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TWO COMPLEX COMBINATIONS AND COMPLEX INTERSECTION BODIES

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Abstract. This paper devotes to establish complex dual Brunn-Minkowski theory. At first, we introduce the concepts of complex radial combination and complex radial-Blaschke combination, and obtain the relations between those two combinations and dual mixed volumes. Then, we extend the properties of real intersection body to the complex case. Finally, we prove some complex geometric inequalities about complex intersection bodies and complex mixed intersection bodies, such as dual Brunn-Minkowski type, dual Aleksandrov-Fenchel type and dual Minkowski type inequality. Moreover, as applications, we get some corollaries including an isoperimetric type inequality and a uniqueness theorem.

1. Introduction

A compact, convex set in $\mathbb{R}^n$ is said to be a convex body if it has non empty interior. A compact set K with non-empty interior in $\mathbb{R}^n$ $(n \geq 2)$ is called a star body if $tK \subseteq K, \forall t \in [0,1]$, and the radial function $\rho_K(\theta) = \sup\{\lambda \geq 0 : \lambda\theta \in K\}$ is continuous on the unit sphere S^{n-1}. The Minkowski functional of a star body K in $\mathbb{R}^n$ is defined by $\|x\|_K = \min\{a \geq 0 : x \in aK\}$, so that $\|\theta\|_K = \rho_K^{-1}(\theta), \theta \in S^{n-1}$.

We use $\mathcal{K}(W)$ to denote the set of convex bodies in W and $\mathcal{K}_c(W)$ to denote the subset of $\mathcal{K}(W)$ that contains the centered (centrally symmetric with respect to the origin) bodies. We denote by $\mathcal{S}(W)$ and $\mathcal{S}_c(W)$ the set of all star bodies and the set of centered star bodies in W. For $W = \mathbb{R}^n$, we shall usually write $\mathcal{K}^n$ and $\mathcal{K}_c^n$ as $\mathcal{K}(W)$ and $\mathcal{K}_c(W)$, and write $\mathcal{S}^n$ and $\mathcal{S}_c^n$ instead of $\mathcal{S}(W)$ and $\mathcal{S}_c(W)$. We shall use $\text{Vol}_i(\cdot)$ to denote the i-dimensional volume function (which is the volume restricted to bodies

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of dimension i). Instead of $\mathrm{Vol}_n(\cdot)$ we usually write $V(\cdot)$. The volume of the standard unit ball B_n and unit sphere S^{n-1} in $\mathbb{R}^n$ are denoted by ω_n and σ_{n-1}, respectively.

At the beginning of the late nineteenth century, the classical Brunn-Minkowski theory was developed by Minkowski, Blaschke, Aleksandrov, Fenchel, and others. The Brunn-Minkowski inequality, combining volume and Minkowski addition, became an extremely powerful tool in convex geometry with important applications to various other areas of mathematics. The concept of projection body was introduced by Minkowski as one of the core concepts within Brunn-Minkowski theory. See Schneider's classic book [36]. One extension of the core Brunn-Minkowski theory, L_p-Brunn-Minkowski theory, was first studied by Lutwak in the 1990's and then by Lutwak, Yang, and Zhang, and many others. It has allowed many of the already potent sharp affine isoperimetric inequalities of the classical theory, as well as related analytic inequalities, to be strengthened. It also provided tools and methods for studying the unsolved problems such as the slicing problem of Bourgain, and connections between convex geometry and information theory (see e.g. [8, 17, 18, 26, 27, 28, 29, 30, 37]). There is more recent extension, Orlicz-Brunn-Minkowski theory, initiated by Lutwak, Yang, and Zhang [31, 32].

The concept of intersection body was introduced by Lutwak in [23], as dual Brunn-Minkowski theory. The family of intersection bodies and mixed intersection bodies are valuable in geometry, especially in the study of the famous Busemann-Petty problem (see books [11, 19, 21] and references therein). In recent years, intersection bodies and their generalizations received a fast growing attention and naturally appeared in various contexts (see [5, 14, 15, 19, 22, 35]).

In real Brunn-Minkowski theory and its dual, the Brunn-Minkowski inequalities and its generalizations are important and have significant applications in convex geometric analysis, information theory, partial differential equations, probability theory and other fields (see e.g. [4, 6, 7, 10, 12, 13, 33, 34, 36]). However, for complex-affine geometric inequalities it seems that not too much work has been done (see e.g. [1, 2, 3, 20]).

Let W be a complex space with complex dimension $m \geq 3$, and $K, K_1, K_2, \cdots, K_{2n-2} \in \mathcal{K}(W)$ and $C \in \mathcal{K}(\mathbb{C})$. In [3], Abardia and Bernig introduced the concepts of complex projection bodies $\Pi_C K$ and complex mixed projection bodies $\Pi_C(K_1, \cdots, K_{2n-2})$ as complex Brunn-Minkowski theory, and established the following three important inequalities.

Theorem A. (Brunn-Minkowski type inequality). *Let $K, L \in \mathcal{K}(W)$, $C \in \mathcal{K}(\mathbb{C})$. We have*

$$V\big(\Pi_C(K+L)\big)^{\frac{1}{2n(2n-1)}} \geq V\big(\Pi_C K\big)^{\frac{1}{2n(2n-1)}} + V\big(\Pi_C L\big)^{\frac{1}{2n(2n-1)}},$$

if K and L have non-empty interior and C is not a point then equality holds if and only if K and L are homothetic.

Theorem B. (Aleksandrov-Fenchel type inequality). *Let $K_1, K_2, \cdots, K_{2n-2} \in$*

$\mathcal{K}(W)$ and $C \in \mathcal{K}(\mathbb{C})$. If $0 \le i \le 2n-1, 2 \le k \le 2n-2$, then

$$W_i\big(\Pi_C(K_1, K_2, \cdots, K_{2n-2})\big)^k \ge \prod_{j=1}^{k} W_i\big(\Pi_C(K_j, \cdots, K_j, K_{k+1}, \cdots, K_{2n-2})\big).$$

Theorem C. (Minkowski type inequality). *Let $K, L \in \mathcal{K}(W)$ and $C \in \mathcal{K}(\mathbb{C})$. If $0 \le i < 2n-1$, then*

$$W_i\big(\Pi_C(K[2n-2], L)\big)^{2n-1} \ge W_i\big(\Pi_C K\big)^{2n-2} W_i\big(\Pi_C L\big),$$

if K and L have non-empty interior and C is not a point then equality holds if and only if K and L are homothetic.

In this paper, we study complex intersection bodies as complex dual Brunn-Minkowski theory, and obtain the dual type of Theorem A, B and C.

Following [23], we say that $K \in \mathcal{S}_c^n$ is the intersection body of $L \in \mathcal{S}_c^n$ and write $K = IL$ if for each $\xi \in S^{n-1}$,

$$\rho_K(\xi) = \mathrm{Vol}_{n-1}(L \cap \xi^{\perp}),$$

here $\xi^{\perp}$ is the central hyperplane which is perpendicular to ξ.

To formulate the complex version, we need some additional definitions.

Let $\xi \in \mathbb{C}^n$ with $|\xi| = 1$. We denote by

$$H_\xi = \left\{ z \in \mathbb{C}^n : (z, \xi) = \sum_{k=1}^{n} z_k \bar{\xi}_k = 0 \right\}$$

the complex central hyperplane which is orthogonal to ξ.

In order to define volume, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the mapping

$$\xi = (\xi_1, \cdots, \xi_n) = (\xi_{11} + i\xi_{12}, \cdots, \xi_{n1} + i\xi_{n2}) \longrightarrow (\xi_{11}, \xi_{12}, \cdots, \xi_{n1}, \xi_{n2})$$

and observe that under this mapping the complex hyperplane H_ξ turns into a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ which is orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \cdots, \xi_{n1}, \xi_{n2}) \text{ and } \xi^{\perp} = (-\xi_{12}, \xi_{11}, \cdots, -\xi_{n2}, \xi_{n1}).$$

Origin symmetric complex convex bodies are defined as those origin symmetric convex bodies K in $\mathbb{R}^{2n}$ that are invariant with respect to any coordinate-wise two-dimensional rotation, i.e., for each $\theta \in [0, 2\pi]$ and each $x = (\xi_{11}, \xi_{12}, \cdots, \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

(1.1) $$\|x\|_K = \|R_\theta(x_{11}, x_{12}), \cdots, R_\theta(x_{n1}, x_{n2})\|_K,$$

here R_θ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle θ with respect to the origin. Origin symmetric complex convex bodies in $\mathbb{C}^n$ are the unit balls of norms on $\mathbb{C}^n$. If a convex body satisfies (1.1), we will say that it is invariant with respect to all R_θ.

In [20], Koldobsky introduced the concept of complex intersection body of origin symmetric complex star body. Here, we remove ”origin symmetric”, and obtain the extended concept.

Definition 1.1. Let $K \in \mathcal{S}_c(\mathbb{C}^n), L \in \mathcal{S}(\mathbb{C}^n)$. We say that K is the complex intersection body of L and write $K = I_cL$ if for every $\xi \in S^{2n-1}$,

$$\text{Vol}_2(K \cap H_\xi^\perp) = \text{Vol}_{2n-2}(L \cap H_\xi). \tag{1.2}$$

Since $K \cap H_\xi^\perp$ is the two-dimensional Euclidean circle with radius $\rho_K(\xi)$, (1.2) can be written as

$$\pi \rho_K(\xi)^2 = \text{Vol}_{2n-2}(L \cap H_\xi).$$

Using the polar coordinates transform, we get

$$\text{Vol}_{2n-2}(L \cap H_\xi) = \frac{1}{2n-2} \int_{S^{2n-1} \cap H_\xi} \rho_L(u)^{2n-2} du.$$

Therefore, the concept of the complex intersection body can be formulated by

$$\rho_{I_cL}(\xi)^2 = \frac{1}{2\pi(n-1)} \int_{S^{2n-1} \cap H_\xi} \rho_L(u)^{2n-2} du. \tag{1.3}$$

We note that from (1.3), we have $I_cB_{2n} = (\frac{\omega_{2n-2}}{\pi})^{\frac{1}{2}} B_{2n}$.

Now, we extend the concept to complex mixed intersection body.

Definition 1.2. Let $K \in \mathcal{S}_c(\mathbb{C}^n), L_1, L_2, \cdots, L_{2n-2} \in \mathcal{S}(\mathbb{C}^n)$. We say that K is the complex mixed intersection body of $L_1, L_2, \cdots, L_{2n-2}$ and write $K = I_c(L_1, L_2, \cdots, L_{2n-2})$ if for every $\xi \in S^{2n-1}$,

$$\begin{aligned} &\rho_{I_c(L_1,L_2,\cdots,L_{2n-2})}(\xi)^2 \\ &= \frac{1}{2\pi(n-1)} \int_{S^{2n-1} \cap H_\xi} \rho_{L_1}(u)\rho_{L_2}(u) \cdots \rho_{L_{2n-2}}(u) du. \end{aligned} \tag{1.4}$$

For $L_1 = L_2 = \cdots = L_{2n-i-2} = K, L_{2n-i-1} = \cdots = L_{2n-2} = L$, $I_c(L_1, L_2, \cdots, L_{2n-2})$ is denoted by $I_{c,i}(K, L)$. And if $L = B_{2n}$ in $I_{c,i}(K, L)$, then we write $I_{c,i}K$.

We now describe our main results.

In order to study the complex intersection body, we introduce two linear combinations, the complex radial combination and complex radial-Blaschke combination, in Section 2.

In Section 3, we obtain the relations of those two combinations to dual mixed volumes. For complex radial combination, we firstly obtain the linear property about dual mixed volume, and use it to obtain the dual Brunn-Minkowski inequality for complex radial addition. Then, we use the complex radial combination to define a centered body $\tilde{\Delta}K$ for K. We show that the volume of $\tilde{\Delta}K$ is not bigger than that of K. However, the dual mixed volume $\tilde{V}_2(C, \tilde{\Delta}K)$ is always equal to $\tilde{V}_2(C, K)$ for $C \in \mathcal{S}_c(\mathbb{C}^n)$. For complex radial Blaschke combination, we obtain the related linear property and dual Knesser-Süss inequality. By defining the centered body $\check{\nabla}K$ for K, we show the corresponding results about the volume and the dual mixed volume. We will also define the complex dual surface area. As an application of dual Brunn-Minkowski inequality for complex radial addition, we obtain the related complex dual isoperimetric inequality.

In Section 4, we study the complex intersection operator. We firstly obtain that the complex intersection body of the complex radial-Blaschke combination is equal to the complex radial combination of complex intersection bodies, i.e., $I_c(\alpha \cdot K \check{+}_c \beta \cdot L) = \alpha \cdot I_c K \tilde{+}_c \beta \cdot I_c L$, and show that $I_c(-K) = I_c K$. Using this property, we obtain the identity $I_c \check{\nabla} K = I_c K$. Combinating this and the fact that the complex intersection operator is injective, we obtain that $I_c \check{\nabla} K$ is the unique centered star body in complex class of K, and it is characterized by having smaller volume. We will also prove an identity which will be useful in the later proof.

In Section 5, we shall establish the complex-affine geometric inequalities for complex intersection bodies and complex mixed intersection bodies.

Theorem 1.3. (dual Brunn-Minkowski type inequality). *Let* $K, L \in \mathcal{S}(\mathbb{C}^n), n \geq 2.$ *We have*

$$V\big(I_c(K \check{+}_c L)\big)^{\frac{1}{n}} \leq V\big(I_c K\big)^{\frac{1}{n}} + V\big(I_c L\big)^{\frac{1}{n}}, \tag{1.5}$$

with equality if and only if K *is a dilation of* L (*with the origin as the center of dilation*).

Theorem 1.4. (dual Aleksandrov-Fenchel type inequality). *Let* $K_1, K_2, \cdots, K_{2n-2} \in \mathcal{S}(\mathbb{C}^n), n \geq 2.$ *If* $0 \leq i \leq 2n-1, 2 \leq k \leq 2n-2$, *then*

$$\begin{aligned} &\tilde{W}_i\big(I_c(K_1, K_2, \cdots, K_{2n-2})\big)^k \\ &\leq \prod_{j=1}^{k} \tilde{W}_i\big(I_c(K_j, \cdots, K_j, K_{k+1}, \cdots, K_{2n-2})\big), \end{aligned} \tag{1.6}$$

with equality if and only if $K_1, K_2, \cdots, K_k$ *are all dilations of each other.*

Theorem 1.5. (dual Minkowski type inequality). *Let* $K, L \in \mathcal{S}(\mathbb{C}^n), n \geq 2$. *If* $0 \leq i \leq 2n-1$, *then*

$$\tilde{W}_i\big(I_{c,1}(K,L)\big)^{2n-2} \leq \tilde{W}_i\big(I_cK\big)^{2n-3}\tilde{W}_i\big(I_cL\big), \tag{1.7}$$

with equality if and only if K *is a dilation of* L.

The proof of Brunn-Minkowski type inequality for complex projection body involves the use of mixed volumes, whereas in our proof of dual Brunn-Minkowski inequality for complex intersection body, we use dual mixed volumes. We will also use Minkowski inequality for dual mixed volume and the identity obtained in Section 4.

In the proof of dual Aleksandrov-Fenchel type inequality, we use Hölder's integral inequality. Theorem 1.5 holds taking $K_1 = K_2 = \cdots = K_{2n-3} = K, K_{2n-2} = L$, and $k = 2n-2$ in Aleksandrov-Fenchel type inequality, and applying the dual Minkowski type inequality.

As applications, we obtain some corollaries including an isoperimetric type inequality and a uniqueness theorem.

2. Notation and Preliminaries

We note that some of our definitions and formulas are given in n-dimensional Euclidean space, however, we will use them in $2n$-dimensional space as n-dimensional complex space.

We extend the domain of the radial function from S^{n-1} to $\mathbb{R}^n$. Let K be a compact star-shaped set (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \to [0,+\infty)$, is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}, x \in \mathbb{R}^n.$$

Let $K \in \mathcal{S}^n$, and c be a real number, the Minkowski scalar multiplication cK is defined by

$$cK = \{cx : x \in K\}.$$

From the definition of the radial function, it is easy to verify that: if K is a star body, and $c \geq 0$, then

$$\rho_{cK}(u) = c\rho_K(u),\ \rho_K(cu) = c^{-1}\rho_K(u). \tag{2.1}$$

We say that two bodies $K, L \in \mathcal{S}^n$ are dilations (of each other) if $\rho(K,u)/\rho(L,u)$ is independent of $u \in S^{n-1}$.

Following [23], for $K, L \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha, \beta \geq 0$, the radial linear combination $\alpha \cdot K \tilde{+} \beta \cdot L$ and the radial-Blaschke linear combination $\alpha \cdot K \check{+} \beta \cdot L$ are defined by:

$$\rho(\alpha \cdot K \tilde{+} \beta \cdot L, \cdot) = \alpha\rho(K,\cdot) + \beta\rho(L,\cdot);$$

$$\rho(\alpha \cdot K \check{+} \beta \cdot L, \cdot)^{n-1} = \alpha\rho(K,\cdot)^{n-1} + \beta\rho(L,\cdot)^{n-1}.$$

Now, we extend those definitions to the complex case. For $K, L \in \mathcal{S}(\mathbb{C}^n)$ and $\alpha, \beta \geq 0$, we define the complex radial linear combination, $\alpha \cdot K \tilde{+}_c \beta \cdot L$, as the star body whose radial function is given by:

$$\rho(\alpha \cdot K \tilde{+}_c \beta \cdot L, \cdot)^2 = \alpha \rho(K, \cdot)^2 + \beta \rho(L, \cdot)^2. \tag{2.2}$$

The addition and scalar multiplication are called complex radial addition and complex scalar multiplication. Obviously, the complex radial scalar multiplication and the Minkowski scalar multiplication are related by $\alpha \cdot K = \alpha^{\frac{1}{2}} K$.

For $K, L \in \mathcal{S}(\mathbb{C}^n)$ and $\alpha, \beta \geq 0$, we define the complex radial-Blaschke linear combination, $\alpha \cdot K \check{+}_c \beta \cdot L$, as the star body whose radial function is given by:

$$\rho(\alpha \cdot K \check{+}_c \beta \cdot L, \cdot)^{2n-2} = \alpha \rho(K, \cdot)^{2n-2} + \beta \rho(L, \cdot)^{2n-2}. \tag{2.3}$$

The addition and scalar multiplication are called complex radial-Blaschke addition and scalar multiplication. It is easy to verify that $\alpha \cdot K = \alpha^{\frac{1}{2n-2}} K$.

Note that " $\cdot$ " rather than "$\tilde{\cdot}_c$" or "$\check{\cdot}_c$" is written for complex scalar multiplication or complex radial-Blaschke scalar multiplication. This create no confusion.

We note that complex radial combination and complex radial-Blaschke combination are associative. From their definitions, it follows that for $K, L \in \mathcal{S}(\mathbb{C}^n)$ and $\alpha, \beta \geq 0$,

$$\alpha \cdot (K \tilde{+}_c L) = \alpha \cdot K \tilde{+}_c \alpha \cdot L, \ (\alpha + \beta) \cdot K = \alpha \cdot K \tilde{+}_c \beta \cdot K; \tag{2.4}$$

$$\alpha \cdot (K \check{+}_c L) = \alpha \cdot K \check{+}_c \alpha \cdot L, \ (\alpha + \beta) \cdot K = \alpha \cdot K \check{+}_c \beta \cdot K. \tag{2.5}$$

In (2.4), " $\cdot$ " denotes complex radial multiplication, however in (2.5), " $\cdot$ " denotes complex radial-Blaschke multiplication.

Let $K_j \in \mathcal{S}^n (1 \leq j \leq n)$. The dual-mixed volume $\tilde{V}(K_1, K_2, \cdots, K_n)$ is defined by Lutwak in [23, 24] by

$$\tilde{V}(K_1, K_2, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}(u) \rho_{K_2}(u) \cdots \rho_{K_n}(u) dS(u). \tag{2.6}$$

For $K_1 = K_2 = \cdots = K_{n-i} = K, K_{n-i+1} = K_{n-i+2} = \cdots = K_n = L$, the dual-mixed volume is written as $\tilde{V}_i(K, L)$. In particular, the dual-mixed volume $\tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$ and is called the ith dual quermassintegral of K. From (2.6), it is obvious that for $K \in \mathcal{S}^n$, $\tilde{V}(K, \cdots, K) = V(K)$.

Let Λ be a nonsingular linear transform. It follows that for $K_1, K_2, \cdots, K_n$,

$$\tilde{V}(\Lambda K_1, \Lambda K_2, \cdots, \Lambda K_n) = |\Lambda| \tilde{V}(K_1, K_2, \cdots, K_n). \tag{2.7}$$

In the following, we need the following dual Minkowski inequality for dual-mixed volumes, which was proved in [24].

Lemma 2.1. *If* $A, B \in \mathcal{S}^n,\ 0 < i < n,\ i \in \mathbb{R}$, *then*

$$\tilde{V}_i(A, B) \leq V^{\frac{n-i}{n}}(A)V^{\frac{i}{n}}(B), \tag{2.8}$$

with equality if and only if K *is a dilation of* L.

We denote by $C_c(S^{2n-1})$ the space of $R_\theta-$invariant continuous functions, i.e., continuous real-functions f on the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ such that $f(\xi) = f(R_\theta(\xi))$ for all $\xi \in S^{2n-1}$ and all $\theta \in [0, 2\pi]$. The complex spherical Radon transform is an operator $\mathcal{R}_c : C_c(S^{2n-1}) \to C_c(S^{2n-1})$ defined by

$$\mathcal{R}_c f(\xi) = \int_{S^{2n-1}\cap H_\xi} f(u)du.$$

It is proved in [19] that the complex spherical Radon transform is self-dual, i.e., for any even functions $f, g \in C_c(S^{2n-1})$

$$\int_{S^{2n-1}} \mathcal{R}_c f(\xi) g(\xi) d\xi = \int_{S^{2n-1}} f(\xi)\mathcal{R}_c g(\xi) d\xi. \tag{2.9}$$

For origin symmetric complex star bodies, we can use complex spherical Radon transform to reformulate the concepts of the complex intersection bodies and complex mixed intersection bodies. From Definition 1.1, we say that $I_c L \in \mathcal{S}_c(\mathbb{C}^n)$ is the complex intersection body of $L \in \mathcal{S}_c(\mathbb{C}^n)$ if for every $\xi \in S^{2n-1}$,

$$\rho_{I_c L}(\xi)^2 = \frac{1}{2\pi(n-1)}\mathcal{R}_c\big(\rho_L(\cdot)^{2n-2}\big)(\xi). \tag{2.10}$$

From Definition 1.2, we say that $I_c(L_1, L_2, \cdots, L_{2n-2}) \in \mathcal{S}_c(\mathbb{C}^n)$ is the complex mixed intersection body of $L_1, L_2, \cdots, L_{2n-2} \in \mathcal{S}_c(\mathbb{C}^n)$ if for every $\xi \in S^{2n-1}$,

$$\rho_{I_c(L_1,L_2,\cdots,L_{2n-2})}(\xi)^2 = \frac{1}{2\pi(n-1)}\mathcal{R}_c\big(\rho_{L_1}(\cdot)\rho_{L_2}(\cdot)\cdots\rho_{L_{2n-2}}(\cdot)\big)(\xi). \tag{2.11}$$

3. Dual Mixed Volume for Complex Combinations

First, we study the complex radial combination. For complex radial combination, we firstly establish the linear property about dual mixed volume, and use it to prove the complex dual Brunn-Minkowski inequality.

From (2.2) and (2.6), it follows that

$$\begin{aligned}\tilde{V}_2(M, \alpha\cdot K \tilde{+}_c \beta\cdot L) &= \frac{1}{2n}\int_{S^{2n-1}} \rho_M^{2n-2}(u)\rho^2_{\alpha\cdot K\tilde{+}_c\beta\cdot L}(u)dS(u)\\ &= \alpha\tilde{V}_2(M, K) + \beta\tilde{V}_2(M, L).\end{aligned}$$

Thus, we obtain the following linear property.

Proposition 3.1. If K, L and M are complex star bodies in $\mathbb{C}^n$ $(n \geq 2)$, and $\alpha, \beta \geq 0$, then

$$\tilde{V}_2(M, \alpha \cdot K \tilde{+}_c \beta \cdot L) = \alpha \tilde{V}_2(M, K) + \beta \tilde{V}_2(M, L).$$

Now we establish the complex version of dual Brunn-Minkowski inequality.

Theorem 3.2. If $K, L \in \mathcal{S}(\mathbb{C}^n)$, then

$$V(K \tilde{+}_c L)^{\frac{1}{n}} \leq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}},$$

with equality if and only if K is a dilation of L.

Proof. From Proposition 3.1 and dual Minkowski inequality (2.8), it follows that

$$\begin{aligned}\tilde{V}_2(M, K \tilde{+}_c L) &= \tilde{V}_2(M, K) + \tilde{V}_2(M, L) \\ &\leq V(M)^{\frac{2n-2}{2n}} \big(V(K)^{\frac{2}{2n}} + V(L)^{\frac{2}{2n}}\big).\end{aligned}$$

Taking $M = K \tilde{+}_c L$, we obtain the desired inequality.

Now suppose that the equality holds. Since equality in dual Minkowski inequality holds if and only if two bodies are dilations of each other, convex bodies $M = K \tilde{+}_c L, K, L$ are all dilations of each other, i.e.,

$$K = \lambda_1(K \tilde{+}_c L), L = \lambda_2(K \tilde{+}_c L),$$

then $K = \lambda L$, as desired. ∎

For $K \in \mathcal{S}(\mathbb{C}^n)$, we define the centered body $\tilde{\Delta} K$ by

$$\tilde{\Delta} K = \frac{1}{2} \cdot K \tilde{+}_c \frac{1}{2} \cdot (-K).$$

The origin is called an $r-$equichordal point of $K \in \mathcal{S}^n$, or K is called $r-$equichordal, if $\rho_K^r(u) + \rho_K^r(-u)$ is independent of $u \in S^{n-1}$. For $K \in \mathcal{S}(\mathbb{C}^n)$, $\rho(-K, u) = \rho(K, -u)$, for all $u \in S^{2n-1}$; hence from (2.2), it follows that K is $2-$equichordal if and only if $\tilde{\Delta} K$ is a centered ball.

For $\tilde{\Delta} K$, we establish the following two propositions. We firstly show that the volume of $\tilde{\Delta} K$ is not bigger than that of K.

Proposition 3.3. *Let K be a complex star body in $\mathbb{C}^n$ $(n \geq 2)$. Then*

$$V(\tilde{\Delta} K) \leq V(K),$$

with equality if and only if K is centered.

Proof. Using Theorem 3.2 and the homogeneity of volume, we have that

$$\begin{aligned} V(\tilde{\Delta}K)^{\frac{1}{n}} &\leq V(2^{-1}\cdot K)^{\frac{1}{n}} + V\big(2^{-1}\cdot(-K)\big)^{\frac{1}{n}} \\ &= 2\Big(V\big((2^{-1})^{\frac{1}{2}}K\big)\Big)^{\frac{1}{n}} = 2\Big((2^{-1})^{\frac{2n}{2}}V(K)\Big)^{\frac{1}{n}} \\ &= 2(2^{-1})^{\frac{2n}{2}\frac{1}{n}}V(K)^{\frac{1}{n}} \ = V(K)^{\frac{1}{n}}. \end{aligned}$$

Theorem 3.2 implies that equality holds if and only if $K = \lambda(-K) = -\lambda K$ for $\lambda > 0$.

From (2.1), we have $\rho_K(u) = \rho_{-\lambda K}(u) = \lambda\rho_{-K}(u)$, and $\rho_K(-u) = \rho_{-\lambda K}(-u) = \lambda\rho_K(u)$. However, $\rho_{-K}(u) = \rho_K(-u)$, hence, $\lambda = 1$. Therefore, we have that K is centered, as desired. ■

The following proposition shows that $\tilde{V}_2(C, \tilde{\Delta}K)$ is always equal to $\tilde{V}_2(C, K)$ for $C \in \mathcal{S}_c(\mathbb{C}^n)$.

Proposition 3.4. *Let K be a complex star body in $\mathbb{C}^n$ $(n \geq 2)$. Then for $C \in \mathcal{S}_c(\mathbb{C}^n)$*

$$\tilde{V}_2(C, \tilde{\Delta}K) = \tilde{V}_2(C, K).$$

Proof. From Proposition 3.1 and (2.7), it follows that

$$\begin{aligned} \tilde{V}_2(C, \tilde{\Delta}K) &= \frac{1}{2}\tilde{V}_2(C, K) + \frac{1}{2}\tilde{V}_2(C, -K) \\ &= \frac{1}{2}\tilde{V}_2(C, K) + \frac{1}{2}|-I|\tilde{V}_2(-C, K) \\ &= \frac{1}{2}\tilde{V}_2(C, K) + \frac{1}{2}\tilde{V}_2(C, K) \\ &= \tilde{V}_2(C, K). \end{aligned}$$ ■

We now study the complex radial-Blaschke combination.

Similar to Proposition 3.1 and Theorem 3.2, it is easy to obtain the related linear property and dual Knesser-Süss inequality for complex radial-Blaschke combination.

Proposition 3.5. If K, L and M are complex star bodies in $\mathbb{C}^n$ $(n \geq 2)$, and $\alpha, \beta \geq 0$, then

$$\tilde{V}_2(\alpha\cdot K \check{+}_c \beta\cdot L, M) = \alpha\tilde{V}_2(K, M) + \beta\tilde{V}_2(L, M).$$

Theorem 3.6. (Dual Knesser-Süss inequality). If K, L are complex star bodies in $\mathbb{C}^n$ $(n \geq 2)$, then

$$V(K \check{+}_c L)^{\frac{n-1}{n}} \leq V(K)^{\frac{n-1}{n}} + V(L)^{\frac{n-1}{n}},$$

where equality holds if and only if K is a dilation of L.

For $K \in \mathcal{S}(\mathbb{C}^n)$, we define the centered body $\check{\nabla} K$ by

$$\check{\nabla} K = \frac{1}{2} \cdot K \check{+}_c \frac{1}{2} \cdot (-K).$$

For $K \in \mathcal{S}(\mathbb{C}^n)$, it follows from (2.3) that K is $(2n-2)-$equichordal if and only if $\check{\nabla} K$ is a centered ball.

For the centered body $\check{\nabla} K$, we have the corresponding results of Proposition 3.3 and Proposition 3.4.

Proposition 3.7. Let K be a complex star body in $\mathbb{C}^n$ $(n \geq 2)$. Then

$$V(\check{\nabla} K) \leq V(K),$$

with equality if and only if K is centered.

Proposition 3.8. Let K be a complex star body in $\mathbb{C}^n$ $(n \geq 2)$. Then for $C \in \mathcal{S}_c(\mathbb{C}^n)$

$$\tilde{V}_2(\check{\nabla} K, C) = \tilde{V}_2(K, C).$$

The surface area $S(K)$ of a convex body K defined by Minkowski (see [12]) is given by

$$S(K) = \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon B_n) - V(K)}{\varepsilon}. \tag{3.1}$$

The isoperimetric inequality for convex bodies in $\mathbb{R}^n$ is the nontrivial statement that if K is a convex body in $\mathbb{R}^n$, then

$$\left(\frac{V(K)}{V(B_n)}\right)^{1/n} \leq \left(\frac{S(K)}{S(B_n)}\right)^{1/(n-1)}, \tag{3.2}$$

with equality if and only if K is a ball.

For $K \in \mathcal{S}(\mathbb{C}^n)$, we define the complex dual surface area $\tilde{S}_c(K)$ of K by

$$\tilde{S}_c(K) = \lim_{\varepsilon \to 0^+} \frac{V(K \tilde{+}_c \varepsilon \cdot B_{2n}) - V(K)}{\varepsilon}. \tag{3.3}$$

Using (2.2) and then the polar coordinate formula for volume, it follows immediately that $\tilde{S}_c(K) = n\tilde{V}_2(K, B_{2n})$. Up to a constant, complex dual surface area is just a special dual mixed volume. We show the complex dual of isoperimetric inequality.

Theorem 3.9. If K is a star body in $\mathbb{C}^n$, then

$$\left(\frac{\tilde{S}_c(K)}{\tilde{S}_c(B_{2n})}\right)^{1/(n-1)} \leq \left(\frac{V(K)}{V(B_{2n})}\right)^{1/n}, \tag{3.4}$$

with equality if and only if K is a centered ball.

Proof. Substituting $\varepsilon = t/(1-t)$ in (3.3) and using (2.4) and the homogeneity of volume, we obtain

$$
\begin{aligned}
nV_2(K, B_{2n}) &= \lim_{t\to 0^+} \frac{V\big(K \tilde{+}_c \frac{t}{1-t} \cdot B_{2n}\big) - V(K)}{\frac{t}{1-t}} \\
&= \lim_{t\to 0^+} \frac{(1-t)^n V(K \tilde{+}_c \frac{t}{1-t} \cdot B_{2n}) - (1-t)^n V(K)}{t(1-t)^{n-1}} \\
&= \lim_{t\to 0^+} \frac{V((1-t)\cdot K \tilde{+}_c t \cdot B_{2n}) - (1-t)^n V(K)}{t(1-t)^{n-1}} \\
&= \lim_{t\to 0^+} \frac{V((1-t)\cdot K \tilde{+}_c t \cdot B_{2n}) - V(K)}{t} + \lim_{t\to 0^+} \frac{(1-(1-t)^n)V(K)}{t} \\
&= \lim_{t\to 0^+} \frac{V((1-t)\cdot K \tilde{+}_c t \cdot B_{2n}) - V(K)}{t} + nV(K).
\end{aligned}
$$

Using this expression for $V_2(K, B_{2n})$ and letting $f(t) = V((1-t)\cdot K \tilde{+}_c t \cdot B_{2n})^{\frac{1}{n}}$ for $0 \le t \le 1$, we see that

$$f'(0) = \frac{V_2(K, B_{2n}) - V(K)}{V(K)^{(n-1)/n}}.$$

Note that from Theorem 3.2 and the homogeneity of volume, it follows that for $0 \le t \le 1$

$$V\big((1-t)\cdot K \tilde{+}_c t \cdot L\big)^{\frac{1}{n}} \le (1-t)V(K)^{\frac{1}{n}} + tV(L)^{\frac{1}{n}},$$

with equality if and only if K is a dilation of L.

It shows that $f(t)$ is convex function, and thus $f'(0) \le f(1) - f(0)$. This is equivalent to

$$nV_2(K, B_{2n}) \le nV(K)^{\frac{n-1}{n}} V(B_{2n})^{\frac{1}{n}}.$$

Inequality (3.4) results from recalling that $\tilde{S}_c(K) = nV_2(K, B_{2n})$ and $\tilde{S}_c(B_{2n}) = nV(B_{2n})$ and rearranging.

Suppose equality holds in (3.4). Then $f'(0) = f(1) - f(0)$. Since f is convex function, we have

$$\frac{f(t) - f(0)}{t} = f(1) - f(0)$$

for $0 < t \le 1$, and this is just equality in Theorem 3.2. Equality condition for (3.4) follows immediately. ■

4. Complex Radial Blaschke Linear Combination for Complex Intersection Bodies

The complex intersection operator will be studied in this section.

Let $K, L \in \mathcal{S}(\mathbb{C}^n)$ and $\alpha, \beta \geq 0$. From Definition 1.1, we see that for $\xi \in S^{2n-1}$

$$\begin{aligned}
&\rho(I_c(\alpha \cdot K \check{+}_c \beta \cdot L), \xi)^2 \\
&= \frac{1}{2\pi(n-1)} \int_{S^{2n-1} \cap H_\xi} \rho(\alpha \cdot K \check{+}_c \beta \cdot L, u)^{2n-2} du \\
&= \frac{\alpha}{2\pi(n-1)} \int_{S^{2n-1} \cap H_\xi} \rho(K, u)^{2n-2} du + \frac{\beta}{2\pi(n-1)} \int_{S^{2n-1} \cap H_\xi} \rho(L, u)^{2n-2} du \\
&= \alpha \rho(I_c K, u)^2 + \beta \rho(I_c L, u)^2 = \rho(\alpha \cdot I_c K \tilde{+}_c \beta \cdot I_c L, \xi)^2.
\end{aligned}$$

Thus, $I_c(\alpha \cdot K \check{+}_c \beta \cdot L) = \alpha \cdot I_c K \tilde{+}_c \beta \cdot I_c L$. And $I_c(-K) = I_c K$ is an immediate consequence of Definition 1.1. Therefore, we have the following proposition.

Proposition 4.1. If $K, L \in \mathcal{S}(\mathbb{C}^n)$ and $\alpha, \beta \geq 0$, then

$$I_c(\alpha \cdot K \check{+}_c \beta \cdot L) = \alpha \cdot I_c K \tilde{+}_c \beta \cdot I_c L, \;\; I_c(-K) = I_c K. \tag{4.1}$$

Using this property and (2.4), it follows that

$$I_c \check{\nabla} K = \frac{1}{2} \cdot I_c K \tilde{+}_c \frac{1}{2} \cdot I_c(-K) = I_c K.$$

Thus, we have:

Proposition 4.2. If $K \in \mathcal{S}(\mathbb{C}^n)$, then

$$I_c \check{\nabla} K = I_c K. \tag{4.2}$$

The previous propositions allow us to find necessary and sufficient conditions for that two complex star bodies have the same complex intersection bodies.

Theorem 4.3. If $K, L \in \mathcal{S}(\mathbb{C}^n)$, then

$$I_c K = I_c L \tag{4.3}$$

if and only if

$$\tilde{V}_2(K, M) = \tilde{V}_2(L, M), \tag{4.4}$$

for all $M \in \mathcal{S}_c(\mathbb{C}^n)$.

Proof. From Proposition 3.8 and Proposition 4.2, we may assume that $K, L \in \mathcal{S}_c(\mathbb{C}^n)$.

Suppose that for all $M \in \mathcal{S}_c(\mathbb{C}^n)$, (4.4) hold. Let $f \in C_c^+(S^{2n-1})$, we define M by

$$\rho_M^2 = \mathcal{R}_c f.$$

From (2.6), we have

$$\begin{aligned}\tilde{V}_2(K,M) &= \frac{1}{2n}\int_{S^{2n-1}}\rho_K^{2n-2}(u)\rho_M^2(u)dS(u)\\ &= \frac{1}{2n}\int_{S^{2n-1}}\rho_K^{2n-2}(u)\mathcal{R}_cf(u)dS(u).\end{aligned}$$

and

$$\begin{aligned}\tilde{V}_2(L,M) &= \frac{1}{2n}\int_{S^{2n-1}}\rho_L^{2n-2}(u)\rho_M^2(u)dS(u)\\ &= \frac{1}{2n}\int_{S^{2n-1}}\rho_L^{2n-2}(u)\mathcal{R}_cf(u)dS(u).\end{aligned}$$

Using (4.4), (2.9), and (2.10), we have

$$\int_{S^{2n-1}}\rho_{I_cK}^2(u)f(u)dS(u) = \int_{S^{2n-1}}\rho_{I_cL}^2(u)f(u)dS(u).$$

Therefore, for all $f \in C_c^+(S^{2n-1})$,

$$\int_{S^{2n-1}}\big(\rho_{I_cK}^2(u) - \rho_{I_cL}^2(u)\big)f(u)dS(u) = 0.$$

But this must hold for all $f \in C_c(S^{2n-1})$, since we can write an arbitrary function in $C_c(S^{2n-1})$ as the difference of two functions in $C_c^+(S^{2n-1})$. Substituting $f = \rho_{I_cL}^2(u) - \rho_{I_cL}^2(u)$, we have $|\rho_{I_cL}^2(u) - \rho_{I_cL}^2(u)|_2 = 0$, and hence, $I_cK = I_cL$.

Now suppose (4.3) holds and $M \in \mathcal{S}_c(\mathbb{C}^n)$. Suppose M satisfies that $\rho_M \in \mathcal{R}_c\big(C_c(S^{2n-1})\big)$ and hence there exists an $f \in C_c(S^{2n-1})$, such that

$$\rho_M^2 = \mathcal{R}_cf.$$

From (2.6), we have

$$\begin{aligned}\tilde{V}_2(K,M) &= \frac{1}{2n}\int_{S^{2n-1}}\rho_K^{2n-2}(u)\rho_M^2(u)dS(u)\\ &= \frac{1}{2n}\int_{S^{2n-1}}\rho_K^{2n-2}(u)\mathcal{R}_cf(u)dS(u).\end{aligned}$$

and

$$\begin{aligned}\tilde{V}_2(L,M) &= \frac{1}{2n}\int_{S^{2n-1}}\rho_L^{2n-2}(u)\rho_M^2(u)dS(u)\\ &= \frac{1}{2n}\int_{S^{2n-1}}\rho_L^{2n-2}(u)\mathcal{R}_cf(u)dS(u).\end{aligned}$$

From (4.3), (2.9), and (2.10), we have $\tilde{V}_2(K,M) = \tilde{V}_2(L,M)$. Since $\mathcal{R}_c\big(C_c(S^{2n-1})\big)$ is dense in $C_c(S^{2n-1})$, and dual mixed volumes are continuous, it follows that (4.4) must hold for all $M \in \mathcal{S}_c(\mathbb{C}^n)$. ■

Now we use Theorem 4.3 to prove that the complex intersection operator is injective.

Theorem 4.4. The mapping $I_c : \mathcal{S}_c(\mathbb{C}^n) \longrightarrow \mathcal{S}(\mathbb{C}^n)$ is injective.

Proof. Suppose that $I_cK = I_cL$. From Theorem 4.3, we know that $\tilde{V}_2(K, M) = \tilde{V}_2(L, M)$ holds for all $M \in \mathcal{S}_c^{2n}$. Substituting $M = K$, and using the dual Minkowski inequality (2.8), we have

$$V(K) = \tilde{V}_2(K, K) = \tilde{V}_2(L, K) \le V(L)^{\frac{n-1}{n}} V(K)^{\frac{1}{n}}.$$

Hence $V(K) \le V(L)$ with equality if and only if K and L are dilations of each other. Similarly substituting $M = L$, we get $V(L) \le V(K)$. Hence, $V(L) = V(K)$ and from equality condition we can conclude that K and L are dilations. However, since they have the same volume they must be equal, i.e., $K = L$, as desired. ∎

We remark that the complex Radon transform is injective. In fact, we can obtain it from Theorem 4.4. Suppose $f, g \in C_c(S^{2n-1})$ and $\mathcal{R}_c f = \mathcal{R}_c g$. We take any $\lambda \in \mathbb{R}$ such that $f + \lambda$ and $g + \lambda$ are both positive functions, and we define $K, L \in \mathcal{S}^{2n}$ by

$$\rho_K^{2n-2} = f + \lambda \text{ and } \rho_L^{2n-2} = g + \lambda.$$

Since $\mathcal{R}_c f = \mathcal{R}_c g$, it follows that $\mathcal{R}_c \rho_K^{2n-2} = \mathcal{R}_c \rho_L^{2n-2}$, thus from (2.10) we have $I_cK = I_cL$. From Theorem 4.4, we have $K = L$. Hence $f = g$.

For $K \in \mathcal{S}(\mathbb{C}^n)$, we shall use $I_c\langle K\rangle$ to denote the set of all star bodies which have the same complex intersection body as K, i.e.,

$$I_c\langle K\rangle = \{L \in \mathcal{S}^{2n} : I_cL = I_cK\}.$$

We shall call $I_c\langle K\rangle$ the complex intersection class of K. From (4.2), $\check{\nabla}K \in I_c\langle K\rangle$.

Since the complex intersection operation is injective, we have that $I_c\check{\nabla}K$ is the unique centered star body in $I_c\langle K\rangle$.

Theorem 4.5. If $K \in \mathcal{S}(\mathbb{C}^n)$, then $I_c\langle K\rangle$ contains a unique centered star body, and this centered body is characterized by having smaller volume than any other star body in $I_c\langle K\rangle$.

Proof. Suppose $K \in \mathcal{S}(\mathbb{C}^n)$, then the centered star body $\check{\nabla}K$ is in $I_c\langle K\rangle$. If $L \in I_c\langle K\rangle$, then $I_cK = I_cL$. Hence, from Proposition 4.2 and Theorem 4.4, we have $\check{\nabla}L = \check{\nabla}K$. But from Theorem 3.6, we obtain

$$V(\check{\nabla}K) = V(\check{\nabla}L) \le V(L),$$

with equality if and only if L is centered. Therefore, we obtain the result. ∎

In order to prove Theorem 1.3, we need the following identity.

Lemma 4.6. If $K, L \in \mathcal{S}(\mathbb{C}^n)$, then

$$\tilde{V}_2(K, I_cL) = \tilde{V}_2(L, I_cK).$$

Proof. From Proposition 3.8 and Proposition 4.2, it follows that

$$\tilde{V}_2(K, I_cL) = \tilde{V}_2(\check{\nabla}K, I_c\check{\nabla}L) \text{ and } \tilde{V}_2(L, I_cK) = \tilde{V}_2(\check{\nabla}L, I_c\check{\nabla}K).$$

Therefore, we need only to prove the lemma for centered bodies. Suppose that $K, L \in S_c(\mathbb{C}^n)$. From (2.6),(2.10), it follows that

$$\begin{aligned}
\tilde{V}_2(K, I_cL) &= \frac{1}{2n}\int_{S^{2n-1}} \rho_K^{2n-2}(u)\rho_{I_cL}^2(u)dS(u)\\
&= \frac{c_0}{2n}\int_{S^{2n-1}} \rho_K^{2n-2}(u)\mathcal{R}_c\big(\rho_L^{2n-2}(u)\big)dS(u),\\
\tilde{V}_2(L, I_cK) &= \frac{1}{2n}\int_{S^{2n-1}} \rho_L^{2n-2}(u)\rho_{I_cK}^2(u)dS(u)\\
&= \frac{c_0}{2n}\int_{S^{2n-1}} \rho_L^{2n-2}(u)\mathcal{R}_c\big(\rho_K^{2n-2}(u)\big)dS(u).
\end{aligned}$$

Thus, the fact that complex Randon transform is self-adjoint implies the result. ■

A body $K \in \mathcal{S}(\mathbb{C}^n)$ will be said to have constant slice if $\mathrm{Vol}_{2n-2}(K \cap H_\xi)$ is independent of $\xi \in S^{2n-1}$. It is easy to obtain that K has constant slice if and only if I_cK is a centered ball.

For $K \in \mathcal{S}(\mathbb{C}^n)$, $\check{\nabla}K \in \mathcal{S}_c(\mathbb{C}^n)$ and since $I_c\check{\nabla}K = I_cK$, and I_cB_{2n} is a centered ball, it follows from Theorem 4.4 that K has constant slice if and only if $\check{\nabla}K$ is a centered ball. But $\check{\nabla}K$ is a centered ball if and only if K is $(2n-2)-$equichordal. Hence, we have:

Proposition 4.7. If $K \in \mathcal{S}(\mathbb{C}^n)$, then K has constant slice if and only if K is $(2n-2)-$equichordal.

5. Geometric Inequalities

Proof of Theorem 1.3. Let $Q \in \mathcal{S}(\mathbb{C}^n)$. Using Lemma 4.6, Proposition 3.5, and dual Minkowski inequality (2.8), we have

$$\begin{aligned}
\tilde{V}_2\big(Q, I_c(K\check{+}_cL)\big) &= \tilde{V}_2\big(K\check{+}_cL, I_c(Q)\big)\\
&= \tilde{V}_2\big(K, I_c(Q)\big) + \tilde{V}_2\big(L, I_c(Q)\big)\\
&= \tilde{V}_2\big(Q, I_c(K)\big) + \tilde{V}_2\big(Q, I_c(L)\big)\\
&\le V^{\frac{n-1}{n}}(Q)\big(V^{\frac{1}{n}}(I_cK) + V^{\frac{1}{n}}(I_cL)\big).
\end{aligned}$$

Taking $Q = I_c(K\check{+}_c L)$ and using (2.6), we have the desired inequality.

Now suppose that equality holds in Theorem 1.3. Since equality in (2.8) holds if and only if two bodies are dilations of each other, the three convex bodies $Q = I_c(K\check{+}_c L), I_c(K), I_c(L)$ are dilations, i.e.,

$$I_c(K) = \lambda_1(I_c(K\check{+}_c L)), I_c(L) = \lambda_2(I_c(K\check{+}_c L)),$$

so using Proposition 4.1, we have

$$I_c(K) = \lambda I_c(L) = I_c(\lambda^{\frac{1}{n-1}} L).$$

Therefore, from the injective property of the mapping I_c, we have that K is a dilation of L. ∎

From Theorem 1.3 and Proposition 4.1, it follows that:

Corollary 5.1. Let K, L be complex star bodies in $\mathbb{C}^n$ $(n \geq 2)$. Then

$$V\big(I_c K\tilde{+}_c I_c L\big)^{\frac{1}{n}} \leq V\big(I_c K\big)^{\frac{1}{n}} + V\big(I_c L\big)^{\frac{1}{n}},$$

with equality if and only if K is a dilation of L.

Note that we can also obtain this corollary from Theorem 3.2.

Similar to the proof of Theorem 3.9, we can obtain a complex dual isoperimetric inequality from Corollary 5.1. In fact, we need only notice the fact that $I_c B_{2n} = (\frac{w_{2n-2}}{\pi})^{\frac{1}{2}} B_{2n}$, and $\tilde{S}_c(\lambda K) = \lambda^{2n-2}\tilde{S}_c(K)$ $\big($from $\tilde{S}_c(K) = n\tilde{V}_2(K, B_{2n})\big)$.

Corollary 5.2. Let K is a complex star bodies in $\mathbb{C}^n$ $(n \geq 2)$. Then

$$\left(\frac{\tilde{S}_c(I_c K)}{\tilde{S}_c(B_{2n})}\right)^{1/(n-1)} \leq \left(\frac{V(I_c K)}{V(B_{2n})}\right)^{1/n},$$

with equality if and only if K is a centered ball.

In order to prove Theorem 1.4, we need the following Hölder integral inequality.

Lemma 5.3. (Hölder's integral inequality, see [9, 16]). Let $f_0, f_1, \cdots, f_k$ be Borel measurable functions on X. Suppose that $p_1, p_2, \cdots, p_k$ are nonzero real numbers with $\sum\limits_{i=1}^{k} \frac{1}{p_i} = 1$. Then

$$\int_X f_0(u) f_1(u) \cdots f_k(u) du \leq \prod_{i=1}^{k} \Big(\int_X f_0(u) f_i(u)^{p_i} du\Big)^{\frac{1}{p_i}},$$

with equality if and only if either (a) there are constants $b_1, b_2, \cdots, b_k$ not all zero, such that $b_1|f_1(u)|^{p_1} = b_2|f_2(u)|^{p_2} = \cdots = b_k|f_k(u)|^{p_k}$, or (b) one of the functions is null.

Proof of Theorem 1.4. From Definition 1.2 and Hölder's integral inequality, it follows that

$$\begin{aligned}\rho^2_{I_c(K_1,K_2,\cdots,K_{2n-2})}(\xi) &= \frac{1}{2\pi(n-1)}\int_{S^{2n-1}\cap H_\xi}\rho_{K_1}(u)\rho_{K_2}(u)\cdots\rho_{K_{2n-2}}(u)du\\ &\le \prod_{j=1}^{k}\left(\frac{1}{2\pi(n-1)}\int_{S^{2n-1}\cap H_\xi}\rho^k_{K_j}(u)\rho_{K_{k+1}}(u)\cdots\rho_{K_{2n-2}}(u)du\right)^{\frac{1}{k}}\\ &= \prod_{j=1}^{k}\rho^{\frac{2}{k}}_{I_c(K_j,\cdots,K_j,K_{k+1}\cdots,K_{2n-2})}(\xi).\end{aligned}$$

Integrating both sides of the above inequality on S^{2n-1}, and then using Hölder's integral inequality again, we get

$$\begin{aligned}\tilde{W}_i\big(I_c(K_1,K_2,\cdots,K_{2n-2})\big) &= \frac{1}{n}\int_{S^{2n-1}}\rho^{2n-i}_{I_c(K_1,K_2,\cdots,K_{2n-2})}(\xi)d\xi\\ &\le \frac{1}{n}\int_{S^{2n-1}}\prod_{j=1}^{k}\rho^{\frac{2n-i}{k}}_{I_c(K_j,\cdots,K_j,K_{k+1}\cdots,K_{2n-2})}(\xi)d\xi\\ &\le \prod_{j=1}^{k}\left(\frac{1}{n}\int_{S^{2n-1}}\rho^{2n-i}_{I_c(K_j,\cdots,K_j,K_{k+1}\cdots,K_{2n-2})}(\xi)d\xi\right)^{\frac{1}{k}}\\ &= \prod_{j=1}^{k}\tilde{W}_i\big(I_c(K_j,\cdots,K_j,K_{k+1},\cdots,K_{2n-2})\big)^{\frac{1}{k}}.\end{aligned}$$

From equality condition in Hölder integral inequality, we obtain that equality in our result holds if and only if $K_1, K_2, \cdots, K_k$ are all dilations of each other. ■

When $i = 0$ in Theorem 1.4, it follows that:

Corollary 5.4. Let $K_1, K_2, \cdots, K_{2n-2} \in \mathcal{S}(\mathbb{C}^n)$. If $2 \le k \le 2n-2$, then

$$V\big(I_c(K_1,K_2,\cdots,K_{2n-2})\big)^k \le \prod_{j=1}^{k} V\big(I_c(K_j,\cdots,K_j,K_{k+1},\cdots,K_{2n-2})\big),$$

with equality if and only if $K_1, K_2, \cdots, K_k$ are all dilations of each other.

If $k = 2n-2, K_1 = K_2 = \cdots = K_{2n-j-2} = K, K_{2n-j-1} = \cdots = K_{2n-2} = B_{2n}$ in Theorem 1.5, and note that $I_cB_{2n} = (\frac{\omega_{2n-2}}{\pi})^{\frac{1}{2}}B_{2n}$, then the following holds.

Corollary 5.5. Let $K \in \mathcal{S}(\mathbb{C}^n), 0 \le i < 2n$ and $0 < j < 2n-2$. Then

$$\tilde{W}_i(I_{c,j}K)^{2n-2} \le \left(\frac{1}{\pi}\omega_{2n-2}\right)^{j(n-\frac{i}{2})}\omega^j_{2n}\tilde{W}_i(I_cK)^{2n-2-j},$$

with equality if and only if K is a centered ball.

This result is the complex dual version of the following important result for mixed projection body, which established in [25].

Let $K \in \mathcal{K}^n, 0 \leq i < n$ and $0 < j < n-1$. Then

$$W_i(\Pi_j K)^{n-1} \leq \omega_{n-1}^{j(n-i)} \omega_n^j W_i(\Pi K)^{n-1-j},$$

with equality if and only if K is a ball.

Theorem 1.4 is the dual of Abardia and Bernig's result about Aleksandrov-Fenchel inequality for complex mixed projection body. In the following, we obtain dual Minkowski inequality for complex mixed intersection body, which is also the dual version of Abardia and Bernig's result about Minkowski inequality for complex mixed projection body. In fact, it is only a corollary of Theorem 1.4.

Putting that $K_1 = K_2 = \cdots = K_{2n-3} = K, K_{2n-2} = L$, and $k = 2n-2$ in Theorem 1.4, Theorem 1.5 holds.

As an application of Theorem 1.4, we establish the following uniqueness theorem. The symbol $I_c(K[2n-3-j], M, N[j])$ denotes that K appears $2n-3-j$ times and N appears j times.

Theorem 5.6. Let $K, L \in \mathcal{S}(\mathbb{C}^n)$. If $0 \leq i \leq 2n-1, 0 \leq j \leq 2n-4$, then for $M, N \in \mathcal{S}(\mathbb{C}^n)$

$$\tilde{W}_i\big(I_c(K[2n-3-j], M, N[j])\big) = \tilde{W}_i\big(I_c(L[2n-3-j], M, N[j])\big), \tag{5.1}$$

or

$$\tilde{W}_i\big(I_c(M[2n-3-j], K, N[j])\big) = \tilde{W}_i\big(I_c(M[2n-3-j], L, N[j])\big), \tag{5.2}$$

implies $K = L$.

Proof. Suppose (5.1) holds. Taking L for M, it follows from (5.1) that

$$\tilde{W}_i\big(I_c(K[2n-3-j], L, N[j])\big) = \tilde{W}_i\big(I_c(L[2n-2-j], N[j])\big). \tag{5.3}$$

From Aleksandrov-Fenchel type inequality and (5.3), we have

$$\begin{aligned} &\tilde{W}_i\big(I_c(L[2n-2-j], N[j])\big)^{2n-2-j} \\ &= \tilde{W}_i\big(I_c(K[2n-3-j], L, N[j])\big)^{2n-2-j} \\ &\leq \tilde{W}_i\big(I_c(K[2n-2-j], N[j]\big)^{2n-3-j} \tilde{W}_i\big(I_c(L[2n-2-j], N[j]\big), \end{aligned} \tag{5.4}$$

thus $\tilde{W}_i\big(I_c(L[2n-2-j], N[j]\big) \leq \tilde{W}_i\big(I_c(K[2n-2-j], N[j]\big)$ with equality if and only if K is a dilation of L.

Taking K for M, we similarly get

$$\tilde{W}_i\big(I_c(L[2n-2-j], N[j]\big) \geq \tilde{W}_i\big(I_c(K[2n-2-j], N[j]\big), \tag{5.5}$$

with equality if and only if K is a dilation of L.

Hence, from (5.4), (5.5), it follows that

$$\tilde{W}_i\big(I_c(L[2n-2-j], N[j]\big) = \tilde{W}_i\big(I_c(K[2n-2-j], N[j]\big),$$

where $K = \lambda L$.

From Definition 1.2 and (2.6), it follows that

$$\begin{aligned}\tilde{W}_i\big(I_c(L[2n-2-j], N[j]\big) &= \tilde{W}_i\big(I_c(K[2n-2-j], N[j]\big)\\ &= \lambda^{\frac{(2n-i)(2n-2-j)}{2}}\tilde{W}_i\big(I_c(L[2n-2-j], N[j]\big).\end{aligned}$$

Hence, $\lambda = 1$.

The fact that (5.2) implies $K = L$ can be established in the same manner. ■

When $i = 0, j = 0$ in uniqueness theorem, it follows that:

Corollary 5.7. If $K, L \in \mathcal{S}(\mathbb{C}^n)$, then for $M \in \mathcal{S}(\mathbb{C}^n)$

$$V\big(I_{c,1}(K, M)\big) = V\big(I_{c,1}(L, M)\big),$$

or

$$V\big(I_{c,1}(M, K)\big) = V\big(I_{c,1}(M, L)\big),$$

implies $K = L$.

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