}|t|^2, \quad \forall\ (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \ \ |t| \leq R_0. \tag{1.12}$$

We further assume that

$(f4)$ $\mathcal{F}(x,t) := \frac{1}{4}tf(x,t) - F(x,t) + \frac{K(x)}{12}u^6 \geq 0$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$, and there exist $a_1 > 0$ and $\kappa > \frac{3}{2}$ such that

$$|f(x,t)|^\kappa \leq a_1|t|^\kappa \mathcal{F}(x,t), \quad \forall\ (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \ \ |t| \geq R_0,$$

where R_0 is the same as in (1.12).

Since $\frac{q-6}{q-2} < 0$, we can let $h_0 > 1$ such that

$$\frac{128\pi}{3a_0} h_0^{-1} \leq \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3 a_1}, \tag{1.13}$$

and

$$\begin{aligned}&\frac{(q-2)\pi a_0}{3}\Big[\frac{3}{(q\pi a_0)}\Big(\frac{32\pi}{3} + \Big(\pi S^2\Big(\frac{32\pi}{3}\Big)^{5/3}\Big)^{\frac{q-2}{q-4}}\Big(\frac{3}{(q-2)\pi a_0}\Big)^{\frac{2}{q-4}}\frac{q-4}{q-2}\Big)\Big]^{\frac{q}{q-2}} h_0^{\frac{q-6}{q-2}} \\ &\leq \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3 a_1}.\end{aligned} \tag{1.14}$$

Assumption (V0) implies that there exists some $\lambda_0 > 1$ such that

$$\sup_{\lambda^{1/2}|x|\le 2h_0} |V(x)| \le h_0^{-2}, \quad \forall\ \lambda \ge \lambda_0. \tag{1.15}$$

Theorem 1.1. *Assume that* (V0), (f1), (f2'), (f3')$_1$ *and* (f4) *satisfy . Then system* $(\mathrm{SM}_\varepsilon)$ *possesses at least one nontrivial solutions* u_ε *such that* $0 < I_\varepsilon(u_\varepsilon) \le \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3 a_1}\varepsilon$ *for* $0 < \varepsilon \le \min\{\lambda_0^{-1/2}, [\frac{3S^2h_0^2}{2a_0}(\frac{32\pi}{3})^{\frac{5}{3}}]^{-1/2}\}$.

Theorem 1.2. *Assume that* (V0), (f1), (f2'), (f3')$_1$ *and* (f4) *satisfy . Then system* (SM_λ) *possesses at least one nontrivial solutions* u_λ *such that* $0 < \Phi_\lambda(u_\lambda) \le \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3 a_1}\lambda^{-1/2}$ *for* $\lambda \ge \max\{\lambda_0, \frac{3S^2h_0^2}{2a_0}(\frac{32\pi}{3})^{\frac{5}{3}}\}$.

Theorem 1.3. *Assume that* (V0), (f1), (f2'), (f3')$_2$ *and* (f4) *satisfy. Then system* $(\mathrm{SM}_\varepsilon)$ *possesses at least one nontrivial solutions* u_ε *such that* $0 < I_\varepsilon(u_\varepsilon) \le \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3 a_1}\varepsilon$ *for* $0 < \varepsilon \le \min\{\lambda_0^{-1/2}, h_0^{\frac{-2(q-3)}{q-2}}\}$.

Theorem 1.4. *Assume that* (V0), (f1), (f2'), (f3')$_2$ *and* (f4) *satisfy . Then system* (SM_λ) *possesses at least one nontrivial solutions* u_λ *such that* $0 < \Phi_\lambda(u_\lambda) \le \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3 a_1}\lambda^{-1/2}$ *for* $\lambda \ge \max\{\lambda_0, h_0^{\frac{4(q-3)}{q-2}}\}$.

Remark 1.1. (i) Let $f(x,u) = 4u^3\ln(2 + |\cos|x|| + u^2) + \frac{2u^5}{2+|\cos|x||+u^2}$, then f satisfies all the conditions in Theorem 1.1, but f does not satisfy assumption (f3).

(ii) In [31], the following fact

$$\inf\{\int_{\mathbb{R}^3} |\nabla\varphi|^2 : \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3), |\varphi|_p = 1\} = 0, \tag{1.16}$$

plays a key role in the proof of Theorem A. But in the present work, as one will find in section 2, we don't need (1.16). Meanwhile to make a complete understanding of [31], one has to get some knowledge about spectral theory. But at the present work, we also do not need spectral theory, so we believe that our approach is much more simple and straightforward.

(iii) By (f1) and (f2'), one can easily show that for any $\varepsilon > 0$, there exists some $C_\varepsilon > 0$ such that (see [24])

$$|f(x,u)| \le \varepsilon|u| + C_\varepsilon|u|^p. \tag{1.17}$$

(1.17) plays an important role in our later discussions.

2. Proof of Theorem 1.2 and Theorem 1.4

Let h_0 and λ_0 be the same as in (1.13)-(1.15). Set

$$\vartheta(x) := \begin{cases} \frac{1}{h_0}, & |x| \le h_0, \\ \frac{1}{h_0}(2 - \frac{|x|}{h_0}), & h_0 < |x| \le 2h_0, \\ 0, & |x| > 2h_0. \end{cases} \tag{2.1}$$

Then $\vartheta \in H^1(\mathbb{R}^3)$, and

$$\|\nabla\vartheta\|_2^2 = \int_{\mathbb{R}^3} |\nabla\vartheta(x)|^2 dx \le \int_{|x|\le 2h_0} \frac{1}{h_0^4} dx = \frac{32\pi}{3} h_0^{-1}. \tag{2.2}$$

$$\|\vartheta\|_2^2 = \int_{\mathbb{R}^3} |\vartheta(x)|^2 dx \le \int_{|x|\le 2h_0} \frac{1}{h_0^2} dx = \frac{32\pi}{3} h_0. \tag{2.3}$$

$$\|\vartheta\|_{\frac{12}{5}}^4 = (\|\vartheta\|_{\frac{12}{5}}^{\frac{12}{5}})^{5/3} = (\int_{\mathbb{R}^3} |\vartheta(x)|^{\frac{12}{5}} dx)^{5/3} \le (\frac{32\pi}{3})^{5/3} h_0. \tag{2.4}$$

$$\|\vartheta\|_4^4 = \int_{\mathbb{R}^3} |\vartheta(x)|^4 dx \ge \frac{4\pi}{3} h_0^{-1}. \tag{2.5}$$

$$\|\vartheta\|_q^q = \int_{\mathbb{R}^3} |\vartheta(x)|^q dx \ge \frac{4\pi}{3} h_0^{3-q}. \tag{2.6}$$

Let $e_\lambda(x) = \vartheta(\lambda^{\frac{1}{2}} x)$, then one can easily show that

$$\|\nabla e_\lambda\|_2^2 = \lambda^{-1/2}\|\nabla\vartheta\|_2^2, \quad \text{and} \quad \|e_\lambda\|_s^s = \lambda^{-3/2}\|\vartheta\|_s^s \quad \forall \ s \in [2,6],$$

and

$$\int \phi_{e_\lambda} e_\lambda^2 dx = \lambda^{-5/2} \int \phi_\vartheta \vartheta^2 dx.$$

Lemma 2.1. *Let* $a > 0$, $t \ge 0$ and $4 < q < 6$. *Then for any given* $M > 0$, *we have*

$$at^4 \le \frac{a^p M^p t^2}{p} + \frac{t^q}{p' M^{p'}}, \tag{2.7}$$

where $p = \frac{q-2}{q-4}$ and $p' = \frac{2}{q-2}$.

Proof. Let $m = \frac{2(q-4)}{q-2}$, $n = \frac{2q}{q-2}$, then $mp = 2$, $np' = q$ and $m + n = 4$. By the well known Young inequality, we obtain

$$at^4 = (aMt^m)(\frac{t^n}{M}) \le \frac{(aMt^m)^p}{p} + \frac{(t^n/M)^{p'}}{p'} = \frac{a^p M^p t^2}{p} + \frac{t^q}{p' M^{p'}}.$$

We remark that the inequality (2.7) plays a key role in the proof of Lemma 2.3 below.

Lemma 2.2. *Suppose that* (V0), (f1), (f2') *and* $(\mathrm{f3'})_1$ *are satisfied, then*

$$\sup\{\Phi_\lambda(se_\lambda) : s \geq 0\} \leq \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3 a_1}\lambda^{-1/2} \quad \forall \ \lambda \geq \max\{\lambda_0, \frac{3S^2h_0^2}{2a_0}(\frac{32\pi}{3})^{\frac{5}{3}}\}. \tag{2.8}$$

Proof. By (1.6), (1.15), (2.2)-(2.5), for $\lambda \geq \max\{\lambda_0, \frac{3S^2h_0^2}{2a_0}(\frac{32\pi}{3})^{\frac{5}{3}}\}$, we have

$$\begin{aligned}
\Phi_\lambda(se_\lambda) &= \frac{s^2}{2}\|e_\lambda\|_{\lambda\dagger}^2 + \frac{\lambda s^4}{4}\int_{\mathbb{R}^3}\phi_{e_\lambda}e_\lambda^2dx - \lambda\int_{\mathbb{R}^3}F(x,se_\lambda)dx - \frac{\lambda}{6}\int_{\mathbb{R}^3}K(x)(se_\lambda)^6dx \\
&\leq \frac{s^2}{2}\|e_\lambda\|_{\lambda\dagger}^2 + \frac{\lambda s^4}{4}\int_{\mathbb{R}^3}\phi_{e_\lambda}e_\lambda^2dx - \lambda\int_{\mathbb{R}^3}F(x,se_\lambda)dx \\
&\leq \lambda^{-\frac{1}{2}}[\frac{s^2}{2}(\|\nabla\vartheta\|_2^2 + \int V(\lambda^{-\frac{1}{2}}x)\vartheta^2(x)dx) + \frac{\lambda^{-1}s^4}{4}\int_{\mathbb{R}^3}\phi_\vartheta\vartheta^2dx - a_0\|\vartheta\|_4^4s^4] \\
&\leq \lambda^{-\frac{1}{2}}[\frac{s^2}{2}(\|\nabla\vartheta\|_2^2 + \sup_{|x|\leq 2h_0}V(\lambda^{-\frac{1}{2}}x)\|\vartheta\|_2^2) + (\lambda^{-1}\pi S^2\|\vartheta\|_{\frac{12}{5}}^4 s^4 - a_0\|\vartheta\|_q^q s^q)] \\
&\leq \lambda^{-\frac{1}{2}}[\frac{s^2}{2}(\frac{32\pi}{3}h_0^{-1} + \frac{32\pi}{3}h_0^{-1}) + (\lambda^{-1}\pi S^2(\frac{32\pi}{3})^{\frac{5}{3}}h_0 - \frac{4a_0\pi}{3}h_0^{-1})s^4] \\
&\leq \lambda^{-\frac{1}{2}}[\frac{32\pi}{3}h_0^{-1}s^2 - \frac{2a_0\pi}{3}h_0^{-1}s^4] \\
&\leq \frac{128\pi h_0^{-1}}{3a_0}\lambda^{-\frac{1}{2}} \\
&\leq \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3a_1}\lambda^{-1/2}.
\end{aligned}$$

Lemma 2.3. *Suppose that* (V0), (f1), (f2') *and* $(\mathrm{f3'})_2$ *are satisfied, then*

$$\sup\{\Phi_\lambda(se_\lambda) : s \geq 0\} \leq \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3a_1}\lambda^{-1/2} \quad \forall \ \lambda \geq \max\{\lambda_0, h_0^{\frac{4(q-3)}{q-2}}\}. \tag{2.9}$$

Proof. Let $M = [\frac{3h_0^{q-3}}{(q-2)\pi a_0}]^{\frac{2}{q-2}}$, by (1.6), (1.15), (2.2)-(2.6) and Lemma 2.1, for $\lambda \geq \max\{\lambda_0, h_0^{\frac{4(q-3)}{q-2}}\}$, we have

$$\begin{aligned}
&\Phi_\lambda(se_\lambda) \\
&= \frac{s^2}{2}\|e_\lambda\|_{\lambda\dagger}^2 + \frac{\lambda s^4}{4}\int_{\mathbb{R}^3}\phi_{e_\lambda}e_\lambda^2dx - \lambda\int_{\mathbb{R}^3}F(x,se_\lambda)dx - \frac{\lambda}{6}\int_{\mathbb{R}^3}K(x)(se_\lambda)^6dx \\
&\leq \frac{s^2}{2}\|e_\lambda\|_{\lambda\dagger}^2 + \frac{\lambda s^4}{4}\int_{\mathbb{R}^3}\phi_{e_\lambda}e_\lambda^2dx - \lambda\int_{\mathbb{R}^3}F(x,se_\lambda)dx
\end{aligned}$$

$$
\begin{aligned}
&\le \lambda^{-\frac{1}{2}}[\frac{s^2}{2}(\|\nabla\vartheta\|_2^2+\int V(\lambda^{-\frac{1}{2}}x)\vartheta^2(x)dx)+\frac{\lambda^{-1}s^4}{4}\int_{\mathbb{R}^3}\phi_\vartheta\vartheta^2dx-a_0\|\vartheta\|_q^qs^q]\\
&\le \lambda^{-\frac{1}{2}}[\frac{s^2}{2}(\|\nabla\vartheta\|_2^2+\sup_{|x|\le 2h_0}V(\lambda^{-\frac{1}{2}}x)\|\vartheta\|_2^2)+(\lambda^{-1}\pi S^2\|\vartheta\|_{\frac{12}{5}}^4s^4-a_0\|\vartheta\|_q^qs^q)]\\
&\le \lambda^{-\frac{1}{2}}[\frac{32\pi}{3}h_0^{-1}s^2+\lambda^{-1}\pi S^2(\frac{32\pi}{3})^{\frac{5}{3}}h_0s^4-\frac{4a_0\pi}{3}h_0^{3-q}s^q]\\
&\le \lambda^{-\frac{1}{2}}[\frac{32\pi}{3}h_0^{-1}s^2+(\lambda^{-1}\pi S^2(\frac{32\pi}{3})^{\frac{5}{3}}h_0)^{\frac{q-2}{q-4}}M^{\frac{q-2}{q-4}}\frac{q-4}{q-2}s^2+\frac{2}{q-2}\frac{s^q}{M^{\frac{q-2}{2}}}-\frac{4a_0\pi}{3}h_0^{3-q}s^q]\\
&\le \lambda^{-\frac{1}{2}}[(\frac{32\pi}{3}+(\pi S^2(\frac{32\pi}{3})^{\frac{5}{3}})^{\frac{q-2}{q-4}}(\frac{3}{(q-2)\pi a_0})^{\frac{2}{q-2}}\frac{q-4}{q-2}h_0^{-1}s^2-\frac{2a_0\pi}{3}h_0^{3-q}s^q]\\
&\le \frac{(q-2)\pi a_0}{3}[\frac{3}{q\pi a_0}(\frac{32\pi}{3}+(\pi S^2(\frac{32\pi}{3})^{5/3})^{\frac{q-2}{q-4}}(\frac{3}{(q-2)\pi a_0})^{\frac{2}{q-4}}\frac{q-4}{q-2})]^{\frac{q}{q-2}}h_0^{\frac{q-6}{q-2}}\lambda^{-1/2}\\
&\le \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3a_1}\lambda^{-1/2}.
\end{aligned}
$$

Remark 2.1. The discussion above implies that there exists some $s_0>0$ such that

$$\Phi_\lambda(se_\lambda)<0 \quad \text{for} \quad \forall \;\; s>s_0. \tag{2.10}$$

In order to prove the main results in the present paper, we need a geometrical result which is an expression of the Ambrosetti-Rabinowitz [5] mountain pass theorem without the (PS)condition:

Lemma 2.4. [15]. *Let Φ be a C^1 function on a Banach space E. Suppose there exists a neighborhood U of 0 in E and a constant ρ such that*

$$\Phi(u)\ge\rho, \qquad \forall \;\; u\in\partial U, \tag{2.11}$$

and

$$\Phi(0)<\rho \quad \text{and} \quad \Phi(v)<\rho \quad \text{for some } v\bar{\in}U. \tag{2.12}$$

Set

$$c=\inf_{\varphi\in\Gamma}\max_{s\in[0,1]}\Phi(\varphi(s))\ge\rho,$$

where $\Gamma=\{\varphi\in C([0,1],E):\varphi(0)=0 \;\text{ and }\; \varphi(1)=v\}$. *Then there is a sequence* (u_i) *such that*

$$\Phi(u_i)\to c \quad \text{and} \quad \Phi'(u_i)\to 0 \;\text{ in } E^*.$$

Lemma 2.5 *Suppose that* (V0),(f1), (f2') *and* (f3')$_1$ (*or* (f3')$_2$) *be satisfied. Then* Φ_λ *satisfies all the assumptions in Lemma 2.4.*

Proof. It is obviously that $\Phi_\lambda(0) = 0$ and (2.10) implies that $\Phi_\lambda(v) \le 0$, where $v = se_\lambda$ for some $s > s_0$. Given $\lambda \ge \lambda_0$, let $0 < \varepsilon < \frac{1}{2\lambda(\gamma_2\gamma_0)^2}$ then by (1.8), (1.11) and (1.17) we obtain

$$\begin{aligned}
\Phi_\lambda(u) &\ge \frac{1}{2}\|u\|^2_{\lambda\dagger} - \lambda\int_{\mathbb{R}^3} F(x,u)dx - \frac{\lambda}{6}\int_{\mathbb{R}^3} K(x)u^6dx \\
&\ge \frac{1}{2}\|u\|^2_{\lambda\dagger} - \frac{\lambda\varepsilon}{2}\int_{\mathbb{R}^3} u^2(x)dx - \frac{\lambda C_\varepsilon}{p+1}\int_{\mathbb{R}^3} u^{p+1}(x)dx - \frac{\lambda}{6}\int_{\mathbb{R}^3} K(x)u^6dx \\
&\ge \frac{1}{2}\|u\|^2_{\lambda\dagger} - \frac{\lambda\varepsilon}{2}\|u\|^2_2 - \frac{\lambda C_\varepsilon}{p+1}\|u\|^{p+1}_{p+1} - \frac{\lambda K_2}{6}\|u\|^6_6 \\
&\ge \frac{1}{2}\|u\|^2_{\lambda\dagger} - \frac{\lambda\varepsilon}{2}(\gamma_2\gamma_0)^2\|u\|^2_{\lambda\dagger} - \frac{\lambda C_\varepsilon}{p+1}(\gamma_{p+1}\gamma_0)^{p+1}\|u\|^{p+1}_{\lambda\dagger} - \frac{\lambda K_2}{6}(\gamma_6\gamma_0)^6\|u\|^6_{\lambda\dagger} \\
&\ge \frac{1}{4}\|u\|^2_{\lambda\dagger} - \frac{\lambda C_\varepsilon}{p+1}(\gamma_{p+1}\gamma_0)^{p+1}\|u\|^{p+1}_{\lambda\dagger} - \frac{\lambda K_2}{6}(\gamma_6\gamma_0)^6\|u\|^6_{\lambda\dagger} \\
&= (\frac{1}{8}\|u\|^2_{\lambda\dagger} - \frac{\lambda C_\varepsilon}{p+1}(\gamma_{p+1}\gamma_0)^{p+1}\|u\|^{p+1}_{\lambda\dagger}) + (\frac{1}{8}\|u\|^2_{\lambda\dagger} - \frac{\lambda K_2}{6}(\gamma_6\gamma_0)^6\|u\|^6_{\lambda\dagger})
\end{aligned}$$

Let $\rho_1 = [\frac{1}{4\lambda C_\varepsilon(\gamma_{p+1}\gamma_0)^{p+1}}]^{\frac{1}{p-1}}$ and $\rho_2 = [\frac{1}{4\lambda K_2(\gamma_6\gamma_0)^6}]^{\frac{1}{4}}$, pick $\rho = \min\{\rho_1, \rho_2\}$, then

$$\Phi_\lambda(u) \ge \min\{\frac{p}{4(p+1)}\rho_1^2, \frac{5}{24}\rho_2^2\} > 0 \quad \text{for all} \quad u \in E \quad \text{with} \quad \|u\|_{\lambda\dagger} = \rho. \tag{2.13}$$

Hence by Lemma 2.5 we obtain

Lemma 2.6. *Suppose that* (V0), (f1), (f2'), *and* (f3'$_1$) (*or* (f3')$_2$) *are satisfied. Then there exist a constant* $c_\lambda \in (0, \sup_{s\ge 0}\Phi_\lambda(se_\lambda)]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi_\lambda(u_n) \to c_\lambda, \qquad \|\Phi'_\lambda(u_n)\| \to 0. \tag{2.14}$$

Lemma 2.7. *Suppose that* (V0), (f1), (f2'), (f3'$_1$) (*or* (f3')$_2$) *and* (f4) *are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying* (2.14) *is bounded in* E.

Proof. By (f4) one has

$$\begin{aligned}
\Phi_\lambda(u_n) - \frac{1}{4}\langle\Phi'_\lambda(u_n), u_n\rangle &= \frac{1}{4}\|u_n\|^2_{\lambda\dagger} + \lambda\int_{\mathbb{R}^3}[\frac{1}{4}f(x,u_n)u_n - F(x,u_n) + \frac{K(x)}{12}u^6]dx \\
&\ge \frac{1}{4}\|u_n\|^2_{\lambda\dagger},
\end{aligned}$$

hence for sufficiently large n,

$$4c_\lambda + \|u_n\|_{\lambda\dagger} + o(1) \ge \|u_n\|^2_{\lambda\dagger}.$$

This implies that $\{u_n\}$ is bounded in E.

Proof of Theorem 1.2 Applying Lemmas 2.2 and 2.7, we deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (2.14) with

$$c_\lambda \leq \frac{b^{(2\kappa-3)/2}}{3^\kappa(\gamma_6\gamma_0)^3 a_1}\lambda^{-1/2} \quad \text{for} \quad \lambda \geq \max\{\lambda_0, \frac{3S^2h_0^2}{2a_0}(\frac{32\pi}{3})^{\frac{5}{3}}\} \tag{2.15}$$

Without loss of generality, by Eberlein-Shmulyan theorem (see for instance in [34]), passing to a subsequence if necessary, there exists a $u_\lambda \in E$ such that

$$u_n \rightharpoonup u_\lambda \quad \text{in} \quad E \tag{2.16}$$

$u_n \rightharpoonup u_\lambda$ in $(E, \|\cdot\|_{\lambda\dagger})$ and and u_λ is a critical point of Φ_λ. Hence $(u_\lambda, \phi_{u_\lambda})$ is a solution of (SM_λ). Now we shall prove $u_\lambda \neq 0$.

Arguing by contradiction, suppose that $u_\lambda = 0$, i.e. $u_n \rightharpoonup 0$ in E, and so $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$, $2 \leq s < 6$ and $u_n \to 0$ a.e. on $\mathbb{R}^3$. Since $\mathcal{V}_b$ is a set of finite measure and $u_n \rightharpoonup 0$ in E, there holds

$$\|u_n\|_2^2 = \int_{\mathbb{R}^3\setminus\mathcal{V}_b} u_n^2 \mathrm{d}x + \int_{\mathcal{V}_b} u_n^2 \mathrm{d}x \leq \frac{1}{\lambda b}\|u_n\|_{\lambda\dagger}^2 + o(1). \tag{2.17}$$

For $s \in (2,6)$, it follows from (1.8), (2.17) and the Hölder inequality that

$$\begin{aligned} \|u_n\|_s^s &\leq \|u_n\|_2^{2(6-s)/(6-2)}\|u_n\|_6^{6(s-2)/(6-2)} \\ &\leq (\gamma_6\gamma_0)^{\frac{3(s-2)}{2}}(\lambda b)^{-\frac{6-s}{4}}\|u_n\|_{\lambda\dagger}^s + o(1). \end{aligned} \tag{2.18}$$

According to (f4) and (2.17), one can get that

$$\lambda \int_{|u_n|\leq R_0} f(x,u_n)u_n \mathrm{d}x \leq \frac{\lambda b}{3}\int_{|u_n|\leq R_0} |u_n|^2 \mathrm{d}x \leq \frac{1}{3}\|u_n\|_{\lambda\dagger}^2 + o(1). \tag{2.19}$$

By virtue of (1.11) and (2.14) , for sufficiently large n, we have

$$\Phi_\lambda(u_n) - \frac{1}{4}\langle \Phi'_\lambda(u_n), u_n\rangle = \frac{1}{4}\|u_n\|_{\lambda\dagger}^2 + \lambda\int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\mathrm{d}x = c_\lambda + o(1).$$

Thus

$$\lambda \int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\mathrm{d}x \leq c_\lambda + o(1). \tag{2.20}$$

Using (f4), (2.15), (2.18) with $s = 2\kappa/(\kappa-1)$ and (2.20) , we obtain

$$
\begin{aligned}
&\lambda \int_{|u_n|>R_0} f(x,u_n)u_n \mathrm{d}x \\
&\le \lambda \left(\int_{|u_n|>R_0} \left(\frac{|f(x,u_n)|}{|u_n|} \right)^{\kappa} \mathrm{d}x \right)^{1/\kappa} \|u_n\|_s^2 \\
&\le \lambda (\gamma_6\gamma_0)^{\frac{3(s-2)}{s}} (\lambda b)^{-\frac{6-s}{2s}} \|u_n\|_{\lambda\dagger}^2 a_1^{1/\kappa} c_\lambda^{1/\kappa} \lambda^{-1/\kappa} \\
&= \frac{a_1^{1/\kappa}(\gamma_6\gamma_0)^{3/\kappa}}{b^{(2\kappa-3)/2\kappa}} \left[\lambda^{1/2} c_\lambda \right]^{1/\kappa} \|u_n\|_{\lambda\dagger}^2 + o(1) \\
&\le \frac{1}{3} \|u_n\|_{\lambda\dagger}^2 + o(1), \qquad (2.21)
\end{aligned}
$$

this, together with (2.14) and (2.19) , implies that

$$
(2.22) \quad o(1) = \langle \Phi_\lambda'(u_n), u_n \rangle = \|u_n\|_{\lambda\dagger}^2 - \lambda \int_{\mathbb{R}^3} f(x,u_n)u_n \mathrm{d}x \ge \frac{1}{3}\|u_n\|_{\lambda\dagger}^2 + o(1).
$$

This results in the fact that $\|u_n\|_{\lambda\dagger} \to 0$. Consequently, it follows from (f1), (1.11) and (2.14) that

$$
0 < c_\lambda = \lim_{n\to\infty} \Phi_\lambda(u_n) = \Phi_\lambda(0) = 0.
$$

This contradiction shows $u_\lambda \neq 0$. By a standard argument, we easily certify that $\Phi_\lambda'(u_\lambda) = 0$ and $\Phi_\lambda(u_\lambda) \le c_\lambda$. Then u_λ is a positive solution of (SM_λ) ,

Remark 2.2. Theorem 1.1 is a direct corollary of Theorem 1.2. By the same method, we can prove Theorem 1.4 and Theorem 1.3, hence we omit the detail of the proof.

Remark 2.3. Let $f(x,u) = 4u^3 \ln(2+|\cos|x||+u^2) + \frac{2u^5}{2+|\cos|x||+u^2}$, then $F(x,u) = u^4 \ln(2+|\cos|x||+u^2)$ and $\mathcal{F}(x,u) := \frac{1}{4}uf(x,u) - F(x,u) + \frac{K(x)}{12}u^6 = \frac{u^6}{2(2+|\cos|x||+u^2)} + \frac{K(x)}{12}u^6$, then it is obvious that f satisfies (f1),(f2'), and (f3')$_1$. Now we shall show that f also satisfies assumption (f4), hence Theorem 1.1 can be applied . Let $G(x,u) = \frac{|f(x,t)|^\kappa}{|t|^\kappa \mathcal{F}(x,t)}$ for $|u| \ge R_0$, choose some $\kappa \in (\frac{3}{2}, 2)$ then by L'Hospital Principle, we have

$$
\begin{aligned}
0 \le \lim_{|u|\to\infty} G(x,u) &= \lim_{|u|\to\infty} \frac{|4u^3 \ln(2+|\cos|x||+u^2) + \frac{2u^5}{2+|\cos|x||+u^2}|^\kappa}{|u|^\kappa [\frac{u^6}{2(2+|\cos|x||+u^2)} + \frac{K(x)u^6}{12}]} \\
&\le \lim_{|u|\to\infty} \frac{|4u^3 \ln(3+u^2) + \frac{2u^5}{2+u^2}|^\kappa}{|u|^\kappa \cdot \frac{u^6}{2(3+u^2)}} \le \lim_{u\to+\infty} \frac{|8u^3 \ln(3+u^2)|^\kappa}{|u|^\kappa \cdot \frac{u^6}{2(3+u^2)}} \\
&= \lim_{u\to+\infty} \frac{|8u^3 \ln(2+u^2)|^\kappa}{|u|^\kappa \cdot \frac{u^4}{2}} = 2 \cdot 8^\kappa \cdot \lim_{u\to+\infty} \frac{\ln^\kappa(3+u^2)}{u^{4-2\kappa}}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\cdot 8^{\kappa}\cdot \lim_{u\to+\infty}[\frac{\ln(3+u^2)}{u^{\frac{4-2\kappa}{\kappa}}}]^{\kappa} = 2\cdot 8^{\kappa}\cdot [\lim_{u\to+\infty}\frac{\ln(3+u^2)}{u^{\frac{4-2\kappa}{\kappa}}}]^{\kappa}\\
&= 2\cdot 8^{\kappa}\cdot [\lim_{u\to+\infty}\frac{2u}{(3+u^2)u^{\frac{4-3\kappa}{\kappa}}\cdot\frac{4-2\kappa}{\kappa}}]^{\kappa}\\
&= 2\cdot 8^{\kappa}\cdot (\frac{\kappa}{4-2\kappa})^{\kappa}\cdot 2^{\kappa}[\lim_{u\to+\infty}\frac{1}{(3+u^2)u^{\frac{4-4\kappa}{\kappa}}}]^{\kappa}\\
&\le 2\cdot 8^{\kappa}\cdot (\frac{\kappa}{4-2\kappa})^{\kappa}\cdot 2^{\kappa}[\lim_{u\to+\infty}\frac{1}{u^{\frac{4-2\kappa}{\kappa}}}]^{\kappa}\\
&= 2\cdot 8^{\kappa}\cdot (\frac{\kappa}{4-2\kappa})^{\kappa}\cdot 2^{\kappa}\lim_{u\to+\infty}\frac{1}{u^{4-2\kappa}}\\
&= 0.
\end{aligned}
$$

This implies that there exists some sufficiently large $R > 0$ such that $0 < G(x,u) \le 1$ for all $|u| \ge R$ and $x \in \mathbb{R}^3$, let $M = \sup_{R_0\le|u|\le R,\ x\in\ R^3} G(x,u)$ and $a_1 = \max\{1, M\}$, then we know that f satisfies the assumption (f4).

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