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THE INFINITE GROWTH OF SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS OF WHICH COEFFICIENT WITH DYNAMICAL PROPERTY

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Abstract. In this paper, we prove that the transcendental entire solution of complex linear differential equation $f^{(k)} - e^{P(z)}f = Q(z)$, where $P(z)$ is a transcendental entire function and $Q(z)$ is a polynomial, is of infinite hyper-order under the hypothesis that the Fatou set of $P(z)$ has a multiply connected component.

1. Introduction and Main Results

In this paper, we will use the standard notations of Nevanlinna's value distribution theory (see [13, 17, 18]) and some knowledges of complex dynamics of entire functions (see [5, 20]). During the last ten years many papers have been devoted to the study of the growth of solutions of complex differential equations (see [14]). By making use of the properties of the logarithmic derivative, it is easy to see that if $A(z)$ is a transcendental entire function, then every nonzero solution f of the equation $f^{(k)} + A(z)f = 0$ is an entire function of order $\sigma(f) = \infty$. For the corresponding nonhomogeneous linear differential equation

$$f^{(k)} + A(z)f = F(z). \tag{1.1}$$

Chen and Gao (see [8]) proved that if A is a transcendental entire function and if $F \not\equiv 0$ is an entire function of finite order, then every solution f is of infinite order, with at most one possible exception. Thus an interesting problem arises: What conditions on A and F guarantee that every solution f of (1.1) has infinite order? Gundersen and Yang obtained the following result related to a conjecture of Brück [6].

Theorem 1.1. ([12]). *Let P be a nonconstant polynomial. Then every solution f of the differential equation $f' + e^{P(z)}f = 1$ is an entire function of infinite order.*

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For more precisely estimation of the growth of function f, the hyperorder ([18]) of a meromorphic function f is defined by

$$\sigma_2(f) := \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

In [15] (see also [16]), Yang raised the question below, and proved that if, in Theorem 1.1, P is a nonconstant entire function then the hyperorder of f is a positive integer or infinity *with at most one exception*. However, whether the exceptional solution exists or not is unclear.

Question. ([15] and [16]). Is it true that if P is a nonconstant entire function then the hyperorder of f satisfying the equation of Theorem 1.1 is a positive integer or infinity?

Later, Cao [7] considered this question. He proved that the answer of the above question is affirmative under the hypothesis that the order of P is less than 1/2. In fact, Cao got the result below.

Theorem 1.2. *Let P be a nonconstant entire function, let Q be a nonzero polynomial, and let f be any entire solution of the differential equation*

$$f^{(k)} + e^{P(z)} f = Q(z) \quad (k \in \mathbb{N}). \tag{1.2}$$

If P is a polynomial, then f has infinite order and its hyperorder $\sigma_2(f)$ is a positive integer not exceeding the degree of P. If P is transcendental with order less than 1/2, then the hyperorder of f is infinite.

In the proof of the above theorem, Cao split into two cases according $P(z)$ be a polynomial or transcendental entire function. In the case $P(z)$ be a transcendental entire function, he used the famous $\cos\pi\alpha$ Theorem, see the following Lemma 2.3, which need the condition *the order of $P(z)$ is less than 1/2*. Thus, the question of Yang is still open. In this note, we change the perspective. We introduce some dynamical properties to the transcendental entire function $P(z)$ and deduce that the conclusion of Theorem 1.2 could be also hold. Actually, we have the following theorem.

Theorem 1.3. *Let P be a transcendental entire function and the Fatou set $F(P)$ has a multiply connected component, let Q be a nonzero polynomial, and let f be any entire solution of the differential equation*

$$f^{(k)} - e^{P(z)} f = Q(z) \quad (k \in \mathbb{N}). \tag{1.3}$$

Then the hyper-order of f is infinite.

In the below, in order to explain the assumption of Theorem 1.3 we give some introduction of complex dynamics, see [5] for example. The Fatou set $F(f)$ of a transcendental entire function f is the subset of the plane $\mathbb{C}$ where the iterates f^n of f form a normal family. The complement of $F(f)$ in $\mathbb{C}$ is called the Julia set $J(f)$ of f. The set $F(f)$ is completely invariant under f in the sense that $z \in F(f)$ if and only if $f(z) \in F(f)$. Therefore, if U is a component of $F(f)$, a so-called Fatou component, then there exists, for some $n = 0, 1, 2, \cdots$, a Fatou component U_n such that $f^n(U) \subset U_n$. If, for some $p \geq 1$, we have $U_p = U_0 = U$, then we say that U is a periodic component of period p, assuming p to be the minimal. If U_n is not eventually periodic, then U is a wandering domain of f. Although some entire functions with only simply connected Fatou component, such as Eremenko-Lyubich class function [11], there are many examples of entire function with multiply connected Fatou components. The first such function was constructed by Baker [1], who proved later [3] that this function has a multiply connected Fatou component that is a wandering domain. Moreover, Baker showed [2] that this is not a special property of this example: if U is any multiply connected Fatou component of a transcendental entire function f, then U is wandering domain which called *Baker wandering domain*. It has the following properties: (1) each U_n is bounded and multiply connected; (2) there exists $N \in \mathbb{N}$ such that U_n and 0 lie in a bounded complementary component of U_{n+1} for $n \geq N$; (3) dis$(U_n, 0) \to \infty$ as $n \to \infty$.

For the remained case $F(P)$ has only simply connected Fatou component, a problem arise: is the hyper-order of entire solutions of equation (1.3) of infinite?

2. Some Lemmas

Lemma 2.1. (see [10]). *Let f be an entire function of infinite order and let $\nu_f(r)$ is the central index of $f(z)$, then the hyper-order*

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log \nu_f(r)}{\log r}. \tag{2.1}$$

Lemma 2.2. (see [9]). *Let f be an entire function of infinite order with $\sigma_2(f) = \alpha (0 \leq \alpha < \infty)$ and there exists a set $E \subset [1, \infty)$ have a finite logarithmic measure. Then there exists a sequence $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n, f), \theta_n \in [0, 2\pi), \lim_{n\to\infty} \theta_n = \theta_0 \in [0, 2\pi), r_n \notin E, r_n \to \infty$ and such that*

(1) *if $\sigma_2(f) = \alpha (0 < \alpha < \infty)$, then for any given $\varepsilon_1 (0 < \varepsilon_1 < \alpha)$,*

$$\exp\{r_n^{\alpha-\varepsilon_1}\} < \nu(r_n) < \exp\{r_n^{\alpha+\varepsilon_1}\}; \tag{2.2}$$

(2) *if $\sigma_2(f) = 0$, then for any given $\varepsilon_2 (0 < \varepsilon_2 < \frac{1}{2})$ and for any large $M_1 (> 0)$,*

$$r_n^{M_1} < \nu(r_n) < \exp\{r_n^{\varepsilon_2}\}. \tag{2.3}$$

Lemma 2.3. ([4]). *Let* $w(z)$ *be an entire function of order* $\rho(w) = \beta < 1/2$, $A(r) = \inf_{|z|=r} \log|w(z)|$ *and* $B(r) = \sup_{|z|=r} \log|w(z)|$. *If* $\beta < \alpha < 1$, *then*

$$\underline{\log \text{dens}}\{r : A(r) > \cos(\pi\alpha)B(r)\} \geq 1 - \frac{\beta}{\alpha}.$$

The following lemma which is due to Zheng is crucial to the proof of the main result. We set $M_c(r,a,f) = \max\{|f(z)| : |z-a| = r|\}$, $m_c(r,a,f) = \min\{|f(z)| : |z-a| = r\}$. When $a = 0$, we simply write $M(r,f)$ for $M_c(r,0,f)$.

Lemma 2.4. ([19, Corollary 1]). *Let* $f(z)$ *be a transcendental meromorphic function with at most finitely many poles. If* $J(f)$ *has only bounded components, then for any complex number* a, *there exists a constant* $0 < d < 1$ *and two sequences* $\{r_n\}$ *and* $\{R_n\}$ *of positive numbers with* $r_n \to \infty$ *and* $R_n/r_n \to \infty (n \to \infty)$ *such that*

$$M_c(r,a,f)^d \leq m_c(r,a,f), \quad r \in G, \tag{2.4}$$

where $G = \bigcup_{n=1}^{\infty}\{r : r_n < r < R_n\}$.

Particularly, we have $M(r,f)^d \leq m(r,f),\ r \in G$. It is obvious that the set G has infinite logarithmic measure.

3. Proof of Theorem

Proof of theorem 1.3. Since $P(z)$ is a transcendental entire function, by equation (1.3), we have

$$e^{P(z)} = \frac{f^{(k)}}{f} - \frac{Q(z)}{f} \tag{3.1}$$

Thus f must be transcendental entire function and of infinite order by observing the growth properties of both sides of (3.1).

By the Wiman-Valiron Theory (see e.g. [14, Page 51]), there exists a set $E_1 \subset [1,+\infty]$ of finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E$ and $|f(z)| = M(r,f)$, we have

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^k (1 + o(1)). \tag{3.2}$$

Substituting (3.1) into (3.2), we have

$$e^{P(z)} = \left(\frac{\nu(r,f)}{z}\right)^k (1 + o(1)) + o(1). \tag{3.3}$$

By the hypothesis we know that $F(P)$, the Fatou set of $P(z)$, has a multiply connected component. It follows from a result of Baker [3] that this multiply connected component must be a Baker wandering domain. By the properties of Baker wandering domain mentioned above, every component of $J(P)$, the Julia set of $P(z)$, is bounded. Thus, applying Lemma 2.4 to $P(z)$, there exists a set $G \subset (1, +\infty)$ of infinite logarithmic measure such that for all $|z| = r \in G$, we have $|P(z)| \geq m(r, P(z)) \geq M(r, P(z))^d$, where $0 < d < 1$ is a constant. Assume that the hyper-order of f, denoted by $\sigma_2(f)$, is finite. Then by Lemma 2.2, we have $\nu(r, f) \geq |z|^M$ for any positive constant M. Taking a principal branch of $\log((\nu(r, f)/z)^k (1 + o(1)) + o(1))$ and by (3.3) we have

$$P(z) = \log\left(\left(\frac{\nu(r, f)}{z}\right)^k (1 + o(1)) + o(1)\right). \tag{3.4}$$

Thus, we deduce

$$\begin{aligned} M(r, P)^d \leq |P(z)| &\leq \left|\log\left|\left(\frac{\nu(r, f)}{z}\right)^k (1 + o(1)) + o(1)\right|\right| \\ &\leq k \log \nu(r, f) + O(1) \leq k r^{\sigma_2(f)+\varepsilon} + O(1) \end{aligned} \tag{3.5}$$

for all z with $|z| = r \in G \setminus ([0, 1] \cup E_1)$ and for any given $\varepsilon > 0$. Since $P(z)$ is transcendental entire function, we have that

$$\frac{M(r.P)^d}{r^{\sigma_2(f)+\varepsilon}} \to \infty \tag{3.6}$$

where $0 < d < 1$ is a constant. Obviously, it is contradict to (3.5). Then, we complete the proof.

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