TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 4, pp. 1293-1328, August 2014
DOI: 10.11650/tjm.18.2014.3897
This paper is available online at http://journal.taiwanmathsoc.org.tw

SOME ESTIMATES FOR SCHRÖDINGER TYPE OPERATORS ON MUSIELAK-ORLICZ-HARDY SPACES

Sibei Yang

Abstract. Let $L := -\mathrm{div}(A\nabla) + V$ be a Schrödinger type operator with the nonnegative potential V belonging to the reverse Hölder class $RH_{q_0}(\mathbb{R}^n)$ for some $q_0 \in [n, \infty)$ with $n \geq 3$, where A satisfies the uniformly elliptic condition. Assume that $\varphi: \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is a function such that $\varphi(x, \cdot)$ is an Orlicz function, $\varphi(\cdot, t) \in \mathbb{A}_\infty(\mathbb{R}^n)$ (the class of uniformly Muckenhoupt weights) and its uniformly critical lower type index $i(\varphi) \in (\frac{n}{n+\alpha_0}, 1]$, where $\alpha_0 \in (0, 1]$ measures the regularity of kernels of the semigroup generalized by $L_0 := -\mathrm{div}(A\nabla)$. In this article, we first prove that operators VL^{-1}, $V^{1/2}\nabla L^{-1}$ and $\nabla^2 L^{-1}$ are bounded from the Musielak-Orlicz-Hardy space associated with L, $H_{\varphi, L}(\mathbb{R}^n)$, to the Musielak-Orlicz space $L^\varphi(\mathbb{R}^n)$. Moreover, we also obtain the boundedness of VL^{-1} and $\nabla^2 L^{-1}$ on $H_{\varphi, L}(\mathbb{R}^n)$. All these results are new even when $\varphi(x, t) := t^p$, with $p \in (\frac{n}{n+\alpha_0}, 1]$, for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$.

1. INTRODUCTION

Let $L := -\Delta + V$ be the Schrödinger operator on the Euclidean space $\mathbb{R}^n$ with $n \geq 3$. When V is a nonnegative polynomial on $\mathbb{R}^n$, the boundedness of $\nabla L^{-1/2}$, $L^{-1/2}\nabla$, $\nabla L^{-1}\nabla$, $V^{1/2}\nabla L^{-1}$ and $\nabla^2 L^{-1}$ on $L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$ was studied in many articles (see, for example, [38, 46]). In particular, Zhong [46] proved that, in this case, $\nabla L^{-1/2}$, $\nabla^2 L^{-1}$ and $\nabla L^{-1}\nabla$ are classical Calderón-Zygmund operators.

Moreover, Shen [37] generalized these results by extending the nonnegative polynomial V to the case that V is nonnegative and belongs to the *reverse Hölder class* $RH_q(\mathbb{R}^n)$ with some $q \in [n/2, \infty]$. Recall that it is said that $f \in RH_q(\mathbb{R}^n)$ with $q \in (1, \infty]$, if, when $q \in (1, \infty)$, $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ and there exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$,

$$\left\{\frac{1}{|B|}\int_B |f(y)|^q\,dy\right\}^{1/q} \leq \frac{C}{|B|}\int_B |f(y)|\,dy, \tag{1.1}$$

Received October 13, 2013, accepted January 20, 2014.
Communicated by Der-Chen Chang.
2010 *Mathematics Subject Classification*: Primary 42B20; Secondary 42B30, 42B35, 42B25, 35J10, 42B37, 46E30.
Key words and phrases: Musielak-Orlicz-Hardy space, Schrödinger type operator, Lusin area function, Atom, Second order Riesz transform, Fundamental solution.

or when $q = \infty$, $f \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and there exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$, $\operatorname{ess\,sup}_{y\in B} |f(y)| \le \frac{C}{|B|}\int_B |f(y)|\,dy$. We remark that $RH_p(\mathbb{R}^n) \subset RH_q(\mathbb{R}^n)$ for any $1 < q < p \le \infty$ and, if V is a nonnegative polynomial, then $V \in RH_\infty(\mathbb{R}^n)$ (see, for example, [16, 37]). Specifically, Shen [37] established the boundedness of $L^{i\gamma}$, $\nabla L^{-1/2}$, $\nabla^2 L^{-1}$ and $\nabla L^{-1}\nabla$ on some Lebesgue spaces $L^p(\mathbb{R}^n)$, where i denotes the unit imaginary number, $\gamma \in \mathbb{R}$ and the ranges of p may depend on n and q. Moreover, the boundedness of these operators implies immediately the Sobolev $W^{2,\,p}(\mathbb{R}^n)$ regularity for the solution u to the equation $-\Delta u + Vu = f$ when $f \in L^p(\mathbb{R}^n)$ with some $p \in (1, \infty)$. Furthermore, Shen [37] also established the boundedness of VL^{-1} on $L^p(\mathbb{R}^n)$, which, together with the boundedness of $\nabla^2 L^{-1}$ on $L^p(\mathbb{R}^n)$, further implies the following *maximal inequality* in $L^p(\mathbb{R}^n)$ (see also [1, 6]):

$$\|-\Delta f\|_{L^p(\mathbb{R}^n)} + \|Vf\|_{L^p(\mathbb{R}^n)} \le C\,\|(-\Delta + V)f\|_{L^p(\mathbb{R}^n)}\,, \tag{1.2}$$

where $f \in C_c^\infty(\mathbb{R}^n)$ and C is a positive constant independent of f. In particular, when $V \in RH_n(\mathbb{R}^n)$, Shen [37, Theorem 0.8] proved that $L^{i\gamma}$, $\nabla L^{-1/2}$ and $\nabla L^{-1}\nabla$ are classical Calderón-Zygmund operators. Moreover, the weighted $L^p(\mathbb{R}^n)$-boundedness of these operators was studied in [42].

Recently, the boundedness of $\nabla^2 L^{-1}$ and VL^{-1} on the Musielak-Orlicz-Hardy space $H_{\varphi,\,L}(\mathbb{R}^n)$, associated with L, was studied in [10]. Recall that the Musielak-Orlicz-Hardy space is a function space of Hardy-type which unify the classical Hardy space, the weighted Hardy space, the Orlicz-Hardy space and the weighted Orlicz-Hardy space, in which the spatial and the time variables may not be separable (see [11, 17, 18, 26, 39, 40, 41, 44] for more details on the developments of Hardy-type spaces and Musielak-Orlicz spaces). We also remark that the Musielak-Orlicz-Hardy space appears naturally in many applications (see, for example, [7, 8, 9, 32]).

We point out that this kind of Musielak-Orlicz-Hardy spaces associated with operators generalizes the (Orlicz-)Hardy space and the (weighted) Hardy space associated with operators, which has attracted great interests in recent years. Such function spaces associated with operators play important roles in the study for the boundedness of singular integrals which may not fall within the scope of the classical Calderón-Zygmund theory (see, for example, [2, 3, 12, 14, 15, 23, 24, 25, 28, 27, 29, 30, 43]).

From now on, let

$$L := -\mathrm{div}(A\nabla) + V \tag{1.3}$$

with some nonnegative potential V on $\mathbb{R}^n$ with $n \ge 3$, where the coefficients matrix $A := \{a_{ij}\}_{1\le i,j\le n}$ satisfies the following assumptions:

$(\mathbf{A_1})$ For any $i, j \in \{1, \ldots, n\}$, a_{ij} is a measurable function on $\mathbb{R}^n$. Moreover, there exists a constant $\lambda \in (0, 1]$ such that, for all $i, j \in \{1, \ldots, n\}$ and $x, \xi \in \mathbb{R}^n$,

$$a_{ij}(x) = a_{ji}(x) \text{ and } \lambda|\xi|^2 \le \sum_{i,\,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2.$$

$(\mathbf{A_2})$ There exist constants $\alpha \in (0,1]$ and $K \in (0,\infty)$ such that, for all $i,\, j \in \{1,\, \ldots,\, n\}$,

$$\|a_{ij}\|_{C^\alpha(\mathbb{R}^n)} \le K,$$

where, for $f \in C^\alpha(\mathbb{R}^n)$, $\|f\|_{C^\alpha(\mathbb{R}^n)} := \sup_{x,\, y\in\mathbb{R}^n,\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$.

$(\mathbf{A_3})$ There exists a constant $\alpha \in (0,1]$ such that, for all $i,\, j \in \{1,\, \ldots,\, n\}$, $x \in \mathbb{R}^n$ and $z \in \mathbb{Z}^n$,

$$a_{ij} \in C^{1+\alpha}(\mathbb{R}^n),\ a_{ij}(x+z) = a_{ij}(x) \text{ and } \sum_{k=1}^n \frac{\partial a_{kj}(x)}{\partial x_k} = 0.$$

Let L be as in (1.3). Kurata and Sugano [31] studied the boundedness of VL^{-1}, $V^{1/2}\nabla L^{-1}$ and $\nabla^2 L^{-1}$ on weighted Lebesgue spaces and Morrey spaces. Denote by $A_q(\mathbb{R}^n)$ with $q \in [1,\infty]$ the class of Muckenhoupt weights (see, for example, [18, 19, 21] for their definitions and properties). Specifically, it was proved in [31] that, when A in (1.3) satisfies the assumption (A_1) and $V \in RH_\infty(\mathbb{R}^n)$, VL^{-1} is bounded on the weighted space $L^p_w(\mathbb{R}^n)$, with $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, and the Morrey space $M^q_p(\mathbb{R}^n)$ with $1 < p < q < \infty$; when A satisfies the assumptions (A_1) and (A_2) and $V \in RH_\infty(\mathbb{R}^n)$, $V^{1/2}\nabla L^{-1}$ is bounded on $L^p_w(\mathbb{R}^n)$, with $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, and $M^q_p(\mathbb{R}^n)$ with $1 < p < q < \infty$; when A satisfies the assumptions (A_1), (A_2) and (A_3) and $V \in RH_\infty(\mathbb{R}^n)$, $\nabla^2 L^{-1}$ is bounded on $L^p_w(\mathbb{R}^n)$, with $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, and $M^q_p(\mathbb{R}^n)$ with $1 < p < q < \infty$.

Motivated by [10, 31], in this article, we establish the boundedness of the operators VL^{-1}, $V^{1/2}\nabla L^{-1}$ and $\nabla^2 L^{-1}$ on the Musielak-Orlicz-Hardy space $H_{\varphi,\, L}(\mathbb{R}^n)$, associated with L, where L is as in (1.3).

In order to state the main results of this article, let us first recall some notation and definitions. Assume that the nonnegative function V on $\mathbb{R}^n$ belongs to the reserve Hölder class $RH_{q_0}(\mathbb{R}^n)$ for some $q_0 \in [n/2,\infty)$ with $n \ge 3$. Denote by $W^{1,\,2}(\mathbb{R}^n)$ the usual *Sobolev space on* $\mathbb{R}^n$ equipped with the norm $(\|f\|^2_{L^2(\mathbb{R}^n)} + \|\nabla f\|^2_{L^2(\mathbb{R}^n)})^{1/2}$, where ∇f denotes the *distributional gradient* of f. Let $V \in RH_{q_0}(\mathbb{R}^n)$ and

$$W^{1,\,2}_V(\mathbb{R}^n) := \left\{u \in W^{1,\,2}(\mathbb{R}^n) :\ \int_{\mathbb{R}^n} |u(x)|^2 V(x)\,dx < \infty\right\}.$$

Denote by L the *maximal-accretive operator* (see [35, p. 23, Definition 1.46] for the definition) on $L^2(\mathbb{R}^n)$ with largest domain $D(L) \subset W^{1,\,2}_V(\mathbb{R}^n)$ such that, for any $f \in D(L)$ and $g \in W^{1,\,2}_V(\mathbb{R}^n)$,

$$\langle Lf, g\rangle := \int_{\mathbb{R}^n} A(x)\nabla f(x)\cdot \overline{\nabla g(x)}\,dx + \int_{\mathbb{R}^n} f(x)\overline{g(x)}V(x)\,dx,$$

where $\langle\cdot,\cdot\rangle$ denotes the interior product in $L^2(\mathbb{R}^n)$ and A satisfies the assumption (A_1). In this sense, for all $f \in D(L)$, we write

$$Lf := -\mathrm{div}(A\nabla)f + Vf. \tag{1.4}$$

Now we recall some notions for function spaces of Musielak-Orlicz type. We first describe the growth function considered in this article. Recall that a function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0,\infty)$ and $\lim_{t\to\infty} \Phi(t) = \infty$ (see, for example, [33, 36]). We point out that, different from the classical definition of Orlicz functions, the *Orlicz functions in this article may not be convex*. Moreover, Φ is said to be of *upper type* p (resp. *lower type* p) for some $p \in [0,\infty)$, if there exists a positive constant C such that, for all $s \in [1,\infty)$ (resp. $s \in [0,1]$) and $t \in [0,\infty)$, $\Phi(st) \le Cs^p\Phi(t)$.

For a given function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ such that, for any $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function, φ is said to be of *uniformly upper type* p (resp. *uniformly lower type* p) for some $p \in (0,\infty)$ if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$, $t \in [0,\infty)$ and $s \in [1,\infty)$ (resp. $s \in [0,1]$), $\varphi(x,st) \le Cs^p\varphi(x,t)$. Let

$$i(\varphi) := \sup\{p \in (0,\infty) : \ \varphi \text{ is of uniformly lower type } p\}. \tag{1.5}$$

Observe that $i(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower type $i(\varphi)$; see below for some examples.

Definition 1.1. Let $\varphi : \mathbb{R}^n\times[0,\infty) \to [0,\infty)$ satisfy that $x \mapsto \varphi(x,t)$ is measurable for all $t \in [0,\infty)$. The function φ is said to satisfy the *uniformly Muckenhoupt condition for some* $q \in [1,\infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, if, when $q \in (1,\infty)$,

$$\mathbb{A}_q(\varphi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^q} \int_B \varphi(x,t)\,dx \left\{\int_B [\varphi(y,t)]^{1-q}\,dy\right\}^{q-1} < \infty,$$

or, when $q = 1$,

$$\mathbb{A}_1(\varphi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x,t)\,dx \left(\operatorname*{ess\,sup}_{y\in B}\,[\varphi(y,t)]^{-1}\right) < \infty.$$

Here the first suprema are taken over all $t \in (0,\infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$.

The function φ is said to satisfy the *uniformly reverse Hölder condition for some* $q \in (1,\infty]$, denoted by $\varphi \in \mathbb{RH}_q(\mathbb{R}^n)$, if, when $q \in (1,\infty)$,

$$\mathbb{RH}_q(\varphi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \left\{\frac{1}{|B|} \int_B [\varphi(x,t)]^q\,dx\right\}^{1/q} \left\{\frac{1}{|B|} \int_B \varphi(x,t)\,dx\right\}^{-1} < \infty,$$

or, when $q = \infty$,

$$\mathbb{RH}_\infty(\varphi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \left\{ \operatorname*{ess\,sup}_{y\in B} \varphi(y,t) \right\} \left\{ \frac{1}{|B|} \int_B \varphi(x,t)\, dx \right\}^{-1} < \infty.$$

Here the first suprema are taken over all $t \in (0,\infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$.

Recall that, in Definition 1.1, $\mathbb{A}_p(\mathbb{R}^n)$, with $p \in [1,\infty)$, and $\mathbb{RH}_q(\mathbb{R}^n)$, with $q \in (1,\infty]$, were respectively introduced by Ky [32] and D. Yang and S. Yang [45].

Let $\mathbb{A}_\infty(\mathbb{R}^n) := \cup_{q\in[1,\infty)}\mathbb{A}_q(\mathbb{R}^n)$. The *critical indices* of $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ are defined as follows:

$$q(\varphi) := \inf \{q \in [1,\infty) : \ \varphi \in \mathbb{A}_q(\mathbb{R}^n)\} \tag{1.6}$$

and

$$r(\varphi) := \sup \{q \in (1,\infty] : \ \varphi \in \mathbb{RH}_q(\mathbb{R}^n)\}. \tag{1.7}$$

Now we recall the notion of growth functions from Ky [32].

Definition 1.2. A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if the following hold:

(i) φ is a *Musielak-Orlicz function*, namely,
   (a) $\varphi(x,\cdot) : \ [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$;
   (b) $\varphi(\cdot,t)$ is a measurable function for all $t \in [0,\infty)$.

(ii) $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$.

(iii) The function φ is of uniformly lower type p for some $p\in(0,1]$ and upper type 1.

Clearly, $\varphi(x,t) := \omega(x)\Phi(t)$ is a growth function if $\omega \in A_\infty(\mathbb{R}^n)$ and Φ is an Orlicz function of lower type p for some $p \in (0,1]$ and upper type 1. A typical example of such Orlicz function Φ is $\Phi(t) := t^p$, with $p \in (0,1]$, for all $t \in [0,\infty)$ (see, for example, [44, 45] for more examples of such Φ). Another typical example of growth function is $\varphi(x,t) := \frac{t^\alpha}{[\ln(e+|x|)]^\beta + [\ln(e+t)]^\gamma}$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$ with any $\alpha \in (0,1]$ and $\beta, \gamma \in [0,\infty)$; more precisely, $\varphi \in \mathbb{A}_1(\mathbb{R}^n)$, φ is of uniformly upper type α, and $i(\varphi) = \alpha$ which is not attainable (see [32]).

Recall that, for a Musielak-Orlicz function φ as in Definition 1.2, a measurable function f on $\mathbb{R}^n$ is said to be in the *Musielak-Orlicz space* $L^\varphi(\mathbb{R}^n)$ if

$$\int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx < \infty.$$

Moreover, for any $f \in L^\varphi(\mathbb{R}^n)$, define

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0,\infty) : \ \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}.$$

Let L and φ be, respectively, as in (1.4) and Definition 1.2. We remark that, by A in (1.4) satisfying the assumption (A_1) and V nonnegative, we know that L is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$. Moreover, the Gaussian upper bound estimate for the kernels of the semigroup $\{e^{-tL}\}_{t>0}$ (see Lemma 2.6(i) below) further implies that the semigroup $\{e^{-tL}\}_{t>0}$ satisfies the Davies-Gaffney estimates (see [45, Assumption (B)] for the definition of the Davies-Gaffney estimate). Thus, L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates. Now we recall the Musielak-Orlicz-Hardy space $H_{\varphi,L}(\mathbb{R}^n)$ associated with L introduced in [45].

For $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *Lusin area function* $S_L(f)(x)$, associated with L, is defined by

$$S_L(f)(x) := \left\{ \int_{\Gamma(x)} \left| t^2 L e^{-t^2 L}(f)(y) \right|^2 \frac{dy\, dt}{t^{n+1}} \right\}^{1/2},$$

where $\Gamma(x)$ is the *cone* defined by $\Gamma(x) := \{(y,t) \in \mathbb{R}^n \times (0,\infty) : \ |y-x| < t\}$. A function $f \in L^2(\mathbb{R}^n)$ is said to be in the set $\widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ if $S_L(f) \in L^{\varphi}(\mathbb{R}^n)$; moreover, define

$$\begin{aligned}\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} &:= \|S_L(f)\|_{L^{\varphi}(\mathbb{R}^n)} \\ &= \inf\left\{\lambda \in (0,\infty) : \ \int_{\mathbb{R}^n} \varphi\left(x, \frac{S_L(f)(x)}{\lambda}\right) dx \le 1\right\}.\end{aligned}$$

The *Musielak-Orlicz-Hardy space* $H_{\varphi,L}(\mathbb{R}^n)$ is defined to be the completion of $\widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ respect with to the *quasi-norm* $\|\cdot\|_{H_{\varphi,L}(\mathbb{R}^n)}$.

Moreover, in order to state the main results of this article, we need another necessary notation. By [4], we know that the following conclusion holds true.

Lemma 1.3. *Let* $L_0 := -\mathrm{div}(A\nabla)$ *with* A *satisfying the assumption* (A_1) *and* p_t *be the kernel of the heat semigroup* e^{-tL_0} *generated by* L_0*. Then, for each* $y \in \mathbb{R}^n$ *and* $t \in (0,\infty)$*,* $x \mapsto p_t(x,y)$ *and* $x \mapsto p_t(y,x)$ *are Hölder continuous functions in* $\mathbb{R}^n$ *and there exists* $\alpha_0 \in (0,1]$ *such that, for any* $\alpha \in (0,\alpha_0)$*, there exist positive constants* $C_{(\alpha)}$ *and* c_0 *such that, for any* $x,\ x+h,\ y \in \mathbb{R}^n$ *satisfying* $|h| \le \sqrt{t}$*,*

$$|p_t(x+h,y) - p_t(x,y)| + |p_t(y,x+h) - p_t(y,x)| \le \frac{C_{(\alpha)}}{t^{n/2}} \left[\frac{|h|}{\sqrt{t}}\right]^{\alpha} e^{-\frac{c_0|x-y|^2}{t}}.$$

The first main result of this article is as follows.

Theorem 1.4. *Let* L *and* φ *be, respectively, as in* (1.4) *and Definition 1.2. Assume that* $V \in RH_{q_0}(\mathbb{R}^n)$ *with* $q_0 \in [n,\infty)$*,* $i(\varphi) \in (\frac{n}{n+\alpha_0}, 1]$ *and*

$$[r(\varphi)]' < \frac{n}{nq(\varphi)/i(\varphi) - \alpha_0}, \tag{1.8}$$

where $i(\varphi)$, $q(\varphi)$, $r(\varphi)$ *and* α_0 *be, respectively, as in* (1.5), (1.6), (1.7) *and Lemma 1.3, and* $[r(\varphi)]' := r(\varphi)/[r(\varphi)-1]$.

(*i*) *If* A *in* (1.4) *satisfies the assumptions* (A_1) *and* (A_2), *then the operator* VL^{-1} *is bounded from* $H_{\varphi,L}(\mathbb{R}^n)$ *to* $L^{\varphi}(\mathbb{R}^n)$.

(*ii*) *If* A *in* (1.4) *satisfies the assumptions* (A_1) *and* (A_2), *then the operator* $V^{1/2}\nabla L^{-1}$ *is bounded from* $H_{\varphi,L}(\mathbb{R}^n)$ *to* $L^{\varphi}(\mathbb{R}^n)$.

(*iii*) *If* A *satisfies the assumptions* (A_1), (A_2) *and* (A_3), *then the operator* $\nabla^2 L^{-1}$ *is bounded from* $H_{\varphi,L}(\mathbb{R}^n)$ *to* $L^{\varphi}(\mathbb{R}^n)$.

To prove Theorem 1.4, we first establish an atomic characterization of the Musielak-Orlicz-Hardy space $H_{\varphi,L}(\mathbb{R}^n)$ (see Theorem 2.3 below). It is worth pointing out that the atom used in this article was introduced in [10], which is different from that in [45], but closer to that in [14, 15] in the spirit; the method used in this article to establish the atomic decomposition for the space $H_{\varphi,L}(\mathbb{R}^n)$ is similar to that used in [10, Theorem 2.3], but quite different from that used in [14, 15] (see the introduction of [10] for more details). Moreover, some upper bound estimates for the fundamental solution of L (see Lemma 3.1 below) and the boundedness of VL^{-1}, $V^{1/2}\nabla L^{-1}$ and $\nabla^2 L^{-1}$ on the Lebesgue space $L^p(\mathbb{R}^n)$ (see Lemmas 3.2 and 3.3 below) are also used in the proof of Theorem 1.4. Furthermore, we also need some properties for the growth function φ (see Lemma 2.7 below) and the auxiliary function $m(\cdot,V)$ determined by the potential V (see (2.1) and Lemmas 2.5 and 3.4 below). In particular, it is worth pointing out that the technique of smooth cut-off functions (see (3.33) below) and the special structure of L also play a key role in the proof of Theorem 1.4(iii). Moreover, different from the proof of [10, Theorem 1.4], the main new ingredient appeared in the proofs for (i) and (ii) of Theorem 1.4 is that we use the different parting ring technique of the entire $\mathbb{R}^n$ for the different $H_{\varphi,L}(\mathbb{R}^n)$-atom (see (3.8), (3.13) and (3.18) below).

Now we recall the definition of the Musielak-Orlicz-Hardy space $H_{\varphi}(\mathbb{R}^n)$ introduced in [32]. We first state some notions. In what follows, we denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *dual space* (namely, the *space of all tempered distributions*). Let $\mathbb{N} := \{1,\ldots\}$ and $\mathbb{Z}_+ := \{0\}\cup\mathbb{N}$. For any $\theta := (\theta_1,\ldots,\theta_n) \in \mathbb{Z}_+^n$, let $|\theta| := \theta_1+\cdots+\theta_n$ and $\partial_x^\theta := \frac{\partial^{|\theta|}}{\partial x_1^{\theta_1}\cdots\partial x_n^{\theta_n}}$. For $m\in\mathbb{N}$, define

$$\mathcal{S}_m(\mathbb{R}^n) := \left\{\phi\in\mathcal{S}(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n}\ \sup_{\beta\in\mathbb{Z}_+^n,\,|\beta|\le m+1} (1+|x|)^{(m+2)(n+1)}|\partial_x^\beta\phi(x)| \le 1\right\}.$$

Then, for all $f\in\mathcal{S}'(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, the *non-tangential grand maximal function* f_m^* of f is defined by setting,

$$f_m^*(x) := \sup_{\phi\in\mathcal{S}_m(\mathbb{R}^n)}\ \sup_{|y-x|<t,\,t\in(0,\infty)} |f*\phi_t(y)|,$$

where, for all $t \in (0,\infty)$, $\phi_t(\cdot) := t^{-n}\phi(\frac{\cdot}{t})$. When $m(\varphi) := \lfloor n[q(\varphi)/i(\varphi) - 1]\rfloor$, where $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (1.6) and (1.5), and $\lfloor s\rfloor$ for $s \in \mathbb{R}$ denotes the *maximal integer not more than* s, we *denote* $f^*_{m(\varphi)}$ *simply by* f^*.

Definition 1.5. Let φ be as in Definition 1.2. The *Musielak-Orlicz-Hardy space* $H_\varphi(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^* \in L^\varphi(\mathbb{R}^n)$ with the *quasi-norm* $\|f\|_{H_\varphi(\mathbb{R}^n)} := \|f^*\|_{L^\varphi(\mathbb{R}^n)}$.

Now we state the second main result of this article as follows.

Theorem 1.6. *Let L and φ be, respectively, as in* (1.4) *and Definition 1.2. Assume that $i(\varphi)$, $q(\varphi)$, $r(\varphi)$ and α_0 are, respectively, as in* (1.5), (1.6), (1.7) *and Lemma 1.3. Let A in* (1.4) *satisfy the assumptions (A_1), (A_2) and (A_3), $V \in RH_{q_0}(\mathbb{R}^n)$ with $q_0 \in [n,\infty)$, $i(\varphi) \in (\frac{n}{n+\alpha_0}, 1]$ and*

$$q(\varphi)[r(\varphi)]' < \frac{n}{nq(\varphi)/i(\varphi) - \alpha_0}. \tag{1.9}$$

Then

- (i) *the operator $\nabla^2 L^{-1}$ is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to $H_\varphi(\mathbb{R}^n)$;*
- (ii) *the operator VL^{-1} is bounded on $H_{\varphi,L}(\mathbb{R}^n)$.*

Similar to the proof of [10, Theorem 1.4], we prove Theorem 1.6(i) by using the atomic characterization of $H_{\varphi,L}(\mathbb{R}^n)$ established in Theorem 2.3 below, some estimates for the fundamental solution of L (see Lemma 3.1 below), the boundedness of VL^{-1} and $\nabla^2 L^{-1}$ on $L^p(\mathbb{R}^n)$ (see Lemmas 3.2 and 3.3 below), and some properties for φ (see Lemma 2.7 below) and the auxiliary function $m(\cdot, V)$ (see Lemmas 2.5 and 3.4 below). Moreover, similar to [10, Theorem 1.5], we prove (ii) of Theorem 1.6 via (i) of this theorem and the atomic characterization of $H_{\varphi,L}(\mathbb{R}^n)$ obtained in Theorem 2.3 below.

Moreover, we also have the following two remarks for Theorems 1.4 and 1.6.

Remark 1.7. Let L and φ be as in Theorem 1.4.

- (i) By Remark 2.4(iii) below, we know that $H_\varphi(\mathbb{R}^n) \subset H_{\varphi,L}(\mathbb{R}^n)$. Thus, the operators VL^{-1}, $V^{1/2}\nabla L^{-1}$ and $\nabla^2 L^{-1}$ are also bounded from the space $H_\varphi(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$.
- (ii) When $\varphi(x,t) := t^p$, with $p \in (\frac{n}{n+\alpha_0}, 1]$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, it is easy to see that (1.8) holds true. In this case, denote by $H^p_L(\mathbb{R}^n)$ the space $H_{\varphi,L}(\mathbb{R}^n)$. Moreover, it is easy to see that $L^\varphi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $H_\varphi(\mathbb{R}^n) = H^p(\mathbb{R}^n)$, where $H^p(\mathbb{R}^n)$ denotes the classical Hardy space. Thus, VL^{-1}, $V^{1/2}\nabla L^{-1}$ and $\nabla^2 L^{-1}$ are bounded from $H^p_L(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, and from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ by (i) of this remark.

(iii) Let $L_I := -\Delta + V$ be the Schrödeger operator on $\mathbb{R}^n$. Recall that, when $L = L_I$, namely, A in (1.4) is just the unit matrix, then $\alpha_0 = 1$. Let $\varphi(x,t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, where $p \in (\frac{n}{n+1}, 1]$. In this case, we denote the space $H_{\varphi, L_I}(\mathbb{R}^n)$ simply by $H^p_{L_I}(\mathbb{R}^n)$. It is worth pointing out that the space $H^p_{L_I}(\mathbb{R}^n)$ has been studied in [14, 15, 28, 43]. Observe that, in this case, by the definitions of $i(\varphi)$, $q(\varphi)$ and $r(\varphi)$, we easily see that $i(\varphi) = p$, $q(\varphi) = 1$ and $r(\varphi) = \infty$ and hence the condition (1.8) automatically holds true. We point out that Theorem 1.4 is new even for $H^p_{L_I}(\mathbb{R}^n)$ with $p \in (\frac{n}{n+1}, 1]$.

(iv) For $\omega \in A_\infty(\mathbb{R}^n)$, we denote by q_ω and r_ω, respectively, the critical indexes of ω defined by a way similar to (1.6) and (1.7). When $\varphi(x,t) := \omega(x)t^p$, with $p \in (\frac{n}{n+\alpha_0}, 1]$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, by the definitions of $i(\varphi)$, $q(\varphi)$ and $r(\varphi)$, we see that $i(\varphi) = p$, $q(\varphi) = q_\omega$ and $r(\varphi) = r_\omega$, and hence the condition (1.8) becomes $r'_\omega < \frac{n}{nq_\omega/p - \alpha_0}$. Theorem 1.4 is also new even in this case.

(v) We also give some examples of growth functions satisfying the assumptions in Theorem 1.6.

(v)$_1$ Assume that $p \in (\frac{n}{n+\alpha_0}, 1]$ and $a \in (\frac{n}{p} - (n+\alpha_0), 0]$. Let $\varphi(x,t) := |x|^a t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$. It is easy to show that $i(\varphi) = p$, $q(\varphi) = 1$ and $r(\varphi) > \frac{n}{n+\alpha_0 - n/p}$. From this, we deduce that the assumption (1.8) holds true. Thus, such φ satisfies the assumptions of Theorem 1.4.

(v)$_2$ Assume that $p \in (\frac{n}{n+\alpha_0}, 1]$, $q \in (1, \frac{(n+\alpha_0)p}{n})$ and $a \in (0, (q-1)n)$. Let $\varphi(x,t) := |x|^a t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$. In this case, it is easy to see that $i(\varphi) = p$, $q(\varphi) < q$ and $r(\varphi) = \infty$. By this and the assumption for q, we see that (1.8) holds true and hence such φ satisfies the assumptions of Theorem 1.4.

(v)$_3$ Let $\varphi(x,t) := \frac{t^\alpha}{[\log(e+|x|)]^\beta + [\log(e+t)]^\gamma}$, with $\alpha \in (\frac{n}{n+\alpha_0}, 1]$ and $\beta,\ \gamma \in (0,\infty)$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$. In this case, it is easy to prove that φ satisfies Definition 1.2, $i(\varphi) = \alpha$, $q(\varphi) = 1$ and $r(\varphi) = \infty$. From this and $\alpha \in (\frac{n}{n+\alpha_0}, 1]$, it follows that the assumption (1.8) automatically holds true and hence such φ satisfies the assumptions of Theorem 1.6. Moreover, it is worth pointing out that such a function φ naturally appears in the study of the pointwise multiplier characterization for the BMO-type space (see [34, 32]).

Remark 1.8. Let L and φ be as in Theorem 1.6.

(i) By Remark 2.4(iii) below, we know that $H_\varphi(\mathbb{R}^n) \subset H_{\varphi, L}(\mathbb{R}^n)$, which, together with Theorem 1.6(i), implies that $\nabla^2 L^{-1}$ is also bounded on $H_{\varphi, L}(\mathbb{R}^n)$.

(ii) Let $\varphi(x,t) := t^p$, with $p \in (\frac{n}{n+\alpha_0}, 1]$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$. Then we know that $\nabla^2 L^{-1}$ is bounded from $H^p_L(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ and on $H^p(\mathbb{R}^n)$; VL^{-1} is bounded on $H^p_L(\mathbb{R}^n)$ and from $H^p(\mathbb{R}^n)$ to $H^p_L(\mathbb{R}^n)$. Theorem 1.6 is new even in this case.

(iii) Let $\varphi(x,t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, where $p \in (\frac{n}{n+\alpha_0}, 1]$. Similar to Remark 1.7(ii), we know that (1.9) automatically holds true. We point out that Theorem 1.6 is new even for $H^p_L(\mathbb{R}^n)$ with $p \in (\frac{n}{n+\alpha_0}, 1]$. Moreover, when $L := L_I$, where L_I is as in Remark 1.7(ii), (i) and (ii) of Theorem 1.6 are, respectively, [10, Theorems 1.4 and 1.5].

(iv) When $\varphi(x,t) := \omega(x)t^p$, with $\omega \in A_\infty(\mathbb{R}^n)$ and $p \in (\frac{n}{n+\alpha_0}, 1]$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$. Similar to Remark 1.7(iii), the condition (1.9) becomes $q_\omega r'_\omega < \frac{n}{nq_\omega/p-\alpha_0}$, where r_ω and q_ω are as in Remark 1.7(iii). Theorem 1.6 is also new even in this case.

(v) Similar to Remark 1.7(v) and [10, Remark 1(v)], we also have the following examples of growth functions satisfying the assumptions of Theorem 1.6.

$(\mathrm{v})_1$ Assume that $p \in (\frac{n}{n+\alpha_0}, 1]$ and $a \in (\frac{n}{p} - (n+\alpha_0), 0]$. Let $\varphi(x,t) := |x|^a t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$.

$(\mathrm{v})_2$ Assume that $p \in (\frac{n}{n+\alpha_0}, 1]$, $q \in (1, \frac{\alpha_0 p}{2n} + [\frac{\alpha_0^2 p^2}{(2n)^2} + p]^{1/2})$ and $a \in (0, (q-1)n)$. Let $\varphi(x,t) := |x|^a t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$.

$(\mathrm{v})_3$ Let $\varphi(x,t) := \frac{t^\alpha}{[\log(e+|x|)]^\beta + [\log(e+t)]^\gamma}$, with $\alpha \in (\frac{n}{n+\alpha_0}, 1]$ and $\beta,\ \gamma \in (0,\infty)$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$.

As a corollary of Theorem 1.6, we have the following maximal inequality.

Corollary 1.9. *Let φ, L and V be the same as in Theorem 1.6. Then there exists a positive constant C such that, for all $f \in C_c^\infty(\mathbb{R}^n)$,*

$$\|-\Delta f\|_{H_{\varphi,L}(\mathbb{R}^n)} + \|Vf\|_{H_{\varphi,L}(\mathbb{R}^n)} \le C\,\|Lf\|_{H_{\varphi,L}(\mathbb{R}^n)}\,. \tag{1.10}$$

Remark 1.10. We point out that (1.10) is a variant of the maximal inequality (1.2) in the space $H_{\varphi,L}(\mathbb{R}^n)$ when $L := L_I$ is the Schrödinger operator, and hence further completes (1.2). Indeed, when $\varphi(x,t) := t^p$, with $p \in (\frac{n}{n+\alpha_0}, 1]$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, (1.10) becomes that there exists a positive constant C such that, for all $f \in C_c^\infty(\mathbb{R}^n)$,

$$\|-\Delta f\|_{H^p_L(\mathbb{R}^n)} + \|Vf\|_{H^p_L(\mathbb{R}^n)} \le C\,\|Lf\|_{H^p_L(\mathbb{R}^n)}\,,$$

which is also new.

The layout of this article is as follows. In Section 2, we establish an atomic characterization of the space $H_{\varphi,L}(\mathbb{R}^n)$ (see Theorem 2.3 below), which completely covers [14, Theorem 1.11] by taking $\varphi(x,t) := t^p$, with $p \in (\frac{n}{n+\mu_0}, 1]$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, and $A := I$ in (1.4), where μ_0 is as in (2.2) below and I denotes the unit matrix. Then, in Sections 3 and 4, we give the proofs of Theorems 1.4 and 1.6, respectively.

Finally we make some conventions on notation. Throughout the whole article, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\gamma,\beta,\ldots)}$ to denote a *positive constant* depending on the indicated parameters γ, β, $\ldots$. The *symbol* $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For any given normed spaces $\mathcal{A}$ and $\mathcal{B}$ with the corresponding norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$, the *symbol* $\mathcal{A} \subset \mathcal{B}$ means that, for all $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. For any measurable subset E of $\mathbb{R}^n$, we denote by $E^{\complement}$ the *set* $\mathbb{R}^n \setminus E$ and by χ_E its *characteristic function*. Moreover, for each ball $B \subset \mathbb{R}^n$, let $S_0(B) := 2B$ and $S_j(B) := 2^{j+1}B \setminus 2^j B$ for $j \in \mathbb{N}$. Finally, for $q \in [1, \infty]$, we denote by q' the *conjugate exponent* of q, namely, $1/q + 1/q' = 1$.

2. An Atomic Characterization of $H_{\varphi,L}(\mathbb{R}^n)$

To prove the main results of this article, similar to [10], in this section, we establish an atomic characterization of $H_{\varphi,L}(\mathbb{R}^n)$. We begin with the definition of $(\varphi, q)_m$-atoms introduced in [10, Definition 2.1]. To this end, we need the following auxiliary function $m(\cdot, V)$ introduced by Shen [37]. More precisely, let V be as in (1.4). For all $x \in \mathbb{R}^n$, the *auxiliary function* $m(x, V)$ is defined by

$$[m(x,V)]^{-1} := \sup\left\{r \in (0,\infty) :\ \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \le 1\right\}. \tag{2.1}$$

Definition 2.1. Let φ and $m(\cdot, V)$ be, respectively, as in Definition 1.2 and (2.1), and $q \in (1, \infty]$. A function a on $\mathbb{R}^n$ is called a $(\varphi, q)_m$*-atom* associated with the ball $B := B(x_0, r_0)$, if

(i) $\operatorname{supp}(a) \subset B$;

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \le |B|^{1/q}\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)}$;

(iii) $\int_{\mathbb{R}^n} a(x)\,dx = 0$ if $r_0 < [m(x_0, V)]^{-1}$.

Now we recall the definition of the atomic Musielak-Orlicz-Hardy space $H^{\varphi,q}_m(\mathbb{R}^n)$ introduced in [10, Definition 2.2].

Definition 2.2. Let φ, $m(\cdot, V)$ and q be as in Definition 2.1. A function $f \in L^2(\mathbb{R}^n)$ is said to be in the set $\widetilde{H}^{\varphi,q}_m(\mathbb{R}^n)$ if $f = \sum_j \lambda_j a_j$ in $L^2(\mathbb{R}^n)$, where, for each j, a_j is a $(\varphi, q)_m$-atom associated with the ball B_j and $\{\lambda_j\}_j \subset \mathbb{C}$ satisfies that $\sum_j \varphi(B_j, \frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{\varphi}(\mathbb{R}^n)}}) < \infty$. Define

$$\Lambda(\{\lambda_j a_j\}_j) := \inf\left\{\lambda \in (0,\infty) :\ \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^{\varphi}(\mathbb{R}^n)}}\right) \le 1\right\}$$

and $\|f\|_{H^{\varphi,q}_m(\mathbb{R}^n)} := \inf \Lambda(\{\lambda_j a_j\}_j)$, where the infimum is taken over all decompositions of f as above. The *atomic Musielak-Orlicz-Hardy space* $H^{\varphi,q}_m(\mathbb{R}^n)$ is then defined to be the completion of $\widetilde{H}^{\varphi,q}_m(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{\varphi,q}_m(\mathbb{R}^n)}$.

Moreover, by [20], we know that $RH_q(\mathbb{R}^n)$ has the property of the self-improvement. Namely, if $V \in RH_q(\mathbb{R}^n)$ for some $q \in (1,\infty)$, then there exists $\epsilon \in (0,\infty)$, depending only on n and the constant C in (1.1), such that $V \in RH_{q+\epsilon}(\mathbb{R}^n)$. Thus, when $V \in RH_q(\mathbb{R}^n)$ for some $q \geq n/2$ with $n \geq 3$, there exists $q_0 > n/2$ such that $V \in RH_{q_0}(\mathbb{R}^n)$. For convenience, in what follows, we *always assume that L is as in* (1.4) *with $V \in RH_{q_0}(\mathbb{R}^n)$, $q_0 \in (n/2,\infty)$ and $n \geq 3$*; moreover, let

$$\mu_0 := \min\{\alpha_0,\, 2 - n/q_0\}\,, \tag{2.2}$$

where α_0 is as in Lemma 1.3.

Now we state the main result of this section as follows.

Theorem 2.3. *Let φ and L be, respectively, as in Definition 1.2 and* (1.4)*. Assume that $i(\varphi)$, $q(\varphi)$ and μ_0 are, respectively, as in* (1.5), (1.6) *and* (2.2)*. Let $q \in (1,\infty)$ satisfy $\mu_0 + n/q > \frac{nq(\varphi)}{i(\varphi)}$. Then the spaces $H_{\varphi,L}(\mathbb{R}^n)$ and $H_m^{\varphi,q}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Remark 2.4.

(i) Theorem 2.3 completely covers [14, Theorem 1.11] by taking $\varphi(x,t) := t^p$, with $p \in (\frac{n}{n+\mu_0}, 1]$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, and $A := I$ in (1.4), where I denotes the unit matrix. Moreover, when $A := I$, $\mu_0 = \min\{1, 2 - n/q_0\}$ and Theorem 2.3 is just [10, Theorem 2.3].

(ii) Let $L_I := -\Delta + V$, where V is as in (1.4). When $L = L_I$, we denote the Musielak-Orlicz-Hardy space $H_{\varphi,L}(\mathbb{R}^n)$ by $H_{\varphi,L_I}(\mathbb{R}^n)$. By Theorem 2.3 and [10, Theorem 2.3], we conclude that, when φ, μ_0 and q are as in Theorem 2.3, $H_{\varphi,L}(\mathbb{R}^n) = H_{\varphi,L_I}(\mathbb{R}^n)$ with equivalent quasi-norms.

(iii) Let φ be as in Theorem 2.3. Similar to [10, Remark 4(ii)], we see that $H_\varphi(\mathbb{R}^n) \subset H_{\varphi,L}(\mathbb{R}^n)$.

To prove Theorem 2.3, we also need some estimates related to L. For the auxiliary function $m(\cdot,V)$, we have the following Lemma 2.5, which is just [37, Lemma 1.4].

Lemma 2.5. *Let V and $m(\cdot,V)$ be, respectively, as in* (1.4) *and* (2.1)*. Then there exist positive constants C_1, C_2, C_3 and k_0 such that, for all x, $y \in \mathbb{R}^n$,*

$$C_2^{-1} m(x,V) \leq m(y,V) \leq C_2 m(x,V) \text{ if } |x-y| \leq C_1[m(x,V)]^{-1}, \tag{2.3}$$

$$m(y,V) \leq C_2[1 + |x-y|m(x,V)]^{k_0} m(x,V)$$

and

$$m(y,V) \geq \frac{C_3 m(x,V)}{[1 + |x-y|m(x,V)]^{k_0/(k_0+1)}}.$$

Moreover, we also need the following estimates for the kernel of e^{-tL}.

Lemma 2.6. *Let L be as in* (1.4) *and K_t the kernel of e^{-tL}. Assume that μ_0 is as in* (2.2).

(i) *For each $t \in (0,\infty)$, K_t is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ and, for any $N \in \mathbb{N}$, there exist positive constants $C_{(N)}$ and α such that, for almost every $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$,*

$$0 \le K_t(x,y) \le \frac{C_{(N)}}{t^{n/2}} e^{-\frac{\alpha|x-y|^2}{t}} \left\{1+\sqrt{t}m(x,V)+\sqrt{t}m(y,V)\right\}^{-N}.$$

Moreover, this estimate holds true also for $t\partial_t K_t$.

(ii) *For each $y \in \mathbb{R}^n$ and $t \in (0,\infty)$, $x \mapsto K_t(x,y)$ and $x \mapsto K_t(y,x)$ are Hölder continuous functions in $\mathbb{R}^n$ and, for any $N \in \mathbb{N}$ and $\mu \in (0,\mu_0)$, there exist positive constants $C_{(N,\mu)}$ and α such that, for any $x,\ x+h,\ y \in \mathbb{R}^n$ satisfying $|h| \le \sqrt{t}$,*

$$\begin{aligned}&|K_t(x+h,y)-K_t(x,y)|+|K_t(y,x+h)-K_t(y,x)|\\ &\le \frac{C_{(N,\mu)}}{t^{n/2}}\left[\frac{|h|}{\sqrt{t}}\right]^{\mu} e^{-\frac{\alpha|x-y|^2}{t}} \left\{1+\sqrt{t}m(x,V)+\sqrt{t}m(y,V)\right\}^{-N}.\end{aligned}$$

Proof. (i) of this lemma was obtained in [13, Theorem 2.1]. Moreover, similar to the proof of [14, Theorem 2.11] and [15, Proposition 4.11], we can prove (ii). We omit the details here. ■

Moreover, we also need some properties of φ in Definition 1.2. In what follows, for any measurable subset E of $\mathbb{R}^n$ and $t \in [0,\infty)$, let $\varphi(E,t) := \int_E \varphi(x,t)\,dx$. We have the following properties for $\mathbb{A}_\infty(\mathbb{R}^n)$, whose proofs are similar to those in [19, 21]; see also [22].

Lemma 2.7.

(i) $\mathbb{A}_1(\mathbb{R}^n) \subset \mathbb{A}_p(\mathbb{R}^n) \subset \mathbb{A}_q(\mathbb{R}^n)$ *for* $1 \le p \le q < \infty$.

(ii) $\mathbb{RH}_\infty(\mathbb{R}^n) \subset \mathbb{RH}_p(\mathbb{R}^n) \subset \mathbb{RH}_q(\mathbb{R}^n)$ *for* $1 < q \le p \le \infty$.

(iii) *If $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in (1,\infty)$, then there exists $q \in (1,p)$ such that $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$.*

(iv) $\mathbb{A}_\infty(\mathbb{R}^n) = \cup_{p\in[1,\infty)}\mathbb{A}_p(\mathbb{R}^n) = \cup_{q\in(1,\infty]}\mathbb{RH}_q(\mathbb{R}^n)$.

(v) *If $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in [1,\infty)$, then there exists a positive constant C such that, for all balls $B_1,\ B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$ and $t \in [0,\infty)$, $\frac{\varphi(B_2,t)}{\varphi(B_1,t)} \le C[\frac{|B_2|}{|B_1|}]^p$.*

(vi) *If $\varphi \in \mathbb{RH}_q(\mathbb{R}^n)$ with $q \in (1,\infty]$, then there exists a positive constant C such that, for all balls $B_1,\ B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$ and $t \in [0,\infty)$, $\frac{\varphi(B_2,t)}{\varphi(B_1,t)} \ge C[\frac{|B_2|}{|B_1|}]^{(q-1)/q}$.*

Remark 2.8. Denote by $A_p(\mathbb{R}^n)$, $p \in [1, \infty]$, and $RH_q(\mathbb{R}^n)$, $q \in (1, \infty]$, respectively, the class of Muckenhoupt weights and the class of weights satisfying the reverse Hölder condition. Then the conclusion of Lemma 2.7 also holds true for the classes $A_p(\mathbb{R}^n)$ and $RH_q(\mathbb{R}^n)$ (see, for example, [19, 21]).

Now we prove Theorem 2.3 by using Lemmas 2.5 through 2.7.

Proof of Theorem 2.3. The proof of Theorem 2.3 is similar to that of [10, Theorem 2.3]. Here we give some necessary details. We first prove

$$H_m^{\varphi,q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{2.4}$$

Let $f \in H_m^{\varphi,q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then there exists a sequence $\{a_j\}_j$ of $(\varphi, q)_m$-atoms and $\{\lambda_j\}_j \subset \mathbb{C}$ such that

$$f = \sum_j \lambda_j a_j \quad \text{in} \quad L^2(\mathbb{R}^n) \tag{2.5}$$

and

$$\|f\|_{H_m^{\varphi,q}(\mathbb{R}^n)} \sim \Lambda(\{\lambda_j a_j\}_j), \tag{2.6}$$

where, for each j, $\operatorname{supp}(a_j) \subset B_j := B(x_j, r_j)$. By using Lemmas 2.5, 2.6 and 2.7, similar to the proof of [22, (4.7)], we know that, for any $(\varphi, q)_m$-atom a with $\operatorname{supp}(a) \subset B$ and $\lambda \in \mathbb{C}$,

$$\int_{\mathbb{R}^n} \varphi(x, S_L(\lambda a)(x))\, dx \lesssim \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right). \tag{2.7}$$

From (2.5) and (2.7), it follows that, for all $\lambda \in (0, \infty)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{S_L(f)(x)}{\lambda}\right) dx \lesssim \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\right),$$

which, together with (2.6), implies that $f \in H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \lesssim \|f\|_{H_m^{\varphi,q}(\mathbb{R}^n)}$. Thus, (2.4) holds true.

Moreover, similar to the proof of [10, (2.9)], we obtain that

$$H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_m^{\varphi,q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{2.8}$$

By (2.4) and (2.8), we conclude that $H_m^{\varphi,q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) = H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with equivalent norms, which, together with the fact that $H_m^{\varphi,q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ are, respectively, dense in $H_m^{\varphi,q}(\mathbb{R}^n)$ and $H_{\varphi,L}(\mathbb{R}^n)$, and a density argument, then implies that the spaces $H_m^{\varphi,q}(\mathbb{R}^n)$ and $H_{\varphi,L}(\mathbb{R}^n)$ coincide with equivalent quasi-norms. This finishes the proof of Theorem 2.3. ■

3. Proof of Theorem 1.4

In this section, we give out the proof of Theorem 1.4. We begin with some useful auxiliary conclusions. In what follows, we always assume that $V \in RH_{q_0}(\mathbb{R}^n)$ with $q_0 \in [n, \infty)$ and $n \geq 3$. Denote by Γ the *fundamental solution* for the operator L as in (1.4). For Γ, we have the following estimates.

Lemma 3.1. *Let L be as in* (1.4) *with $V \in RH_{q_0}(\mathbb{R}^n)$ and $q_0 \in [n, \infty)$. Denote the fundamental solution of L by Γ.*

(i) *Assume that the matrix A in* (1.4) *satisfies the assumption (A_1). Then, for any $k \in \mathbb{N}$, there exists a positive constant $C_{(k)}$ such that, for any $x,\ y \in \mathbb{R}^n$ with $x \neq y$, $|\Gamma(x,y)| \leq \frac{C_{(k)}}{[1+|x-y|m(x,V)]^k} \frac{1}{|x-y|^{n-2}}$.*

Assume further that A in (1.4) *satisfies the assumptions (A_1) and (A_2). Then, for any $k \in \mathbb{N}$, there exists a positive constant $C_{(k)}$ such that, for any $x,\ y \in \mathbb{R}^n$ with $x \neq y$,*

(ii) $|\nabla_x \Gamma(x,y)| \leq \frac{C_{(k)}}{[1+|x-y|m(x,V)]^k} \frac{1}{|x-y|^{n-1}}$ *and* $|\nabla_y \Gamma(x,y)| \leq \frac{C_{(k)}}{[1+|x-y|m(x,V)]^k} \frac{1}{|x-y|^{n-1}}$;

(iii) $|\nabla_x \nabla_y \Gamma(x,y)| \leq \frac{C_{(k)}}{[1+|x-y|m(x,V)]^k} \frac{1}{|x-y|^{n}}$.

Proof. (i) and (ii) of this lemma were established in [31, Theorem 2.5]. Moreover, by the fact that $\nabla_y \Gamma(x,y)$ is the solution of $Lu = 0$, [31, (2.5)] and (ii), we conclude that (iii) holds true, which completes the proof of Lemma 3.1. ■

Moreover, we also need the following $L^p(\mathbb{R}^n)$-boundedness of VL^{-1}, $V^{1/2}\nabla L^{-1}$ and $\nabla^2 L^{-1}$.

Lemma 3.2. *Let L be as in* (1.4) *with $V \in RH_{q_0}(\mathbb{R}^n)$ and $q_0 \in [n, \infty)$.*

(i) *Assume that A in* (1.4) *satisfies the assumption (A_1). Then, for any $p \in (1, q_0]$, there exists a positive constant $C_{(p)}$ such that, for all $f \in L^p(\mathbb{R}^n)$, $\|VL^{-1}(f)\|_{L^p(\mathbb{R}^n)} \leq C_{(p)}\|f\|_{L^p(\mathbb{R}^n)}$.*

(ii) *Assume that A in* (1.4) *satisfies the assumptions (A_1) and (A_2). Then, for any $p \in (1, 2q_0]$, there exists a positive constant $C_{(p)}$ such that, for all $f \in L^p(\mathbb{R}^n)$,*

$$\left\| V^{1/2}\nabla L^{-1}(f) \right\|_{L^p(\mathbb{R}^n)} \leq C_{(p)}\|f\|_{L^p(\mathbb{R}^n)}.$$

The proofs of (i) and (ii) of Lemma 3.2 are, respectively, similar to that of [37, Theorems 3.1 and 4.13] and we omit the details.

Lemma 3.3. *Let L be as in* (1.4) *with $V \in RH_{q_0}(\mathbb{R}^n)$ and $q_0 \in [n, \infty)$. Assume that A satisfies the assumptions (A_1), (A_2) and (A_3). Then, for any $p \in (1, q_0]$, there exists a positive constant $C_{(p)}$ such that, for all $f \in L^p(\mathbb{R}^n)$, $\left\|\nabla^2 L^{-1}(f)\right\|_{L^p(\mathbb{R}^n)} \leq C_{(p)}\|f\|_{L^p(\mathbb{R}^n)}$.*

Proof. Let $L_0 := -\mathrm{div}(A\nabla)$. Then $L = L_0 + V$. It was proved in [5, Theorem B] (see also [31, Theorem 2.7]) that $\nabla^2 L_0^{-1}$ is bounded on $L^p(\mathbb{R}^n)$ for any $p \in (1,\infty)$, which, together with Lemma 3.2(i), implies that, for all $f \in L^p(\mathbb{R}^n)$ with $p \in (1, q_0]$,

$$\|\nabla^2 L^{-1} f\|_{L^p(\mathbb{R}^n)} \lesssim \|L_0 L^{-1} f\|_{L^p(\mathbb{R}^n)} \sim \|(L-V)L^{-1} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

This finishes the proof of Lemma 3.3. ∎

Furthermore, we need the following estimates for the potential V, which were established in [37, Lemma 1.2].

Lemma 3.4. *Let $V \in RH_{q_0}(\mathbb{R}^n)$ with $q_0 \in [n/2, \infty)$. Then there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $0 < r < R < \infty$,*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\,dy \le C \left(\frac{R}{r}\right)^{\frac{n}{q_0}-2} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)\,dy.$$

Moreover, if $r := [m(x,V)]^{-1}$ with $x \in \mathbb{R}^n$, then $\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\,dy = 1$.

Now we prove Theorem 1.4 by using Lemmas 3.1 through 3.4.

Proof of Theorem 1.4. We first prove (i) of Theorem 1.4. Let $f \in H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Recall that, in this case, $\mu_0 = \alpha_0$. By the assumption $[r(\varphi)]' < \frac{n}{nq(\varphi)/i(\varphi) - \alpha_0}$, we see that there exists

$$q \in \left([r(\varphi)]', \frac{n}{nq(\varphi)/i(\varphi) - \alpha_0}\right). \tag{3.1}$$

Thus, $q > [r(\varphi)]'$ and $\alpha_0 + n/q > nq(\varphi)/i(\varphi)$. From this and Theorem 2.3, it follows that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $(\varphi, q)_m$-atoms such that

$$f = \sum_i \lambda_i a_i \text{ in } L^2(\mathbb{R}^n) \text{ and } \|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \Lambda(\{\lambda_i a_i\}_i). \tag{3.2}$$

To finish the proof of Theorem 1.4(i), it suffices to prove that, for all $\lambda \in \mathbb{C}$ and $(\varphi, q)_m$-atoms a supported in the ball $B := B(x_0, r_0)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, |VL^{-1}(\lambda a)(x)|\right) dx \lesssim \varphi\left(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right). \tag{3.3}$$

If (3.3) holds true, from this, (3.2) and Lemma 3.2, we further deduce that, for all $\lambda \in (0,\infty)$,

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\left(x, \frac{|VL^{-1}(f)(x)|}{\lambda}\right) dx &\lesssim \sum_i \int_{\mathbb{R}^n} \varphi\left(x, \frac{|VL^{-1}(\lambda_i a_i)(x)|}{\lambda}\right) dx \\ &\lesssim \sum_i \varphi\left(B_i, \frac{|\lambda_i|}{\lambda \|\chi_{B_i}\|_{L^\varphi(\mathbb{R}^n)}}\right),\end{aligned}$$

which implies that $\|VL^{-1}(f)\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H_{\varphi,L}(\mathbb{R}^n)}$. By this and the fact that $H_{\varphi,L}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ is dense in $H_{\varphi,L}(\mathbb{R}^n)$, we further conclude that VL^{-1} is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$.

Now we prove (3.3). We first write

$$\begin{aligned}(3.4)\int_{\mathbb{R}^n} \varphi\left(x, |VL^{-1}(\lambda a)(x)|\right)\,dx &= \int_{4B} \varphi\left(x, |VL^{-1}(\lambda a)(x)|\right)\,dx + \int_{\mathbb{R}^n\setminus 4B}\cdots\\ &=: \mathrm{I}_{1,1} + \mathrm{I}_{1,2}.\end{aligned}$$

Moreover, from (3.1) and the definitions of $i(\varphi)$, $q(\varphi)$ and $r(\varphi)$, we deduce that there exist $p_0\in(0,(i(\varphi))$ and $\widetilde{q}\in(q(\varphi),\infty)$ such that φ is of uniformly lower type p_0, $\varphi\in\mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)\cap\mathbb{RH}_{q'}(\mathbb{R}^n)$ and

$$\alpha_0+\frac{n}{q}>\frac{n\widetilde{q}}{p_0}. \tag{3.5}$$

For $\mathrm{I}_{1,1}$, by the uniformly upper type 1 and lower type p_0 properties of φ, Hölder's inequality, Lemma 3.2(i), $\varphi\in\mathbb{RH}_{q'}(\mathbb{R}^n)\subset\mathbb{RH}_{(q/p_0)'}(\mathbb{R}^n)$ and Lemma 2.7(v), we know that

$$\begin{aligned}\mathrm{I}_{1,1} &\lesssim \int_{4B}\varphi\left(x,|\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\left[|VL^{-1}(a)(x)|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\right.\\ &\qquad\left.+\,|VL^{-1}(a)(x)|^{p_0}\|\chi_B\|^{p_0}_{L^\varphi(\mathbb{R}^n)}\right]\,dx\\ &\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|VL^{-1}(a)\|_{L^q(4B)}\left\|\varphi\left(\cdot,|\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right\|_{L^{q'}(4B)}\\ &\qquad+\|\chi_B\|^{p_0}_{L^\varphi(\mathbb{R}^n)}\|VL^{-1}(a)\|^{p_0}_{L^q(4B)}\left\|\varphi\left(\cdot,|\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right\|_{L^{(q/p_0)'}(4B)}\\ &\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|a\|_{L^q(\mathbb{R}^n)}|B|^{-1/q}\varphi\left(4B,|\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\\ &\qquad+\|\chi_B\|^{p_0}_{L^\varphi(\mathbb{R}^n)}\|a\|^{p_0}_{L^q(\mathbb{R}^n)}|B|^{-p_0/q}\varphi\left(4B,|\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\\ &\lesssim \varphi\left(B,|\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right).\end{aligned}\tag{3.6}$$

Now we estimate the term $\mathrm{I}_{1,2}$ by considering the following two cases for r_0.

Case 1. $r_0\ge[m(x_0,V)]^{-1}$. In this case, for any given $x\in S_j(B):=2^{j+1}B\setminus(2^jB)$ with $j\in\mathbb{N}$ and $j\ge2$, from Lemma 2.5, we deduce that, for all $y\in B$,

$$|x-y|m(x,V)\gtrsim\frac{2^jr_0m(x_0,V)}{[1+2^jr_0m(x_0,V)]^{k_0/(1+k_0)}}\gtrsim 2^{j/(1+k_0)}r_0m(x_0,V),$$

where k_0 is as in Lemma 2.5, which, together with Lemma 3.1(i), implies that, for all

$x \in S_j(B)$ with $j \geq 2$ and any $k \in \mathbb{N}$,

$$\begin{aligned}\left|VL^{-1}(a)(x)\right| &\lesssim V(x)\int_B \frac{|a(y)|}{[1+|x-y|m(x,V)]^k|x-y|^{n-2}}\,dy \\ &\lesssim (2^j r_0)^{2-n}2^{-jk/(1+k_0)}[r_0 m(x_0,V)]^{-k}\|a\|_{L^1(B)}V(x) \\ &\lesssim 2^{-[k/(1+k_0)+n-2]j}[r_0 m(x_0,V)]^{-k}r_0^2\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}V(x).\end{aligned} \tag{3.7}$$

By (3.7) and the uniformly upper type 1 and lower type p_0 properties of φ, we conclude that, for any $j \in \mathbb{N}$ with $j \geq 2$,

$$\begin{aligned}\mathrm{H}_j &:= \int_{S_j(B)} \varphi\left(x, |VL^{-1}(\lambda a)(x)|\right)\,dx \\ &\lesssim 2^{-[k/(1+k_0)+n-2]j}[r_0 m(x_0,V)]^{-k}r_0^2 \\ &\quad\times \int_{S_j(B)} \varphi\left(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)V(x)\,dx \\ &\quad+2^{-[k/(1+k_0)+n-2]jp_0}[r_0 m(x_0,V)]^{-kp_0}r_0^{2p_0} \\ &\quad\times \int_{S_j(B)} \varphi\left(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)[V(x)]^{p_0}\,dx =: \mathrm{E}_j + \mathrm{F}_j.\end{aligned} \tag{3.8}$$

Now we estimate the term E_j. From $[r(\varphi)]' < \frac{n}{nq(\varphi)/i(\varphi)-\alpha_0} < n/(n-1)$, $n \geq 3$ and $q_0 \in [n,\infty)$, it follows that $[r(\varphi)]' < q_0$ and hence $\varphi \in \mathbb{RH}_{q_0'}(\mathbb{R}^n)$. Moreover, by $V \in RH_{q_0}(\mathbb{R}^n)$, Remark 2.8 and Lemma 2.7(iv), we conclude that there exists $\widetilde{q}_0 \in (1,\infty)$ such that $V \in A_{\widetilde{q}_0}(\mathbb{R}^n)$. Thus, from $\varphi \in \mathbb{RH}_{q_0'}(\mathbb{R}^n) \cap \mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$, $V \in RH_{q_0}(\mathbb{R}^n) \cap A_{\widetilde{q}_0}(\mathbb{R}^n)$, Lemma 2.7(v) and Remark 2.8, we deduce that, for any $k \in \mathbb{N}$,

$$\begin{aligned}\mathrm{E}_j &\lesssim 2^{-[k/(1+k_0)+n-2]j}[r_0 m(x_0,V)]^{-k}r_0^2 \\ &\quad\times\|V\|_{L^{q_0}(S_j(B))}\left\|\varphi\left(\cdot, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right\|_{L^{q_0'}(S_j(B))} \\ &\lesssim 2^{-[k/(1+k_0)+2n-2]j}[r_0 m(x_0,V)]^{-k}r_0^{(2-n)}V(2^jB)\varphi\left(2^jB, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) \\ &\lesssim 2^{-[k/(1+k_0)+2n-n(\widetilde{q}_0+\widetilde{q})-2]j}[r_0 m(x_0,V)]^{-k}r_0^{(2-n)}V(B)\varphi\left(B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) \\ &\lesssim 2^{-[k/(1+k_0)+2n-n(\widetilde{q}_0+\widetilde{q})-2]j}[r_0 m(x_0,V)]^{-(k+n-n\widetilde{q}_0-2)}\varphi\left(B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right),\end{aligned} \tag{3.9}$$

where k_0 is as in Lemma 2.5 and $V(B) := \int_B V(x)\,dx$.

For F_j, similar to (3.9), we know that, for any $k \in \mathbb{N}$,

$$\begin{aligned}\mathrm{F}_j &\lesssim 2^{-[k/(1+k_0)+2n-n(\widetilde{q}_0+\widetilde{q}/p_0)-2]jp_0} \\ &\quad\times[r_0 m(x_0,V)]^{-(k+n-n\widetilde{q}_0-2)p_0}\varphi\left(B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right).\end{aligned} \tag{3.10}$$

Taking k enough large in (3.8), (3.9) and (3.10) and then using (3.8), (3.9), (3.10) and the fact that $r_0 m(x_0, V) \geq 1$, we conclude that

$$
\begin{aligned}
\mathrm{I}_{1,2} &\lesssim \sum_{j=2}^{\infty} \mathrm{H}_j \lesssim \sum_{j=2}^{\infty} 2^{-[k/(1+k_0)+2n-n(\widetilde{q}_0+\widetilde{q}/p_0)-2]jp_0} \varphi\left(B, |\lambda| \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\right) \\
&\lesssim \varphi\left(B, |\lambda| \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\right).
\end{aligned} \tag{3.11}
$$

Case 2. $r_0 \in (0, [m(x_0, V)]^{-1})$. In this case, let $j_0 \in \mathbb{N}$ such that

$$2^{j_0-1} r_0 < [m(x_0, V)]^{-1} \leq 2^{j_0} r_0.$$

For any $x \in S_j(B)$ with $j \in \{2, \ldots, j_0+2\}$, by $\int_{\mathbb{R}^n} a(x)\,dx = 0$, the mean valued theorem, Lemma 3.1(ii) and Hölder's inequality, we conclude that, for any $k \in \mathbb{N}$,

$$
\begin{aligned}
\left|VL^{-1}(a)(x)\right| &= V(x) \left| \int_{\mathbb{R}^n} [\Gamma(x,y) - \Gamma(x, x_0)] a(y)\,dy \right| \\
&\leq V(x) \int_B |\nabla_y \Gamma(x, y_1)| |(y - x_0) a(y)|\,dy \\
&\lesssim (2^j r_0)^{-(n-1)} r_0^{n+1} [1 + 2^j r_0 m(x_0, V)]^{-k} \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1} V(x),
\end{aligned} \tag{3.12}
$$

where $y_1 := x_0 + \theta(y - x_0)$ with $\theta \in (0,1)$. From this, the uniformly upper type 1 and lower type p_0 properties of φ, it follows that, for any $j \in \{2, \ldots, j_0+2\}$,

$$
\begin{aligned}
\mathrm{J}_j :=& \int_{S_j(B)} \varphi\left(x, |VL^{-1}(\lambda a)(x)|\right)\,dx \\
\lesssim&\ (2^j r_0)^{-(n-1)} r_0^{n+1} [1 + 2^j r_0 m(x_0, V)]^{-k} \\
&\times \int_{S_j(B)} \varphi\left(x, |\lambda| \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\right) V(x)\,dx \\
&+ (2^j r_0)^{-(n-1)p_0} r_0^{(n+1)p_0} [1 + 2^j r_0 m(x_0, V)]^{-kp_0} \\
&\times \int_{S_j(B)} \varphi\left(x, |\lambda| \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\right) [V(x)]^{p_0}\,dx =: \mathrm{K}_j + \mathrm{M}_j.
\end{aligned} \tag{3.13}
$$

For K_j, by Hölder's inequality, $\varphi \in \mathbb{RH}_{q_0'}(\mathbb{R}^n) \cap \mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$, $V \in RH_{q_0}(\mathbb{R}^n)$, Lemma

2.7(v) and Lemma 3.4, we see that, for any $k\in\mathbb{N}$,

$$
\begin{aligned}
\mathrm{K}_j &\le (2^j r_0)^{-(n-1)} r_0^{n+1}[1+2^j r_0 m(x_0,V)]^{-k}\|V\|_{L^{q_0}(S_j(B))}\\
&\quad\times\left\|\varphi\left(\cdot,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\right\|_{L^{q_0'}(S_j(B))}\\
&\lesssim (2^j r_0)^{-(2n-1)} r_0^{n+1}[1+2^j r_0 m(x_0,V)]^{-k}\varphi\left(2^jB,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)V(2^jB)\\
&\lesssim (2^j r_0)^{-(2n-1)} r_0^{n+1}[1+2^j r_0 m(x_0,V)]^{-k}\varphi\left(2^jB,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)V(2^jB)\\
&\lesssim 2^{-(n+k+1-n\widetilde{q})}[r_0 m(x_0,V)]^{-k}\left[\frac{1}{2^j r_0 m(x_0,V)}\right]^{n/q_0-2}\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\\
&\lesssim 2^{-(n+k+n/q_0-1-n\widetilde{q})}[r_0 m(x_0,V)]^{-(k+n/q_0-2)}\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right).
\end{aligned}\tag{3.14}
$$

For M_j, similar to (3.14), we have

$$
\begin{aligned}
\mathrm{M}_j &\lesssim 2^{-(n+k+n/q_0-1-n\widetilde{q}/p_0)p_0}[r_0 m(x_0,V)]^{-(k+n/q_0-2)p_0}\\
&\quad\times\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right).
\end{aligned}\tag{3.15}
$$

Taking $k=2-n/q_0$ in (3.14) and (3.15) and then using (3.13), (3.14), (3.15), (3.5), $\alpha_0\in(0,1]$ and $q>1$, we further conclude that

$$
\begin{aligned}
\sum_{j=2}^{j_0+2}\mathrm{J}_j &\lesssim \sum_{j=2}^{j_0+2}2^{-(n+1-n\widetilde{q}/p_0)p_0}\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\\
&\lesssim \varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right).
\end{aligned}\tag{3.16}
$$

From the definition of j_0, we deduce that $2^j r_0 m(x_0,V)\ge 8$ for any $j\in\mathbb{N}$ with $j\ge j_0+3$. Let $\widetilde{B}_0:=B(x_0,\widetilde{r}_0)$ with $\widetilde{r}_0:=[m(x_0,V)]^{-1}$. Then $\cup_{j=j_0+3}^\infty S_j(B)\subset\cup_{j=2}^\infty S_j(\widetilde{B}_0)$. By Lemma 2.5, we see that, for any $x\in S_j(\widetilde{B}_0)$ with $j\ge 2$ and $y\in B$,

$$
|x-y|m(x,V)\gtrsim\frac{2^j[m(x_0,V)]^{-1}m(x_0,V)}{[1+2^j[m(x_0,V)]^{-1}m(x_0,V)]^{k_0/(1+k_0)}}\sim 2^{-j/(1+k_0)},
$$

where k_0 is as in Lemma 2.5. From this, $\int_{\mathbb{R}^n}a(x)\,dx=0$, the mean valued theorem, Lemma 3.1(ii) and Hölder's inequality, it follows that, for any $x\in S_j(\widetilde{B}_0)$ with $j\in\{2,\ldots\}$ and $k\in\mathbb{N}$,

$$
\begin{aligned}
\left|VL^{-1}(a)(x)\right| &= V(x)\left|\int_{\mathbb{R}^n}[\Gamma(x,y)-\Gamma(x,x_0)]a(y)\,dy\right|\\
&\le V(x)\int_B|\nabla_y\Gamma(x,y_1)||(y-x_0)a(y)|\,dy\\
&\lesssim 2^{-[k/(1+k_0)+n-1]j}r_0^{n+1}[m(x_0,V)]^{n-1}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}V(x),
\end{aligned}\tag{3.17}
$$

where $y_1 := x_0 + \theta(y - x_0)$ with $\theta \in (0,1)$. By (3.17), the uniformly upper type 1 and lower type p_0 properties of φ, we conclude that, for any $j \in \{2, \ldots\}$,

$$\begin{aligned}
\widetilde{\mathrm{J}}_j &:= \int_{S_j(\widetilde{B}_0)} \varphi\left(x, |VL^{-1}(\lambda a)(x)|\right) dx \\
&\lesssim 2^{-[k/(1+k_0)+n-1]j} \widetilde{r}_0^{1-n} r_0^{n+1} \int_{S_j(\widetilde{B}_0)} \varphi\left(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) V(x)\, dx \\
&\quad + 2^{-[k/(1+k_0)+n-1]jp_0} \widetilde{r}_0^{(n-1)p_0} r_0^{(n+1)p_0} \\
&\quad \times \int_{S_j(\widetilde{B}_0)} \varphi\left(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) [V(x)]^{p_0}\, dx \\
&=: \widetilde{\mathrm{K}}_j + \widetilde{\mathrm{M}}_j.
\end{aligned} \tag{3.18}$$

For $\widetilde{\mathrm{K}}_j$, from Hölder's inequality, $\varphi \in \mathbb{RH}_{q_0'}(\mathbb{R}^n) \cap \mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$, $V \in RH_{q_0}(\mathbb{R}^n) \cap A_{\widetilde{q}_0}(\mathbb{R}^n)$, Lemma 2.7(v), (3.5) and $\widetilde{r}_0 \geq r_0$, we deduce that, for any $k \in \mathbb{N}$,

$$\begin{aligned}
\widetilde{\mathrm{K}}_j &\leq 2^{-[k/(1+k_0)+n-1]j} \widetilde{r}_0^{1-n} r_0^{n+1} \|V\|_{L^{q_0}(S_j(\widetilde{B}_0))} \\
&\quad \times \left\| \varphi\left(\cdot, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) \right\|_{L^{q_0'}(S_j(\widetilde{B}_0))} \\
&\lesssim 2^{-[k/(1+k_0)+2n-1]j} \widetilde{r}_0^{1-2n} r_0^{n+1} \varphi\left(2^j \widetilde{B}_0, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) V(2^j \widetilde{B}_0) \\
&\lesssim 2^{-[k/(1+k_0)+2n-n(\widetilde{q}+\widetilde{q}_0)-1]j} \varphi\left(\widetilde{B}_0, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) V(\widetilde{B}_0) \\
&\lesssim 2^{-[k/(1+k_0)+2n-n(\widetilde{q}+\widetilde{q}_0)-1]j} \left[\frac{r_0}{\widetilde{r}_0}\right]^{n+1-n\widetilde{q}} \varphi\left(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) \\
&\lesssim 2^{-[k/(1+k_0)+2n-n(\widetilde{q}+\widetilde{q}_0)-1]j} \varphi\left(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)
\end{aligned} \tag{3.19}$$

For $\widetilde{\mathrm{M}}_j$, similar to (3.19), we obtain that

$$\widetilde{\mathrm{M}}_j \lesssim 2^{-[k/(1+k_0)+2n-n(\widetilde{q}/p_0+\widetilde{q}_0)-1]jp_0} \varphi\left(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right). \tag{3.20}$$

Taking k large enough in (3.19) and (3.20) and then using (3.18), (3.19) and (3.20), we know that

$$\begin{aligned}
\sum_{j=2}^{\infty} \widetilde{\mathrm{J}}_j &\lesssim \sum_{j=2}^{\infty} 2^{-[k/(1+k_0)+2n-n(\widetilde{q}/p_0+\widetilde{q}_0)-1]jp_0} \varphi\left(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) \\
&\lesssim \varphi\left(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right),
\end{aligned} \tag{3.21}$$

which, together with (3.16) and $\cup_{j=j_0+3}^{\infty} S_j(B) \subset \cup_{j=2}^{\infty} S_j(\widetilde{B}_0)$, implies that, in this case,

$$\mathrm{I}_{1,2} \lesssim \sum_{j=2}^{j_0} \mathrm{J}_j + \sum_{j=2}^{\infty} \widetilde{\mathrm{J}}_j \lesssim \varphi\left(B, |\lambda| \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\right).$$

From this, (3.4), (3.6) and (3.11), it follows that (3.3) holds true, which completes the proof of Theorem 1.4(i).

Now we prove (ii) of this theorem. Let q be as in the proof of (i). Similar to the proof of (i), it suffices to prove that, for all $\lambda \in \mathbb{C}$ and $(\varphi, q)_m$-atoms a supported in the ball $B := B(x_0, r_0)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \left|V^{1/2}\nabla L^{-1}(\lambda a)(x)\right|\right) dx \lesssim \varphi\left(B, |\lambda| \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\right). \tag{3.22}$$

Now we prove (3.22). We first write

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\left(x, \left|V^{1/2}\nabla L^{-1}(\lambda a)(x)\right|\right) dx &= \int_{4B} \varphi\left(x, \left|V^{1/2}\nabla L^{-1}(\lambda a)(x)\right|\right) dx \\ &\quad + \int_{\mathbb{R}^n \setminus 4B} \cdots =: \mathrm{I}_{2,1} + \mathrm{I}_{2,2}.\end{aligned} \tag{3.23}$$

For $\mathrm{I}_{2,1}$, similar to (3.6), we see that

$$\mathrm{I}_{2,1} \lesssim \varphi\left(B, |\lambda| \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\right). \tag{3.24}$$

Now we estimate the term $\mathrm{I}_{2,2}$ by considering the following two cases for r_0.

Case 1. $r_0 \geq [m(x_0, V)]^{-1}$. In this case, by using Lemmas 2.5 and 3.1, similar to (3.7), we conclude that, for any given $x \in S_j(B)$ with $j \in \mathbb{N}$ and $j \geq 2$ and $k \in \mathbb{N}$,

$$\left|V^{1/2}\nabla L^{-1}(a)(x)\right| \lesssim 2^{-[k/(1+k_0)+n-1]j} r_0 [r_0 m(x_0, V)]^{-k} \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}, \tag{3.25}$$

where k_0 is as in Lemma 2.2. From (3.25), the uniformly upper type 1 and lower type p_0 properties of φ, Hölder's inequality and Lemma 2.7(v), and similar to (3.8), we deduce that, for any $k \in \mathbb{N}$,

$$\begin{aligned}\mathrm{J}_j &:= \int_{S_j(B)} \varphi\left(x, \left|V^{1/2}\nabla L^{-1}(\lambda a)(x)\right|\right) dx \\ &\lesssim 2^{-[\frac{k}{1+k_0}+\frac{3n}{2}-\frac{n\tilde{q}}{p_0}-\frac{n\tilde{q}_0}{2}-1]p_0 j} \varphi\left(B, |\lambda| \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\right).\end{aligned} \tag{3.26}$$

Taking k enough large in (3.26), then we see that

$$\begin{aligned}\mathrm{I}_{2,2} &\lesssim \sum_{j=2}^{\infty} \mathrm{J}_j \lesssim \sum_{j=2}^{\infty} 2^{-[\frac{k}{1+k_0}+\frac{3n}{2}-\frac{n\tilde{q}}{p_0}-\frac{n\tilde{q}_0}{2}-1]p_0 j} \varphi\left(B, |\lambda| \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\right) \\ &\lesssim \varphi\left(B, |\lambda| \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\right).\end{aligned} \tag{3.27}$$

Case 2. $r_0 \in (0, [m(x_0,V)]^{-1})$. In this case, similar to Case 2 of the proof of (i), let $j_0 \in \mathbb{N}$ such that $2^{j_0-1}r_0 < [m(x_0,V)]^{-1} \le 2^{j_0}r_0$. For any $x \in S_j(B)$ with $j \in \{2, \ldots, j_0+2\}$, by $\int_{\mathbb{R}^n} a(x)\,dx = 0$, the mean valued theorem, Lemma 3.1(iii) and Hölder's inequality, similar to (3.12), we see that, for any $k \in \mathbb{N}$,

$$\left|V^{1/2}\nabla L^{-1}(a)(x)\right| \lesssim (2^j r_0)^{-n} r_0^{n+1}[1+2^j r_0 m(x_0,V)]^{-k}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}[V(x)]^{1/2}.$$

From this, the uniformly upper type 1 and lower type p_0 properties of φ, Hölder's inequality, $\varphi \in \mathbb{RH}_{q_0'}(\mathbb{R}^n) \cap \mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$, $V \in RH_{q_0}(\mathbb{R}^n)$, Lemma 2.7(v) and Lemma 3.4, where $\widetilde{q}$ is as (3.5), similar to (3.16), it follows that, for any $j \in \{2, \ldots, j_0+2\}$,

$$\sum_{j=2}^{j_0+2} \mathrm{J}_j \lesssim \varphi\left(B, |\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right). \tag{3.28}$$

Moreover, by $\int_{\mathbb{R}^n} a(x)\,dx = 0$, the mean valued theorem, Lemma 3.1(iii) and Hölder's inequality, similar to (3.17), we conclude that, for any $x \in S_j(\widetilde{B}_0)$ with $j \in \{2, \ldots\}$ and $k \in \mathbb{N}$,

$$\left|V^{1/2}\nabla L^{-1}(a)(x)\right| \lesssim 2^{-[k/(1+k_0)+n]j} r_0^{n+1}\widetilde{r}_0^{\,n}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}[V(x)]^{1/2}, \tag{3.29}$$

where $\widetilde{B}_0 := B(x_0, \widetilde{r}_0)$ with $\widetilde{r}_0 := [m(x_0,V)]^{-1}$, and k_0 is as in Lemma 2.5. By (3.29), the uniformly upper type 1 and lower type p_0 properties of φ, Hölder's inequality, $\varphi \in \mathbb{RH}_{q_0'}(\mathbb{R}^n) \cap \mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$, $V \in RH_{q_0}(\mathbb{R}^n) \cap A_{\widetilde{q}_0}(\mathbb{R}^n)$ and Lemma 2.7(v), similar to (3.21), we conclude that

$$\begin{aligned}\sum_{j=2}^{\infty} \widetilde{\mathrm{J}}_j &:= \sum_{j=2}^{\infty} \int_{S_j(\widetilde{B}_0)} \varphi\left(x, \left|V^{1/2}\nabla L^{-1}(a)\right|\right)\,dx \\ &\lesssim \sum_{j=2}^{\infty} 2^{-[\frac{k}{1+k_0}+\frac{3n}{2}-\frac{n\widetilde{q}_0}{2}-\frac{n\widetilde{q}}{p_0}]jp_0}\varphi\left(B, |\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) \\ &\lesssim \varphi\left(B, |\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right),\end{aligned}$$

which, together with (3.28) and $\cup_{j=j_0+3}^{\infty} S_j(B) \subset \cup_{j=2}^{\infty} S_j(\widetilde{B}_0)$, implies that, in this case,

$$\mathrm{I}_{2,2} \lesssim \sum_{j=2}^{j_0+2} \mathrm{J}_j + \sum_{j=2}^{\infty} \widetilde{\mathrm{J}}_j \lesssim \varphi\left(B, |\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right).$$

From this, (3.23), (3.24) and (3.27), we further deduce that (3.22) holds true, which completes the proof of Theorem 1.4(ii).

Finally, we prove (iii) of this theorem. Let q be as in the proof of (i). Similar to the proof of (i), it suffices to prove that, for all $\lambda \in \mathbb{C}$ and $(\varphi, q)_m$-atoms a supported in the ball $B := B(x_0, r_0)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \left|\nabla^2 L^{-1}(\lambda a)(x)\right|\right)\, dx \lesssim \varphi\left(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right). \tag{3.30}$$

Now we prove (3.30). We first write

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\left(x, \left|\nabla^2 L^{-1}(\lambda a)(x)\right|\right)\, dx &= \int_{16B} \varphi\left(x, \left|\nabla^2 L^{-1}(\lambda a)(x)\right|\right)\, dx \\ &\quad + \int_{\mathbb{R}^n \setminus 16B} \cdots =: \mathrm{I}_{3,1} + \mathrm{I}_{3,2}.\end{aligned} \tag{3.31}$$

For $\mathrm{I}_{3,1}$, similar to (3.6), we see that

$$\mathrm{I}_{3,1} \lesssim \varphi\left(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right). \tag{3.32}$$

Now we estimate the term $\mathrm{I}_{3,2}$ by considering the following two cases for r_0.

Case 1. $r_0 \geq [m(x_0, V)]^{-1}$. In this case, let $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(t) \equiv 0$ when $t \in (-\infty, 1/2] \cup [4, \infty)$, and $\psi(t) \equiv 1$ when $t \in [1, 2]$. For any $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let $\psi_j(x) := \psi(\frac{|x - x_0|}{2^j r_0})$. Then, when $j \in \mathbb{N}$ with $j \geq 5$, $\psi_j a \equiv 0$, which, together with the definition of L, shows that

$$\begin{aligned}&L(\psi_j L^{-1}(a)) \\ &= V\psi_j L^{-1}(a) - \mathrm{div}(A\nabla\psi_j)L^{-1}(a) - A\nabla\psi_j \cdot \nabla L^{-1}(a) \\ &\quad -\mathrm{div}(A\nabla L^{-1}(a))\psi_j - A\nabla L^{-1}(a) \cdot \nabla\psi_j \\ &= LL^{-1}(a)\psi_j - \mathrm{div}(A\nabla\psi_j)L^{-1}(a) \\ &\quad -A\nabla\psi_j \cdot \nabla L^{-1}(a) - A\nabla L^{-1}(a) \cdot \nabla\psi_j \\ &= -\mathrm{div}(A\nabla\psi_j)L^{-1}(a) - A\nabla\psi_j \cdot \nabla L^{-1}(a) - A\nabla L^{-1}(a) \cdot \nabla\psi_j.\end{aligned} \tag{3.33}$$

This, together with the fact that $\psi_j \equiv 1$ on $S_j(B)$ with $j \in \mathbb{N}$ and $j \geq 5$, Lemma 3.3, the assumption (A_1) and Hölder's inequality, implies that

$$\begin{aligned}&\int_{S_j(B)} \left|\nabla^2 L^{-1}(a)\right|^q\, dx \\ &= \int_{S_j(B)} \left|\nabla^2 L^{-1}(L(\psi_j L^{-1}(a)))(x)\right|^q\, dx \lesssim \int_{\mathbb{R}^n} \left|L(\psi_j L^{-1}(a))(x)\right|^q\, dx \\ &\lesssim \int_{\mathbb{R}^n} \left|L^{-1}(a)(x)\mathrm{div}(A\nabla\psi_j)(x)\right|^q\, dx + \int_{\mathbb{R}^n} \left|\nabla\psi_j(x) \cdot \nabla L^{-1}(a)(x)\right|^q\, dx.\end{aligned} \tag{3.34}$$

Moreover, from the assumption (A_3), we deduce that

$$-\mathrm{div}(A\nabla) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}\partial_{ik}, \tag{3.35}$$

which, together with the assumption (A_1) and (3.34), implies that

$$\begin{aligned}&\int_{S_j(B)} \left|\nabla^2 L^{-1}(a)\right|^q\,dx \\ &\lesssim \int_{\mathbb{R}^n} \left|L^{-1}(a)(x)\nabla^2\psi_j(x)\right|^q\,dx + \int_{\mathbb{R}^n} \left|\nabla\psi_j(x)\cdot\nabla L^{-1}(a)(x)\right|^q\,dx.\end{aligned} \tag{3.36}$$

For all $x \in S_j(B)$ with $j \geq 5$, by Lemma 3.1(i), (2.3) and Hölder's inequality, we conclude that, for any $k \in \mathbb{N}$,

$$\begin{aligned}|L^{-1}(a)(x)| &\lesssim \int_B \frac{|a(y)|}{[1+|x-y|m(y,V)]^k|x-y|^{n-2}}\,dy \\ &\lesssim \int_B \frac{|a(y)|}{[1+2^j r_0 m(x_0,V)]^k(2^j r_0)^{n-2}}\,dy \\ &\lesssim 2^{-jk}(2^j r_0)^{-(n-2)}\|a\|_{L^q(\mathbb{R}^n)}|B|^{1/q'} \lesssim 2^{-j(k+n-2)}r_0^2\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)},\end{aligned}$$

which, together with the fact that, for all $z \in \mathbb{R}^n$, $|\nabla^2\psi_j(z)| \lesssim (2^j r_0)^{-2}$, implies that

$$\int_{\mathbb{R}^n} \left|L^{-1}(a)(x)\nabla^2\psi_j(x)\right|^q\,dx \lesssim 2^{-jq(k+n-n/q)}|B|\|\chi_B\|^{-q}_{L^\varphi(\mathbb{R}^n)}. \tag{3.37}$$

Furthermore, by using Lemma 3.1(ii) and the fact that, for all $z \in \mathbb{R}^n$, $|\nabla\psi_j(z)| \lesssim (2^j r_0)^{-1}$, similar to the proof of [10, (3.11)], we see that

$$\int_{\mathbb{R}^n} \left|\nabla L^{-1}(a)(x)\cdot\nabla\psi_j(x)\right|^q\,dx \lesssim 2^{-jq(k+n-n/q)}|B|\|\chi_B\|^{-q}_{L^\varphi(\mathbb{R}^n)}. \tag{3.38}$$

Thus, from (3.36), (3.37) and (3.38), it follows that, for all $j \in \mathbb{N}$ with $j \geq 5$,

$$\int_{S_j(B)} \left|\nabla^2 L^{-1}(a)\right|^q\,dx \lesssim 2^{-jq(k+n-n/q)}|B|\|\chi_B\|^{-q}_{L^\varphi(\mathbb{R}^n)}. \tag{3.39}$$

Then, by using the uniformly upper type 1 and lower type p_0 properties of φ, Hölder's inequality, (3.39), $\varphi \in \mathbb{RH}_{q'}(\mathbb{R}^n) \cap \mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$ and Lemma 2.7(v), similar to (3.6), we know that, for any $j \in \mathbb{N}$ with $j \geq 5$ and any given $k \in \mathbb{N}$,

$$\int_{S_j(B)} \varphi\left(x, \left|\nabla^2 L^{-1}(\lambda a)(x)\right|\right)\,dx \lesssim 2^{-jp_0(n+k-n\widetilde{q}/p_0)}\varphi\left(B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right),$$

which further implies that

$$\mathrm{I}_{3,2}=\sum_{j=5}^{\infty}\int_{S_j(B)}\varphi\left(x,\left|\nabla^2L^{-1}(\lambda a)(x)\right|\right)\,dx\lesssim\varphi\left(B,|\lambda|\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)}\right). \tag{3.40}$$

Case 2. $r_0\in(0,[m(x_0,V)]^{-1})$. In this case, similar to the proof of [10, (3.14)], we know that, for all $x\in S_j(B)$ with $j\in\mathbb{N}$ and $j\ge 5$,

$$\left|L^{-1}(a)(x)\right|\lesssim 2^{j(1-n)}r_0^2\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)}\text{ and }\left|\nabla L^{-1}(a)(x)\right|\lesssim 2^{-jn}r_0\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)}.$$

By this, similar to the proof of (3.39), we conclude that

$$\int_{S_j(B)}\left|\nabla^2L^{-1}(a)(x)\right|^q\,dx\lesssim 2^{-jq(n+1-n/q)}|B|\|\chi_B\|^{-q}_{L^{\varphi}(\mathbb{R}^n)}. \tag{3.41}$$

Then, from the uniformly upper type 1 and lower type p_0 properties of φ, Hölder's inequality, (3.41), $\varphi\in\mathbb{RH}_{q'}(\mathbb{R}^n)\cap\mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$ and Lemma 2.7(v), similar to (3.6), it follows that, for any $j\in\mathbb{N}$ with $j\ge 5$,

$$\int_{S_j(B)}\varphi\left(x,\left|\nabla^2L^{-1}(\lambda a)(x)\right|\right)\,dx\lesssim 2^{-jp_0(n+1-n\widetilde{q}/p_0)}\varphi\left(B,|\lambda|\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)}\right),$$

which, together with (3.5) and $\alpha_0\in(0,1]$, further implies that, in this case,

$$\mathrm{I}_{3,2}=\sum_{j=5}^{\infty}\int_{S_j(B)}\varphi\left(x,\left|\nabla^2L^{-1}(\lambda a)(x)\right|\right)\,dx\lesssim\varphi\left(B,|\lambda|\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)}\right).$$

By this, (3.31), (3.32) and (3.40), we conclude that (3.30) holds true, which completes the proof of (iii) and hence the proof of Theorem 1.4. ■

4. Proof of Theorem 1.6

In this section, we give out the proof of Theorem 1.6. To this end, we need the molecular characterization of $H_{\varphi}(\mathbb{R}^n)$ established in [22, Theorem 4.13]. To state the molecular characterization of the space $H_{\varphi}(\mathbb{R}^n)$, we first recall the definitions of $(\varphi,\,q,\,s,\,\varepsilon)$-molecules and molecular Musielak-Orlicz-Hardy spaces $H^{q,\,s,\,\varepsilon}_{\varphi,\,\mathrm{mol}}(\mathbb{R}^n)$.

Definition 4.1. Let φ be as in Definition 1.2, $q\in(1,\infty)$, $s\in\mathbb{Z}_+$ and $\varepsilon\in(0,\infty)$. A function $\alpha\in L^q(\mathbb{R}^n)$ is called a $(\varphi,\,q,\,s,\,\varepsilon)$-*molecule* associated with the ball B, if

- (i) for each $j\in\mathbb{Z}_+$, $\|\alpha\|_{L^q(S_j(B))}\le 2^{-j\varepsilon}|2^jB|^{1/q}\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)}$, where $S_0(B):=2B$ and $S_k(B):=2^{k+1}B\setminus(2^kB)$ when $k\in\mathbb{N}$;
- (ii) $\int_{\mathbb{R}^n}\alpha(x)x^{\beta}\,dx=0$ for all $\beta\in\mathbb{Z}^n_+$ with $|\beta|\le s$.

Definition 4.2. Let φ be as in Definition 1.2, $q \in (1, \infty)$, $s \in \mathbb{Z}_+$ and $\varepsilon \in (0, \infty)$. The *molecular Musielak-Orlicz-Hardy space*, $H^{q,\,s,\,\varepsilon}_{\varphi,\,\mathrm{mol}}(\mathbb{R}^n)$, is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that $f = \sum_j \lambda_j \alpha_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{\lambda_j\}_j \subset \mathbb{C}$, $\{\alpha_j\}_j$ is a sequence of $(\varphi,\, q,\, s,\, \varepsilon)$-molecules, respectively, associated to the balls $\{B_j\}_j$, and

$$\sum_j \varphi\left(B_j,\, \frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\right) < \infty,$$

where, for each j, the molecule α_j is associated with the ball B_j. Moreover, define

$$\|f\|_{H^{q,\,s,\,\varepsilon}_{\varphi,\,\mathrm{mol}}(\mathbb{R}^n)} := \inf\left\{\Lambda\left(\{\lambda_j \alpha_j\}_j\right)\right\},$$

where the infimum is taken over all the decompositions of f as above and

$$\Lambda\left(\{\lambda_j \alpha_j\}_j\right) := \inf\left\{\lambda \in (0, \infty):\ \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\right) \le 1\right\}.$$

Then we have the following conclusion, which is just [22, Theorem 4.11].

Lemma 4.3. *Let φ be as in Definition 1.2. Assume that $s \in \mathbb{Z}_+$ with $s \ge \lfloor n(q(\varphi)/i(\varphi) - 1)\rfloor$, $\epsilon \in (\max\{n + s, nq(\varphi)/i(\varphi)\}, \infty)$ and $p \in (q(\varphi)[r(\varphi)]', \infty)$, where $q(\varphi)$, $i(\varphi)$ and $r(\varphi)$ are, respectively, as in* (1.5), (1.6) *and* (1.7). *Then $H_\varphi(\mathbb{R}^n)$ and $H^{p,\,s,\,\epsilon}_{\varphi,\,\mathrm{mol}}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Now we prove Theorem 1.6 by using Theorem 2.3, Lemmas 3.1, 3.3 and 4.3.

Proof of Theorem 1.6. We first prove (i) of this theorem. Let $f \in H_{\varphi,\,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. We first recall that, in this case, $\mu_0 = \alpha_0$. From $q(\varphi)[r(\varphi)]' < \frac{n}{nq(\varphi)/i(\varphi) - \alpha_0}$, it follows that there exists

$$q \in \left(q(\varphi)[r(\varphi)]', \frac{n}{nq(\varphi)/i(\varphi) - \alpha_0}\right). \tag{4.1}$$

Thus, $q > q(\varphi)[r(\varphi)]'$ and $\alpha_0 + n/q > nq(\varphi)/i(\varphi)$. By this and Theorem 2.3, we know that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $(\varphi, q)_m$-atoms such that $f = \sum_j \lambda_j a_j$ in $L^2(\mathbb{R}^n)$ and $\|f\|_{H_{\varphi,\,L}(\mathbb{R}^n)} \sim \Lambda(\{\lambda_j a_j\}_j)$, which, together with Lemma 3.3, $q_0 \in [n, \infty)$ and $n \ge 3$, implies that

$$\nabla^2 L^{-1}(f) = \sum_j \lambda_j \nabla^2 L^{-1}(a_j) \tag{4.2}$$

in $L^2(\mathbb{R}^n)$. Let $p \in (q(\varphi)[r(\varphi)]', q]$. To finish the proof of Theorem 1.6, it suffices to prove that, for any $(\varphi, q)_m$-atom a supported in the ball $B := B(x_0, r_0)$, $\nabla^2 L^{-1}(a)$

is a harmless constant multiple of a $(\varphi,\, p,\, 0,\, \varepsilon)$-molecule associated with the ball B for some $\varepsilon > nq(\varphi)/i(\varphi)$. If this claim holds true, from this, (4.2), Lemma 4.3 and $\|f\|_{H_{\varphi,\,L}(\mathbb{R}^n)} \sim \Lambda(\{\lambda_j a_j\}_j)$, we further deduce that $f \in H_\varphi(\mathbb{R}^n)$ and

$$\left\|\nabla^2 L^{-1}(f)\right\|_{H_\varphi(\mathbb{R}^n)} \lesssim \Lambda(\{\lambda_j a_j\}_j) \lesssim \|f\|_{H_{\varphi,\,L}(\mathbb{R}^n)},$$

which is desired.

Now we prove that $\nabla^2 L^{-1}(a)$ is a harmless constant multiple of a $(\varphi,\, p,\, 0,\, \varepsilon)$-molecule by considering the following two cases for r_0.

Case 1. $r_0 \in [[m(x_0, V)]^{-1}, \infty)$. In this case, to prove that $\int_{\mathbb{R}^n} \nabla^2 L^{-1}(a)\, dx = 0$, we first prove that $\nabla L^{-1}(a),\ \nabla^2 L^{-1}(a) \in L^1(\mathbb{R}^n)$. Write

$$\int_{\mathbb{R}^n} \left|\nabla L^{-1}(a)(x)\right|\, dx = \sum_{j=0}^{\infty} \int_{S_j(B)} \left|\nabla L^{-1}(a)(x)\right|\, dx =: \sum_{j=0}^{\infty} \mathrm{I}_j. \tag{4.3}$$

When $j \in \{0,\, 1,\, 2\}$, by Fubini's theorem, Lemma 3.1(ii) and Hölder's inequality, we conclude that

$$\begin{aligned}
\mathrm{I}_j &\le \int_{S_j(B)} \int_B |\nabla_x \Gamma(x,y) a(y)|\, dy\, dx \\
&\le \int_B \left\{\int_{\widetilde{S}_j(B)} |\nabla_x \Gamma(x,y)|\, dx\right\} |a(y)|\, dy \\
&\le \int_B \left\{\int_{\widetilde{S}_j(B)} |x-y|^{1-n}\, dx\right\} |a(y)|\, dy \\
&\lesssim r_0 \|a\|_{L^1(\mathbb{R}^n)} \lesssim r_0 \|a\|_{L^q(\mathbb{R}^n)} |B|^{1/q'} \lesssim r_0 |B| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1},
\end{aligned} \tag{4.4}$$

where $\widetilde{S}_j(B) := 2^{j+2}B \setminus (2^{j-1}B)$ with $j \in \mathbb{Z}_+$, and $2^{-1}B := \emptyset$. When $j \in \mathbb{N}$ with $j \ge 3$, from Lemma 3.1(ii), (2.3) and $r_0 \ge [m(x_0, V)]^{-1}$, it follows that, for all $k \in \mathbb{N}$ and $x \in S_j(B)$,

$$\begin{aligned}
\left|\nabla L^{-1}(a)(x)\right| &\lesssim \int_B \frac{|a(y)|}{[1+|x-y|m(y,V)]^k |x-y|^{n-1}}\, dy \\
&\lesssim \int_B \frac{|a(y)|}{[2^j r_0 m(x_0,V)]^k (2^j r_0)^{n-1}}\, dy \\
&\lesssim 2^{-kj} (2^j r_0)^{1-n} \|a\|_{L^1(\mathbb{R}^n)} \lesssim 2^{-j(k+n-1)} r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1},
\end{aligned}$$

which implies that, for all $j \in \mathbb{N}$ with $j \ge 3$,

$$\mathrm{I}_j \lesssim 2^{-(k-1)j} r_0 |B| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}. \tag{4.5}$$

By (4.3), (4.4) and (4.5), we know that $\nabla L^{-1}(a) \in L^1(\mathbb{R}^n)$.

Now we prove that $\nabla^2 L^{-1}(a) \in L^1(\mathbb{R}^n)$. Write

$$\int_{\mathbb{R}^n} \left|\nabla^2 L^{-1}(a)(x)\right| dx = \sum_{j=0}^{\infty} \int_{S_j(B)} \left|\nabla^2 L^{-1}(a)(x)\right| dx =: \sum_{j=0}^{\infty} \mathrm{II}_j. \tag{4.6}$$

By the choice of q and $q_0 \geq n$, we see that $q < \frac{n}{nq(\varphi)/i(\varphi)-\alpha_0} \leq n/(n-\alpha_0) < n \leq q_0$, which, together with $p \leq q$, implies that $p < q_0$. When $j \in \{0, 1, \ldots, 4\}$, from $p < q_0$, Lemma 3.3 and Hölder's inequality, it follows that

$$\begin{aligned} \mathrm{II}_j &\leq \left\{\int_{S_j(B)} \left|\nabla^2 L^{-1}(a)(x)\right|^p dx\right\}^{1/p} |S_j(B)|^{1/p'} \\ &\lesssim \|a\|_{L^p(\mathbb{R}^n)} |S_j(B)|^{1/p'} \lesssim |B| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \end{aligned} \tag{4.7}$$

For $j \geq 5$, by (3.41) and Hölder's inequality, we see that, for any given $k \in \mathbb{N}$,

$$\begin{aligned} \int_{S_j(B)} \left|\nabla^2 L^{-1}(a)(x)\right| dx &\leq \left\{\int_{S_j(B)} \left|\nabla^2 L^{-1}(a)(x)\right|^q dx\right\}^{1/q} |2^j B|^{1/q'} \\ &\lesssim 2^{-jk} |B| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \end{aligned}$$

From this, (4.6) and (4.7), we deduce that $\nabla^2 L^{-1}(a) \in L^1(\mathbb{R}^n)$.

To prove that $\int_{\mathbb{R}^n} \nabla^2 L^{-1}(a)(x)\, dx = 0$, we borrow some ideas from the proof of [27, Theorem 7.4]. Take a family of functions, $\{\phi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$, such that

(i) $\sum_{j=1}^\infty \phi_j(x) = 1$ for almost every $x \in \mathbb{R}^n$;

(ii) for each $j \in \mathbb{N}$, $\operatorname{supp}(\phi_j) \subset 2B_j$, $\phi_j \equiv 1$ on B_j and $0 \leq \phi_j \leq 1$;

(iii) there exists a positive constant C such that, for all $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$, $|\phi_j(x)| + |\nabla\phi_j(x)| \leq C$;

(iv) there exists $N_0 \in \mathbb{N}$ such that $\sum_{j=1}^\infty \chi_{2B_j} \leq N_0$.

Using the properties of $\{\phi_j\}_{j\in\mathbb{N}}$ and the facts that $\nabla L^{-1}(a),\ \nabla^2 L^{-1}(a) \in L^1(\mathbb{R}^n)$, we see that

$$\int_{\mathbb{R}^n} \nabla^2 L^{-1}(a)(x)\, dx = \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \nabla(\phi_j \nabla L^{-1}(a))(x)\, dx.$$

For each $j \in \mathbb{N}$, let $\eta_j \in C_c^\infty(\mathbb{R}^n)$ such that $\eta_j \equiv 1$ on $2B_j$ and $\operatorname{supp}(\eta_j) \subset 4B_j$. Then, by integral by parts, we conclude that, for each $i \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\partial(\phi_j \nabla L^{-1}(a))(x)}{\partial x_i}\, dx &= \int_{\mathbb{R}^n} \eta_j(x) \frac{\partial(\phi_j \nabla L^{-1}(a))(x)}{\partial x_i}\, dx \\ &= -\int_{\mathbb{R}^n} \phi_j(x) \nabla L^{-1}(a)(x) \frac{\partial \eta_j(x)}{\partial x_i}\, dx = 0, \end{aligned} \tag{4.8}$$

which implies that $\int_{\mathbb{R}^n} \nabla^2 L^{-1}(a)(x)\,dx = 0$. Moreover, from the boundedness of $\nabla^2 L^{-1}$ on $L^p(\mathbb{R}^n)$ and (3.39), we deduce that, for each $j \in \mathbb{Z}_+$ and any given $k \in \mathbb{N}$,

$$\left\|\nabla^2 L^{-1}(a)\right\|_{L^p(S_j(B))} \lesssim 2^{-j(k+n)}|2^j B|^{1/p}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.$$

Thus, $\nabla^2 L^{-1}(a)$ is a harmless constant multiple of a $(\varphi,\, p,\, 0,\, k+n)$-molecule, which is desired.

Case 2. $r_0 \in (0, [m(x_0, V)]^{-1})$. In this case, similar to the proof of Case 1, we need to prove that $\nabla L^{-1}(a),\ \nabla^2 L^{-1}(a) \in L^1(\mathbb{R}^n)$. We first prove $\nabla L^{-1}(a) \in L^1(\mathbb{R}^n)$. From the proof of (4.4), it follows that (4.4) is also valid in this case. When $j \in \mathbb{N}$ with $j \geq 5$, by using Lemma 3.1(iii), $\int_B a(x)\,dx = 0$, the mean valued theorem and Hölder's inequality, similar to [10, (3.14)], we know that, for all $k \in \mathbb{N}$ and $x \in S_j(B)$,

$$\left|\nabla L^{-1}(a)(x)\right| \lesssim 2^{-j(k+n)} r_0 [r_0 m(x_0, V)]^{-k}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}, \tag{4.9}$$

which further implies that

$$\int_{S_j(B)} \left|\nabla L^{-1}(a)(x)\right|\,dx \lesssim 2^{-jk}[r_0 m(x_0, V)]^{-k} r_0 |B| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.$$

From this, (4.3) and (4.4), it follows that $\nabla L^{-1}(a) \in L^1(\mathbb{R}^n)$.

Now we prove that $\nabla^2 L^{-1}(a) \in L^1(\mathbb{R}^n)$. In this case, we see that, for each $j \in \{0, \ldots, 4\}$, (4.7) also holds true. When $j \in \mathbb{N}$ with $j \geq 5$, similar to the proof of [10, (3.14)], we know that, for all $x \in S_j(B)$,

$$\left|L^{-1}(a)(x)\right| \lesssim 2^{j(1-n)} r_0^2 \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)} \text{ and } \left|\nabla L^{-1}(a)(x)\right| \lesssim 2^{-jn} r_0 \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.$$

By this, similar to the proof of (3.39), we conclude that

$$\left\{\int_{S_j(B)} \left|\nabla^2 L^{-1}(a)(x)\right|^p\,dx\right\}^{1/p} \lesssim 2^{-j(n+1)}|2^j B|^{1/p}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}, \tag{4.10}$$

which, together with Hölder's inequality, implies that, for each $j \in \mathbb{N}$ with $j \geq 5$,

$$\int_{S_j(B)} \left|\nabla^2 L^{-1}(a)(x)\right|\,dx \lesssim 2^{-j}|B|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.$$

From this, (4.6) and (4.7), it follows that $\nabla^2 L^{-1}(a) \in L^1(\mathbb{R}^n)$. Then, by (4.8), we see that $\int_{\mathbb{R}^n} \nabla^2 L^{-1}(a)(x)\,dx = 0$. From this and (4.10), we deduce that $\nabla^2 L^{-1}(a)$ is a harmless constant multiple of a $(\varphi,\, p,\, 0,\, n+1)$-molecule, which is desired. This finishes the proof of Theorem 1.6(i).

Now we prove (ii) of this theorem by using Theorem 2.3 and (i). Let $f \in H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and q be as in (4.1). By Theorem 2.3, we conclude that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $(\varphi, q)_m$-atoms such that $f = \sum_j \lambda_j a_j$ in $L^2(\mathbb{R}^n)$ and $\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \Lambda(\{\lambda_j a_j\}_j)$, which, together with Lemma 3.2(i), $q_0 \in [n, \infty)$ and $n \geq 3$, implies that

$$VL^{-1}(f) = \sum_j \lambda_j VL^{-1}(a_j) \tag{4.11}$$

in $L^2(\mathbb{R}^n)$. Let $p \in (1, q]$. To finish the proof of Theorem 1.6(ii), it suffices to prove that, for any $(\varphi, q)_m$-atom a supported in the ball $B := B(x_0, r_0)$, there exist $\{\mu_i\}_i \subset \mathbb{C}$ and a sequence $\{\alpha_i\}_i$ of $(\varphi, p)_m$-atoms such that

$$VL^{-1}(a) = \sum_i \mu_i \alpha_i \tag{4.12}$$

in $L^2(\mathbb{R}^n)$ and, for all $\lambda \in (0, \infty)$,

$$\sum_i \varphi\left(B_i, \frac{|\mu_i|}{\lambda\|\chi_{B_i}\|_{L^\varphi(\mathbb{R}^n)}}\right) \lesssim \varphi\left(B, \frac{1}{\lambda\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right), \tag{4.13}$$

where, for each i, $\operatorname{supp}(\alpha_i) \subset B_i$. If (4.12) and (4.13) hold true, from these, (4.11) and $\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \Lambda(\{\lambda_j a_j\}_j)$, it follows that $VL^{-1}(f) \in H_{\varphi,L}(\mathbb{R}^n)$ and

$$\left\|VL^{-1}(f)\right\|_{H_{\varphi,L}(\mathbb{R}^n)} \lesssim \Lambda(\{\lambda_j a_j\}_j) \lesssim \|f\|_{H_{\varphi,L}(\mathbb{R}^n)},$$

which is desired.

Now we prove (4.12) and (4.13) by considering the following two cases for r_0.

Case 1. $r_0 \in [[2m(x_0, V)]^{-1}, \infty)$. In this case, let $S_j(B)$ with $j \in \mathbb{Z}_+$ be as in the proof of (i). Then,

$$VL^{-1}(a) = \sum_{i=0}^{\infty} VL^{-1}(a)\chi_{S_j(B)} =: \sum_{i=0}^{\infty} b_i. \tag{4.14}$$

By Lemma 3.2(i), we know that

$$\|b_0\|_{L^p(\mathbb{R}^n)} \leq \|VL^{-1}(a)\|_{L^p(\mathbb{R}^n)} \lesssim \|a\|_{L^p(\mathbb{R}^n)} \lesssim |B|^{1/p}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.$$

From this, $\operatorname{supp}(b_1) \subset 2B$ and $2r_0 \geq [m(x_0, V)]^{-1}$, we deduce that there exists a positive constant $\widetilde{C}_0$ such that $b_0/\widetilde{C}_0$ is a (φ, p)-atom. Let $\mu_0 := \widetilde{C}_0$ and $\alpha_0 := b_0/\widetilde{C}_0$. Then, $b_0 = \mu_0\alpha_0$ and, for all $\lambda \in (0, \infty)$,

$$\varphi\left(2B, \frac{\mu_0}{\lambda\|\chi_{2B}\|_{L^\varphi}}\right) \lesssim \varphi\left(B, \frac{1}{\lambda\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right). \tag{4.15}$$

Moreover, by (3.35), (3.39) and Hölder's inequality, we conclude that, for any $i \in \mathbb{N}$ and $k \in (n[q(\varphi)/i(\varphi) - \alpha_0], \infty)$,

$$\begin{aligned}\|b_i\|_{L^p(\mathbb{R}^n)} &= \left\|\operatorname{div}(A\nabla L^{-1}(a))\right\|_{L^p(S_i(B))} \lesssim \left\|\nabla^2 L^{-1}(a)\right\|_{L^p(S_i(B))} \\ &\lesssim 2^{-i(k+n)}|2^{i+1}B|^{1/p}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)},\end{aligned}$$

which, together with $\operatorname{supp}(b_i) \subset 2^{i+1}B$, implies that there exists a positive constant $\widetilde{C}_1$ such that $\frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)} b_i}{\widetilde{C}_1 2^{-i(k+n)}\|\chi_{2^{i+1}B}\|_{L^\varphi(\mathbb{R}^n)}}$ is a $(\varphi, p)_m$-atom. Let

$$\mu_i := \frac{\widetilde{C}_1 2^{-i(k+n)}\|\chi_{2^{i+1}B}\|_{L^\varphi(\mathbb{R}^n)}}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \quad \text{and} \quad \alpha_i := \frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)} b_i}{\widetilde{C}_1 2^{-i(k+n)}\|\chi_{2^{i+1}B}\|_{L^\varphi(\mathbb{R}^n)}}.$$

Then, α_i is a $(\varphi, p)_m$-atom and $b_i = \mu_i \alpha_i$. Moreover, from $k > n[q(\varphi)/i(\varphi) - \alpha_0]$ and the definitions of $i(\varphi)$ and $q(\varphi)$, we deduce that there exist $p_0 \in (0, i(\varphi))$ and $\widetilde{q} \in (q(\varphi), \infty)$ such that φ is of uniformly lower type p_0, $\varphi \in \mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$ and $k + n\alpha_0 > n\widetilde{q}/p_0$. By this, the definition of μ_i and Lemma 2.7(v), we further conclude that, for all $\lambda \in (0, \infty)$,

$$\begin{aligned}&\sum_{i=1}^{\infty} \varphi\left(2^{i+1}B, \frac{|\mu_i|}{\lambda\|\chi_{2^{i+1}B}\|_{L^\varphi(\mathbb{R}^n)}}\right) \\ &\lesssim \sum_{i=1}^{\infty} 2^{-i[(k+n)p_0 - n\widetilde{q}]}\varphi\left(B, \frac{1}{\lambda\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right) \lesssim \varphi\left(B, \frac{1}{\lambda\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\right),\end{aligned} \tag{4.16}$$

which, together with (4.14) and (4.15), implies that (4.12) and (4.13) hold true in this case.

Case 2. $r_0 \in (0, [2m(x_0, V)]^{-1})$. In this case, from Case 2 of the proof of Theorem 1.6(i), it follows that $\nabla L^{-1}(a)$, $\nabla^2 L^{-1}(a) \in L^1(\mathbb{R}^n)$, which, together with (3.35) and the assumption (A_1), implies that $\nabla L^{-1}(a)$, $\operatorname{div}(A\nabla L^{-1}(a)) \in L^1(\mathbb{R}^n)$. Then, by the fact that $\int_{\mathbb{R}^n} a(x)\,dx = 0$ and repeating the proof of (4.8), we conclude that

$$\int_{\mathbb{R}^n} \operatorname{div}(A\nabla L^{-1}(a))(x)\,dx = 0,$$

which, together with $-\operatorname{div}(A\nabla L^{-1}(a)) + VL^{-1}(a) = a$ and $\int_{\mathbb{R}^n} a(x)\,dx = 0$, implies that

$$\int_{\mathbb{R}^n} VL^{-1}(a)(x)\,dx = \int_{\mathbb{R}^n} \left[a(x) + \operatorname{div}(A\nabla L^{-1}(a))(x)\right]\,dx = 0.$$

For $i \in \mathbb{Z}_+$, let $m_i := \int_{S_i(B)} \alpha(x)\,dx$, $\widetilde{\chi}_i := \frac{\chi_{S_i(B)}}{|S_i(B)|}$ and $M_i := \alpha\chi_{S_i(B)} - m_i\widetilde{\chi}_i$. Then

$$VL^{-1}(a) = \sum_{i=0}^{\infty} M_k + \sum_{i=0}^{\infty} m_i\widetilde{\chi}_i.$$

For any $j \in \mathbb{N}_+$, let $N_j := \sum_{k=j}^{\infty} m_k$. From $\int_{\mathbb{R}^n} VL^{-1}(a)(x)\,dx = 0$, it follows that

$$VL^{-1}(a) = \sum_{i=0}^{\infty} M_i + \sum_{i=0}^{\infty} N_{i+1}(\widetilde{\chi}_{i+1} - \widetilde{\chi}_i).$$

It is obvious that, for each $i \in \mathbb{Z}_+$, $\int_{\mathbb{R}^n} M_i(x)\,dx = 0$ and $\int_{\mathbb{R}^n} N_{i+1}[\widetilde{\chi}_{i+1}(x) - \widetilde{\chi}_i(x)]\,dx = 0$. By using Lemma 3.2(i) and (4.10), similar to the proofs of (4.15) and (4.16), we know that there exist $\{\mu_{1,\,i}\}_{i\in\mathbb{Z}_+}$, $\{\mu_{2,\,i}\}_{i\in\mathbb{Z}_+} \subset \mathbb{C}$ and two sequences $\{\alpha_{1,\,i}\}_{i\in\mathbb{Z}_+}$ and $\{\alpha_{2,\,i}\}_{i\in\mathbb{Z}_+}$ of $(\varphi,\,p)_m$-atoms such that $M_i = \mu_{1,\,i}\alpha_{1,\,i}$, $N_{i+1}(\widetilde{\chi}_{i+1} - \widetilde{\chi}_i) = \mu_{2,\,i}\alpha_{2,\,i}$ and, for all $\lambda \in (0,\infty)$,

$$\sum_{i\in\mathbb{Z}_+} \varphi\left(2^{i+1}B, \frac{|\mu_{1,\,i}|}{\lambda\|\chi_{2^{i+1}B}\|_{L^{\varphi}(\mathbb{R}^n)}}\right) + \sum_{i\in\mathbb{Z}_+} \varphi\left(2^{i+1}B, \frac{|\mu_{2,\,i}|}{\lambda\|\chi_{2^{i+1}B}\|_{L^{\varphi}(\mathbb{R}^n)}}\right)$$
$$\lesssim \varphi\left(B, \frac{1}{\lambda\|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}}\right),$$

which, together with (4.16), implies that (4.12) and (4.13) hold true in this case. This finishes the proof of (ii) and hence the proof of Theorem 1.6.

Acknowledgments

The author is very grateful to his advisor Professor Dachun Yang for his guidance and encouragements.

References

1. P. Auscher and B. Ben Ali, Maximal inequalities and Riesz transform estimates on L^p spaces for Schrödinger operators with nonnegative potentials, *Ann. Inst. Fourier* (*Grenoble*), **57** (2007), 1975-2013.
2. P. Auscher, X. T. Duong and A. McIntosh, *Boundedness of Banach Space Valued Singular Integral Operators and Hardy Spaces*, Unpublished Manuscript, 2005.
3. P. Auscher, A. McIntosh and E. Russ, Hardy spaces of differential forms on Riemannian manifolds, *J. Geom. Anal.*, **18** (2008), 192-248.
4. P. Auscher and Ph. Tchamitchian, Square root problem for divergence operators and related topics, *Astérisque*, **249** (1998), viii+172 pp.
5. M. Avellaneda and F. Lin, L^p bounds on singular in homogenization, *Comm. Pure Appl. Math.*, **44** (1991), 897-910.
6. N. Badr and B. Ben Ali, L^p boundedness of the Riesz transform related to Schrödinger operators on a manifold, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **8(5)** (2009), 725-765.
7. A. Bonami, J. Feuto and S. Grellier, Endpoint for the DIV-CURL lemma in Hardy spaces, *Publ. Mat.*, **54** (2010), 341-358.

8. A. Bonami, S. Grellier and L. D. Ky, Paraproducts and products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ through wavelets, *J. Math. Pures Appl. (9)*, **97** (2012), 230-241.

9. A. Bonami, T. Iwaniec, P. Jones and M. Zinsmeister, On the product of functions in BMO and H^1, *Ann. Inst. Fourier* $($*Grenoble*$)$, **57** (2007), 1405-1439.

10. J. Cao, D.-C. Chang, D. Yang and S. Yang, Boundedness of second order Riesz transforms associated to Schrödinger operators on Musielak-Orlicz-Hardy spaces, *Comm. Pure Appl. Anal.*, **13** (2014), 1435-1463.

11. R. R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes, Compensated compactness and Hardy spaces, *J. Math. Pures Appl. (9)*, **72** (1993), 247-286.

12. X. T. Duong and L. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds, *J. Amer. Math. Soc.*, **18** (2005), 943-973.

13. J. Dziubański, Note on H^1 spaces related to degenerate Schrödinger operators, *Illinois J. Math.*, **49** (2005), 1271-1297.

14. J. Dziubański and J. Zienkiewicz, H^p *spaces for Schrödinger operators*, Fourier analysis and related topics, Banach Center Publ., 56, Polish Acad. Sci., Warsaw, 2002, 45-53.

15. J. Dziubański and J. Zienkiewicz, H^p spaces associated with Schrödinger operators with potential from reverse Hölder classes, *Colloq. Math.*, **98** (2003), 5-38.

16. C. Fefferman, The uncertainly principle, *Bull. Amer. Math. Soc.*, **9** (1983), 129-206.

17. C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.*, **129** (1972), 137-193.

18. J. García-Cuerva, Weighted H^p spaces, *Dissertationes Math.* $($*Rozprawy Mat.*$)$, **162** (1979), 1-63.

19. J. García-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, Amsterdam, North-Holland, 1985.

20. F. Gehring, The L^p-integrability of the partial derivatives of a quasiconformal mapping, *Acta Math.*, **130** (1973), 265-277.

21. L. Grafakos, *Modern Fourier Analysis*, 2nd edition, Graduate Texts in Mathematics 250, Springer, New York, 2009.

22. S. Hou, D. Yang and S. Yang, Lusin area function and molecular characterizations of Musielak-Orlicz Hardy spaces and their applications, *Commun. Contemp. Math.*, **15(6)** (2013), 1350029, 37 pp.

23. S. Hofmann, G. Lu, D. Mitrea, M. Mitrea and L. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates, *Mem. Amer. Math. Soc.*, **214** (2011), no. 1007, vi+78 pp.

24. S. Hofmann and S. Mayboroda, Hardy and BMO spaces associated to divergence form elliptic operators, *Math. Ann.*, **344** (2009), 37-116.

25. S. Hofmann, S. Mayboroda and A. McIntosh, Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces, *Ann. Sci. École Norm. Sup. (4)*, **44** (2011), 723-800.

26. S. Janson, Generalizations of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation, *Duke Math. J.*, **47** (1980), 959-982.

27. R. Jiang and D. Yang, New Orlicz-Hardy spaces associated with divergence form elliptic operators, *J. Funct. Anal.*, **258** (2010), 1167-1224.

28. R. Jiang and D. Yang, Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates, *Commun. Contemp. Math.*, **13** (2011), 331-373.

29. R. Jiang, Da. Yang and Do. Yang, Maximal function characterizations of Hardy spaces associated with magnetic Schrödinger operators, *Forum Math.*, **24** (2012), 471-494.

30. R. Jiang, D. Yang and Y. Zhou, Orlicz-Hardy spaces associated with operators, *Sci. China Ser. A*, **52** (2009), 1042-1080.

31. K. Kurata and S. Sugano, A remark on estimates for uniformly elliptic operators on weighted L^p spaces and Morrey spaces, *Math. Nachr.*, **209** (2000), 137-150.

32. L. D. Ky, New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators, *Integral Equations Operator Theory*, **78** (2014), 115-150.

33. J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. 1034, Springer-Verlag, Berlin, 1983.

34. E. Nakai and K. Yabuta, Pointwise multipliers for functions of bounded mean oscillation, *J. Math. Soc. Japan*, **37** (1985), 207-218.

35. E. M. Ouhabaz, *Analysis of Heat Equations on Domains*, Princeton University Press, Princeton, N. J., 2005.

36. M. Rao and Z. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, Inc., New York, 1991.

37. Z. Shen, L^p estimates for Schrödinger operators with certain potential, *Ann. Inst. Fourier (Grenoble)*, **45** (1995), 513-546.

38. H. Smith, Parametrix construction for a class of subelliptic differential operators, *Duke Math. J.*, **63** (1991), 343-354.

39. E. M. Stein and G. Weiss, On the theory of harmonic functions of several variables. I. The theory of H^p-spaces, *Acta Math.*, **103** (1960), 25-62.

40. J.-O. Strömberg, Bounded mean oscillation with Orlicz norms and duality of Hardy spaces, *Indiana Univ. Math. J.*, **28** (1979), 511-544.

41. J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer-Verlag, Berlin, 1989.

42. L. Tang, *Weighted norm inequalities for commutators of Littlewood-Paley functions related to Schrödinger operators*, arXiv: 1109.0100.

43. L. Yan, Classes of Hardy spaces associated with operators, duality theorem and applications, *Trans. Amer. Math. Soc.*, **360** (2008), 4383-4408.

44. D. Yang and S. Yang, Local Hardy spaces of Musielak-Orlicz type and their applications, *Sci. China Math.*, **55** (2012), 1677-1720.

45. D. Yang and S. Yang, Musielak-Orlicz Hardy spaces associated with operators and their applications, *J. Geom. Anal.*, **24** (2014), 495-570.

46. J. Zhong, The Sobolev estimates for some Schrödinger type operators, *Math. Sci. Res. Hot-Line*, **3** (1999), 1-48.

Sibei Yang
School of Mathematics and Statistics
Lanzhou University
Lanzhou, Gansu 730000
People's Republic of China
E-mail: yangsb@lzu.edu.cn