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APPROXIMATE CONTROLLABILITY OF FRACTIONAL ORDER STOCHASTIC VARIATIONAL INEQUALITIES DRIVEN BY POISSON JUMPS

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Abstract. This paper proposes the sufficient conditions of approximate controllability for a class of fractional order stochastic variational inequalities driven by Poisson jumps. The possibilities of finding the approximate controllability of a given problem of this type introduce the smoothing system corresponding to the fractional order stochastic variational inequalities driven by Poisson jumps. The results are achieved upon the Moreau-Yosida approximation of subdifferential operator. Sufficient conditions for the approximate controllability of smoothing system are discussed under the boundedness condition on control operator. The results are formulated and proved by using the fractional calculus, semigroup theory, stochastic analysis techniques. An example is provided to illustrate the obtained theory.

1. INTRODUCTION

The theory of nonlinear fractional differential or integro-differential equations has become an active area of investigation due to their applications in nonlinear oscillations of earthquakes, viscoelasticity, electrochemistry, electromagnetic theory, and in fluid dynamic traffic models (see [11] and references therein). The qualitative properties of fractional differential equations have been done in [14]. It is well known that the issue of controllability plays an important role in control theory and engineering (see [6]) because they have close connections to pole assignment, structural decomposition, quadratic optimal control and observer design etc. Recently, the controllability for different kinds of fractional differential systems in abstract spaces have been generated with considerable interest among researchers. Ahmed [1] studied controllability of

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fractional stochastic delay equations. If the semigroup is compact, then the assumption (iv) in section 2 of [1] is valid only if the state space is finite dimensional [26]. On the other hand, in infinite dimensional spaces, the concept of exact controllability is usually too strong and the approximate controllability is more appropriate for these control systems. Therefore, approximate controllability problems for deterministic and stochastic dynamical systems in infinite dimensional spaces is well developed using different kind of approaches [8, 19]. Kumar et al. [13] discussed the sufficient conditions of approximate controllability of deterministic fractional order neutral control systems with delay. Sakthivel et al. [23] studied the approximate controllability results of fractional stochastic evolution equations by using Banach contraction mapping principle. Wang et al. [28] proved the adaptive neural tracking control for a class of stochastic nonlinear systems with unknown dead-zone.

The earlier reports are on the qualitative properties of stochastic differential equations and their applications on systems driven by a Brownian motion (or Wiener process) (see [19, 23]). However, the stochastic differential equations driven by a Poisson process can be widely found in applications from various fields such as storage systems, queueing systems, economic systems and neurophysiology systems, etc, (see [4, 24, 27]). Thus, it is very important to study the qualitative properties of stochastic differential equations driven by a Poisson process in both theoretic and application aspects (see [29]). Knoche [12] discussed the existence and uniqueness of the mild solutions to stochastic evolution equations driven by Poisson jump processes. Cui et al. [5] proved the existence and uniqueness of mild solutions of neutral stochastic evolution equations with infinite delay and Poisson jumps by using successive approximation. Li et al. [15] studied the robust quantized H_∞ control for network control systems with markovian jumps and time delays. Ren et al. [21] discussed the existence, uniqueness and stability of mild solutions for time dependent stochastic evolution equations with Poisson jumps and infinite delay under non-Lipschitz condition with Lipschitz condition being considered as a special case. Sakthivel et al. [22] studied the complete controllability of stochastic evolution equations with jumps without assuming the compactness of the semigroup property. Long et al. [16] proved the sufficient condition for the approximate controllability of SPDE with infinite delays driven by Poisson jumps by using the Krasnoselski-Schaefer fixed point theorem. Luo et al. [17] discussed sufficient conditions for the existence and uniqueness for non-Lipschitz stochastic neutral delay evolution equations driven by Poisson jumps.

The study of evolution problems where the state of the system is subject to some set of constraints has a long history and its beginnings are nearly simultaneous to the early studies of variational inequalities. An elementary example of variational inequality problems are the simple deformation of a beam constrained by an obstacle, nonlinear obstacle problem and describing diffusion in a domain with a semipermeable boundary (see [2]). Huyen Dam [7] discussed the variable fractional delay filter with sub-expression coefficients. Bensoussan et al. [3] investigated the existence of solutions for stochastic variational inequalities in infinite dimensional spaces. Rascanu [20] studied the existence for a class of stochastic parabolic variational inequalities. Recently, Jeong et al. [9, 10] studied the approximate controllability for the nonlinear functional differential control problem governed by variational inequality. To the best of our knowledge, the approximate controllability for a class of fractional order stochastic variational inequalities driven by Poisson jumps is an untreated topic in the literature and this fact is the motivation of the present work.

2. Preliminaries

The purpose of this paper is to investigate the approximate controllability of fractional order stochastic variational inequalities driven by Poisson jumps of the form

$$\begin{aligned} {}^cD_t^\alpha x(t) + Ax(t) + \partial\phi(x(t)) \ni Bu(t) + f(t, x(t)) \\ + \int_Z g(t, x(t), \eta)\tilde{N}(dt, d\eta), t \in J = [0, b], x(0) = x_0, \end{aligned} \tag{1}$$

where ${}^cD_t^\alpha$ denotes the Caputo fractional derivative operator of order $0 < \alpha < 1$. Let H and V be two real separable Hilbert spaces such that V is a dense subspace in H and the injection of V into H is continuous. The norms on V and H will be denoted by $\|\cdot\|_V$ and $\|\cdot\|_H$. Identifying the antidual of H with H, we may consider $V \subset H \subset V^*$. Let A be the operator associated with a sequilinear form defined on $V \times V$ satisfying Gardings inequality. $-A$ generates an analytic semigroup $T(t)$ in both of H and V^* as is seen in [25]. The realization for the operator A in H which is the restriction of A to $D(A) = \{x \in V; Ax \in H\}$ be also denoted by A. Let $\phi : V \to (-\infty, \infty]$ be a lower semicontinuous, proper convex function. Then the subdifferential operator $\partial\phi$ of ϕ is denoted by $\partial\phi(x) = \{x^* \in V^*; \phi(x) \leq \phi(y) + (x^*, x - y), y \in V\}$. The control function $u(\cdot)$ is given in $L_2([0, b], U)$ of admissible control functions, U is a Hilbert space. B be a bounded linear operator from U to H. Let K be an another separable Hilbert space. Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, P)$ be a complete probability space with the filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is the filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ is increasing and right continuous, and $\mathfrak{F}_0$ contains all $P-$ null sets of $\mathfrak{F}$. Let $q = (q(t)), t \in D_q$ be a stationary $\mathfrak{F}_t-$ Poisson point process with a characteristic measure λ. Let $N(dt, d\eta)$ be the Poisson counting measure associated with q. Thus, we have $N(t, Z) = \sum_{s\in D_q, s\leq t} I_Z(q(s))$ with measurable set $Z \in \overline{B}(K - \{0\})$, which denotes the Borel $\sigma-$ field of $K - \{0\}$. Let $\tilde{N}(dt, d\eta) = N(dt, d\eta) - dt\lambda(d\eta)$ be the compensated Poisson measure that is independent of Brownian motion. Let $p_2([0, b] \times Z; H)$ be the space of all predictable mappings $\chi : [0, b] \times Z \to H$ for which $\int_0^b \int_Z E\|\chi(t, \eta)\|_H^2 dt\lambda(d\eta) < \infty$. Then, we can define the $H-$ valued stochastic integral $\int_0^b \int_Z \chi(t, \eta)\tilde{N}(dt, d\eta)$, which is a centred square-integrable martingale. $f : J \times V \to H$ and $g : J \times V \times Z \to H$ are Borel measurable functions.

Definition 2.1. The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \quad t>0, \quad \alpha>0,$$

provided the right hand side is point wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Caputo derivative of order α with the lower limit 0 for a function f can be written as

$${}^{c}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}ds=I^{n-\alpha}f^n(t), \quad t>0, \quad 0\le n-1<\alpha<n.$$

The Caputo derivative of a constant equals to zero. If f is an abstract function with values in H, then the integrals which appear in the above definitions are taken in Bochner's sense (see [18]).

For every $\epsilon>0$, define the Moreau-Yosida approximation of ϕ as

$$\phi_{\epsilon}(x)=\inf\{\|x-y\|_*^2/2\epsilon+\phi(y); y\in H\}.$$

Then the function ϕ_ϵ is Frechet differentiable on H and its Frechet differential $\partial\phi_\epsilon$ is Lipschitz continuous on H with Lipschitz constant ϵ^{-1} where $\partial\phi_\epsilon=\epsilon^{-1}(I-(I+\epsilon\partial\phi)^{-1}]$ as seen in Corollary 2.2 in chapter II of [2]. It is also well known that the result $\lim_{\epsilon\to 0}\phi_\epsilon=\phi$ and $\lim_{\epsilon\to 0}\partial\phi_\epsilon(x)=(\partial\phi)^0(x)$ for every $x\in D(\partial\phi)$, where $(\partial\phi)^0: H\to H$ is the minimum element of $\partial\phi$. Now, we introduce the smoothing system corresponding to (1) as follows

$$\begin{aligned}{}^{c}D_t^{\alpha}x(t)+Ax(t)+\partial\phi_\epsilon(x(t)) &= Bu(t)+f(t,x(t))\\ &\quad+\int_Z g(t,x(t),\eta)\tilde{N}(dt,d\eta), t\in J,\ x(0)=x_0.\end{aligned} \tag{2}$$

Definition 2.3. An H- valued stochastic process $\{x_\epsilon(t), t\in J\}$ is a mild solution of (2) if for each $u\in L_2^{\mathfrak{F}}(J,U)$, it satisfies the following integral equation

$$\begin{aligned}x_\epsilon(t)=&\widehat{T}_\alpha(t)x_0+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)[Bu(s)+f(s,x_\epsilon(s))-\partial\phi_\epsilon(x_\epsilon(s))]ds\\ &+\int_0^t\int_Z(t-s)^{\alpha-1}T_\alpha(t-s)g(s,x_\epsilon(s),\eta)\tilde{N}(ds,d\eta),\end{aligned} \tag{3}$$

where $\widehat{T}_\alpha(t)x=\int_0^\infty \eta_\alpha(\theta)T(t^\alpha\theta)xd\theta$, $T_\alpha(t)x=\alpha\int_0^\infty\theta\eta_\alpha(\theta)T(t^\alpha\theta)xd\theta$, $\eta_\alpha(\theta)=\frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\overline{w}_\alpha(\theta^{-\frac{1}{\alpha}})\ge 0$, $\overline{w}_\alpha(\theta)=\frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}sin(n\pi\alpha)$, $\theta\in(0,\infty)$, η_α is a probability density function defined on $(0,\infty)$, that is $\eta_\alpha(\theta)\ge 0$, $\theta\in(0,\infty)$ and $\int_0^\infty\eta_\alpha(\theta)d\theta=1$.

Lemma 2.4. (see [30]). *For any fixed $t \geq 0$, the operator $\widehat{T}_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators, i.e., for any $x \in H$, $\|\widehat{T}_\alpha(t)x\| \leq M\|x\|$ and $\|T_\alpha(t)x\| \leq \frac{M\alpha}{\Gamma(1+\alpha)}\|x\|$.*

Let $x_\epsilon(b; x_0, u)$ be the state value of the system (2) at terminal time b corresponding to the control u and the initial value x_0. Introduce the set $\mathcal{R}(b; x_0, u) = \{x_\epsilon(b; x_0, u), u \in L_2^{\mathfrak{F}}(J, U)\}$, which is called the reachable set of the system (2) at terminal time b.

Definition 2.5. The system (2) is said to be approximately controllable on J if $\overline{\mathcal{R}(b; x_0, u)} = L_2(\Omega, H)$, where $\overline{\mathcal{R}(b; x_0, u)}$ is the closure of $\mathcal{R}(b; x_0, u)$.

To prove our results, we impose the following hypotheses:

(H_1) The nonlinear function $f : J \times V \to H$ satisfies the Lipschitz condition and there exists a positive constant $M_f > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\|_H^2 \leq M_f \|x_1 - x_2\|_V^2,$$

for all $x_1, x_2 \in V$.

(H_2) The nonlinear function $g : J \times V \times Z \to H$ satisfies the Lipschitz condition and there exists positive constants $M_g, L_g > 0$ such that

$$\int_Z \|g(t, x_1, \eta) - g(t, x_2, \eta)\|_H^2 \lambda(d\eta) \leq M_g \|x_1 - x_2\|_V^2,$$

$$\int_Z \|g(t, x_1, \eta) - g(t, x_2, \eta)\|_H^4 \lambda(d\eta) \leq L_g \|x_1 - x_2\|_V^4,$$

for all $x_1, x_2 \in V$.

(H_3) $(\partial\phi)^0$ is uniformly bounded, i.e., $\|(\partial\phi)^0 x\|^2 \leq M_1$, $x \in H$.

Consider the following L_2- regularity for the abstract fractional linear parabolic equation

$$\begin{aligned} {}^cD_t^\alpha x(t) + Ax(t) &= k(t), \quad t \in J, \\ x(0) &= x_0, \end{aligned} \tag{4}$$

has a unique solution $x(t)$ in $[0, b]$ for each $b > 0$ and the bounded linear operator $k(t) \in L_2(0, b; H)$ is taken instead of the control term $Bu(t)$ (see [10]).

Lemma 2.6. (see [9, 10]). *Let $x_0 \in H$ and $k \in L_2(0, b; V^*)$, $b > 0$. Then there exists a unique solution $x(t)$ of (4) belonging to $C([0, b]; H) \cap L_2(0, b; V)$ and satisfying*

$$\|x(t)\|_{C([0,b];H)\cap L_2(0,b;V)} \leq M_2\Big(\|x_0\| + \|k\|_{L_2(0,b;V^*)}\Big),$$

where M_2 is a constant depending on b.

Lemma 2.7. *Let x_ϵ and x_ϑ be the solutions of (2) with same control u, then there exists a non-negative constant M_3 independent of ϵ and ϑ such that*

$$E\|x_\epsilon - x_\vartheta\|^2_{C([0,b];H)\cap L_2(0,b;V)} \leq M_3, \quad 0 \leq t \leq b.$$

Proof. For given $\epsilon, \vartheta > 0$, let x_ϵ and x_ϑ be the solutions of (2) corresponding to ϵ and ϑ, respectively. Then from the equation (3), we have

$$\begin{aligned}
&E\|x_\epsilon(t) - x_\vartheta(t)\|^2 \\
&\leq 3E\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(s,x_\epsilon(s)) - f(s,x_\vartheta(s))]ds\|^2 \\
&\quad + 3E\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[\partial\phi_\epsilon(x_\epsilon(s)) - \partial\phi_\vartheta(x_\vartheta(s))]ds\|^2 \\
&\quad + 3E\|\int_0^t\int_Z (t-s)^{\alpha-1}T_\alpha(t-s)[g(s,x_\epsilon(s),\eta) - g(s,x_\vartheta(s),\eta)]\tilde{N}(ds,d\eta)\|^2, \\
&\leq 3\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1}\int_0^t E\|f(s,x_\epsilon(s)) - f(s,x_\vartheta(s))\|^2 ds + 3\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1} \\
&\quad \times \int_0^t E\|(\partial\phi)^0x_\epsilon(s) - (\partial\phi)^0x_\vartheta(s)\|^2 ds + 3\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1} \\
&\quad \times \Big\{\int_0^t\int_Z E\|g(s,x_\epsilon(s),\eta) - g(s,x_\vartheta(s),\eta)\|^2\lambda(d\eta)ds \\
&\quad + \Big(\int_0^t\int_Z E\|g(s,x_\epsilon(s),\eta) - g(s,x_\vartheta(s),\eta)\|^4\lambda(d\eta)ds\Big)^{1/2}\Big\}, \\
&\leq 3\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1}(M_f + M_g + \sqrt{L_g})\int_0^t E\|x_\epsilon(s) - x_\vartheta(s)\|^2 ds \\
&\quad + 3\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1}\int_0^t \|(\partial\phi)^0(x_\epsilon(s) - x_\vartheta(s))\|^2 ds, \\
&\leq 3\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1}(M_f + M_g + \sqrt{L_g})\int_0^t E\|x_\epsilon(s) - x_\vartheta(s)\|^2 ds \\
&\quad + 3\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1}M_1 b.
\end{aligned}$$

Here, we used $\partial\phi_\epsilon(x_\epsilon(t)) = \epsilon^{-1}(I - (I + \epsilon\partial\phi)^{-1})x_\epsilon(t)$ and $\|\partial\phi_\epsilon(x)\| \leq \|(\partial\phi)^0 x\|$ for every $x \in D(\partial\phi)$ and by using the Grownwall inequality

$$\begin{aligned}
&E\|x_\epsilon(t) - x_\vartheta(t)\|^2 \\
&\leq 3\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}M_1\exp\Big\{3\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}(M_f + M_g + \sqrt{L_g})\Big\} = M_3. \quad \blacksquare
\end{aligned}$$

Theorem 2.8. *Assume that the hypotheses* $(H_1)-(H_3)$ *hold. Then* $x=\lim_{\epsilon\to 0}x_\epsilon$ *in* $L_2(0,b;V)\cap C([0,b];H)$ *is a solution of (1), where* x_ϵ *is the solution of (2).*

Proof. From Theorem 3.2 in [9], Theorem 3.1 in [10] and Lemma 2.7 in this section, there exists $x(\cdot)\in L_2(0,b;V)$ such that $x_\epsilon(\cdot)\to x(\cdot)$ in $L_2(0,b;V)\cap C([0,b];H)$.

(5) From (H_1), $\quad f(\cdot,x_\epsilon)\to f(\cdot,x) \qquad$ strongly in $\quad L_2(0,b;H)$.

(6) From (H_2), $\quad \int_Z g(t,x_\epsilon(t),\eta)\tilde{N}(dt,d\eta)\to\int_Z g(t,x(t),\eta)\tilde{N}(dt,d\eta)$ stronglyin $\quad L_2(0,b;H)$,

(7) and $\qquad Ax_n\to Ax \qquad$ strongly in $\quad L_2(0,b;V^*)$.

From (H_3), $\partial\phi_\epsilon(x_\epsilon)$ is uniformly bounded, also by using (5), (6), (7), We have

$$^cD_t^\alpha x_\epsilon\to{}^c D_t^\alpha x,\qquad \text{weakly in}\quad L_2(0,b;V^*).$$

Therefore

$$\partial\phi_\epsilon(x_\epsilon)\to f(\cdot,x)+\int_Z g(t,x(t),\eta)\tilde{N}(dt,d\eta)+k-{}^c D_t^\alpha x-Ax$$
$$\text{weakly in}\quad L_2(0,b;V^*).$$

Note that $\partial\phi_\epsilon(x_\epsilon(t))=\epsilon^{-1}(I-(I+\epsilon\partial\phi)^{-1})x_\epsilon(t)$. Since $(I+\epsilon\partial\phi)^{-1}x_\epsilon\to x$ strongly and $\partial\phi$ is demiclosed, we have

$$f(\cdot,x)+\int_Z g(t,x(t),\eta)\tilde{N}(dt,d\eta)+k-{}^c D_t^\alpha x-Ax\in\partial\phi(x)\qquad\text{in}\quad L_2(0,b;V^*).$$

Thus, we have proved that $x(t)$ satisfies a.e on $(0,b)$ the equation (1). ■

3. Approximate Controllability

In this section sufficient conditions are established for the approximate controllability of the system (2) under the boundedness condition on the control operator. First, we consider the sufficient conditions for the approximate controllability of the following linear deterministic system associated with (2) with $f,g\equiv 0$ have discussed in [8] and [13],

$$^cD_t^\alpha x(t)+\partial\phi_\epsilon(x(t))=-Ax(t)+Bu(t),\quad t\in J,$$
$$x(0)=x_0.$$

Let us define the linear operator $\widehat{S}$ from $L_2(0,b;H)$ to H by

$$\widehat{S}p=\int_0^b(b-s)^{\alpha-1}T_\alpha(b-s)p(s)ds,\qquad\text{for}\quad p\in L_2(0,b;H)$$

The system (2) is approximately controllable on J if for any $\epsilon > 0$ and $\xi_b \in H$, there exists a control $u \in L_2(0, b; U)$ such that

$$E\|\xi_b - \widehat{T}_\alpha(b)x_0 - \widehat{S}\{f(\cdot, x_\epsilon(\cdot)) - \partial\phi_\epsilon(x_\epsilon(\cdot))\}$$
$$- \int_0^b \int_Z (b-s)^{\alpha-1} T_\alpha(b-s) g(s, x_\epsilon(s), \eta)\tilde{N}(ds, d\eta) - \widehat{S}Bu\|^2 < \epsilon.$$

To this purpose, the following hypothesis is needed;

(H_4) For any $\epsilon > 0$ and $p \in L_2(0, b; H)$, there exists a $u \in L_2(0, b; U)$ such that
$$E\|\widehat{S}p - \widehat{S}Bu\|^2 < \epsilon,$$
$$\|Bu\|^2_{L_2(0,t;H)} \leq N\|p\|^2_{L_2(0,t;H)}, \quad 0 \leq t \leq b,$$
where N is a constant independent of p.

To prove the approximate controllability of the system (2), the following lemma is required.

Lemma 3.1. *Let u_1 and u_2 be in $L_2(0, b; U)$. Then, under the hypotheses $(H_1) - (H_2)$, we have*

$$E\|x_{\epsilon 1}(t; u_1) - x_{\epsilon 2}(t; u_2)\|^2$$
$$\leq 4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2 \frac{b^{2\alpha}}{2\alpha - 1}\|Bu_1 - Bu_2\|^2_{L_2(0,b;H)}$$
$$\exp\Big\{4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2 \frac{b^{2\alpha}}{2\alpha-1}(M_f + \epsilon^{-1} + M_g + \sqrt{L_g})\Big\}$$

Proof.

$$E\|x_{\epsilon 1}(t; u_1) - x_{\epsilon 2}(t; u_2)\|^2$$
$$\leq 4E\|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu_1(s) - Bu_2(s)]ds\|^2$$
$$+4E\|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[f(s, x_{\epsilon 1}(s; u_1)) - f(s, x_{\epsilon 2}(s; u_2))]ds\|^2$$
$$+4E\|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[\partial\phi_\epsilon(x_{\epsilon 1}(s; u_1)) - \partial\phi_\epsilon(x_{\epsilon 2}(s; u_2))]ds\|^2$$
$$+4E\|\int_0^t\int_Z (t-s)^{\alpha-1} T_\alpha(t-s)[g(s, x_{\epsilon 1}(s; u_1), \eta) - g(s, x_{\epsilon 2}(s; u_2), \eta)]\tilde{N}(ds, d\eta)\|^2,$$
$$\leq 4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2 \frac{b^{2\alpha-1}}{2\alpha - 1}\int_0^t E\|Bu_1(s) - Bu_2(s)\|^2 ds$$

$$+4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1}(M_f+\epsilon^{-1})$$
$$\times\int_0^t E\|x_{\epsilon 1}(s;u_1)-x_{\epsilon 2}(s;u_2)\|^2ds+4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1}$$
$$\times\Big(\int_0^t\int_Z E\|g(s,x_{\epsilon 1}(s;u_1),\eta)-g(s,x_{\epsilon 2}(s;u_2),\eta)\|^2\lambda(d\eta)ds$$
$$+\Big(\int_0^t\int_Z E\|g(s,x_{\epsilon 1}(s;u_1),\eta)-g(s,x_{\epsilon 2}(s;u_2),\eta)\|^4\lambda(d\eta)ds\Big)^{1/2}\Big),$$
$$\leq 4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1}\int_0^t E\|Bu_1(s)-Bu_2(s)\|^2ds+4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha-1}}{2\alpha-1}$$
$$\times(M_f+\epsilon^{-1}+M_g+\sqrt{L_g})\int_0^t E\|x_{\epsilon 1}(s;u_1)-x_{\epsilon 2}(s;u_2)\|^2ds,$$

by using the Gronwall's inequality

$$\leq 4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}\|Bu_1-Bu_2\|^2_{L_2(0,b;H)}$$
$$\exp\Big\{4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}(M_f+\epsilon^{-1}+M_g+\sqrt{L_g})\Big\}.\qquad\blacksquare$$

Theorem 3.2. *Under the hypotheses* $(H_1),(H_2)$ *and* (H_4)*, the system (2) is approximately controllable on* $[0,b]$.

Proof. Let us show that $D(A)\subset\overline{\mathcal{R}(\cdot)}$, i.e, for given $\epsilon>0$ and $\xi_b\in D(A)$, there exists a $u\in L_2(0,b;U)$ such that

$$E\|\xi_b-x_\epsilon(b;u)\|^2<\epsilon,$$

where $x_\epsilon(b;u)=\widehat{T}_\alpha(b)x_0+\int_0^b(b-s)^{\alpha-1}T_\alpha(b-s)[Bu(s)+f(s,x_\epsilon(s;u))-\partial\phi_\epsilon(x_\epsilon(s;u))]ds+\int_0^b\int_Z(b-s)^{\alpha-1}T_\alpha(b-s)g(s,x_\epsilon(s;u),\eta)\tilde{N}(ds,d\eta)$. Because $\xi_b\in D(A)$, there exists a $q_1\in L_2(0,b;H)$ such that $\widehat{S}q_1=\xi_b-\widehat{T}_\alpha(b)x_0$, for instance, take $q_1(s)=(\xi_b-sA\xi_b)-\widehat{T}_\alpha(s)x_0/b$. Let $u_1\in L_2(0,b;U)$ be arbitrary fixed. Because by (H_4), there exists a $u_2\in L_2(0,b;U)$ such that

$$E\|\widehat{S}\Big(q_1-f(s,x_\epsilon(s;u_1))+\partial\phi_\epsilon(x_\epsilon(s;u_1))\Big)$$
$$-\int_0^b\int_Z(b-s)^{\alpha-1}T_\alpha(b-s)g(s,x_\epsilon(s;u_1))\tilde{N}(ds,d\eta)-\widehat{S}Bu_2\|^2<\frac{\epsilon}{2^2},$$

it follows that

$$
\begin{aligned}
& E\|\xi_b - \widehat{T}_\alpha(b)x_0 - \widehat{S}f(s, x_\epsilon(s; u_1)) + \widehat{S}\partial\phi_\epsilon(x_\epsilon(s; u_1)) \\
& \quad - \int_0^b \int_Z (b-s)^{\alpha-1} T_\alpha(b-s) g(s, x_\epsilon(s; u_1)) \tilde{N}(ds, d\eta) - \widehat{S}Bu_2\|^2 < \frac{\epsilon}{2^2}.
\end{aligned} \tag{8}
$$

It can be chosen that $v_2 \in L_2(0, b; U)$ by the hypothesis (H_4) such that

$$
\begin{aligned}
& E\|\widehat{S}\Big(f(\cdot, x_\epsilon(\cdot; u_2)) - f(\cdot, x_\epsilon(\cdot; u_1))\Big) + \widehat{S}\Big(\partial\phi_\epsilon(x_\epsilon(\cdot; u_2)) - \partial\phi_\epsilon(x_\epsilon(\cdot; u_1))\Big) \\
& \quad + \int_0^b \int_Z (b-s)^{\alpha-1} T_\alpha(b-s)[g(s, x_\epsilon(s; u_2), \eta) \\
& \quad - g(s, x_\epsilon(s; u_1), \eta)] \tilde{N}(ds, d\eta) - \widehat{S}Bv_2\|^2 < \frac{\epsilon}{2^3},
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
\|Bv_2\|^2_{L_2(0,t;H)} \le N\Big\{ & \|f(\cdot, x_\epsilon(\cdot; u_2)) - f(\cdot, x_\epsilon(\cdot; u_1))\|^2_{L_2(0,t;H)} \\
& + \|\partial\phi_\epsilon(x_\epsilon(\cdot; u_2)) - \partial\phi_\epsilon(x_\epsilon(\cdot; u_1))\|^2_{L_2(0,t;H)} \\
& + \|\int_0^t \int_Z [g(s, x_\epsilon(s; u_2), \eta) - g(s, x_\epsilon(s; u_1), \eta)] \tilde{N}(ds, d\eta)\|^2_{L_2(0,t;H)} \Big\},
\end{aligned}
$$

for $0 \le t \le b$. Therefore, in view of Hypotheses (H_1), (H_2) and (H_4) and Lemma 3.1

$$
\begin{aligned}
& \|Bv_2\|^2_{L_2(0,t;H)} \\
& \le N\Big\{ \int_0^t E\|f(\tau, x_\epsilon(\tau; u_2)) - f(\tau, x_\epsilon(\tau; u_1))\|^2 d\tau \\
& \quad + \int_0^t E\|\partial\phi_\epsilon(x_\epsilon(\tau; u_2)) - \partial\phi_\epsilon(x_\epsilon(\tau; u_1))\|^2 d\tau \\
& \quad + \int_0^t \int_Z E\|g(\tau, x_\epsilon(\tau; u_2), \eta) - g(\tau, x_\epsilon(\tau; u_1), \eta)\|^2 \lambda(d\eta) d\tau \\
& \quad + \Big(\int_0^t \int_Z E\|g(\tau, x_\epsilon(\tau; u_2), \eta) - g(\tau, x_\epsilon(\tau; u_1), \eta)\|^4 \lambda(d\eta) d\tau \Big)^{1/2} \Big\} \\
& \le N\{M_f + \epsilon^{-1} + M_g + \sqrt{L_g}\} \int_0^t E\|x_\epsilon(\tau; u_2) - x_\epsilon(\tau; u_1)\|^2 d\tau, \\
& \le 4N\{M_f + \epsilon^{-1} + M_g + \sqrt{L_g}\} \Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2 \frac{1}{2\alpha - 1} \\
& \quad \times \exp\Big\{ 4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2 \frac{b^{2\alpha}}{2\alpha - 1} (M_f + \epsilon^{-1} + M_g + \sqrt{L_g}) \Big\} \\
& \quad \int_0^t \tau^{2\alpha} \|Bu_2 - Bu_1\|^2_{L_2(0,t;H)} d\tau,
\end{aligned}
$$

$$\leq 4N\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{1}{2\alpha-1}\{M_f+\epsilon^{-1}+M_g+\sqrt{L_g}\}$$
$$\times \exp\Big\{4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}(M_f+\epsilon^{-1}+M_g+\sqrt{L_g})\Big\}$$
$$\frac{t^{2\alpha+1}}{2\alpha+1}\|Bu_2-Bu_1\|^2_{L_2(0,t;H)}.$$

Put $u_3=u_2-v_2$, we determine v_3 such that

$$E\|\widehat{S}\Big(f(\cdot,x_\epsilon(\cdot;u_3))-f(\cdot,x_\epsilon(\cdot;u_2))\Big)$$
$$+\widehat{S}\Big(\partial\phi_\epsilon(x_\epsilon(\cdot;u_3))-\partial\phi_\epsilon(x_\epsilon(\cdot;u_2))\Big)+\int_0^b\int_Z(b-s)^{\alpha-1}$$
$$\times T_\alpha(b-s)[g(s,x_\epsilon(s;u_3),\eta)-g(s,x_\epsilon(s;u_2),\eta)]\tilde{N}(ds,d\eta)-\widehat{S}Bv_3\|^2<\frac{\epsilon}{2^3},$$

and

$$\|Bv_3\|^2_{L_2(0,t;H)}\leq N\Big\{\|f(\cdot,x_\epsilon(\cdot;u_3))-f(\cdot,x_\epsilon(\cdot;u_2))\|^2_{L_2(0,t;H)}$$
$$+\|\partial\phi_\epsilon(x_\epsilon(\cdot;u_3))-\partial\phi_\epsilon(x_\epsilon(\cdot;u_2))\|^2_{L_2(0,t;H)}$$
$$+\|\int_0^t\int_Z[g(s,x_\epsilon(s;u_3),\eta)-g(s,x_\epsilon(s;u_2),\eta)]\tilde{N}(ds,d\eta)\|^2_{L_2(0,t;H)}\Big\},$$

for $0\leq t\leq b$. We have

$$\|Bv_3\|^2_{L_2(0,t;H)}$$
$$\leq N\Big\{\int_0^t E\|f(\tau,x_\epsilon(\tau;u_3))-f(\tau,x_\epsilon(\tau;u_2))\|^2d\tau$$
$$+\int_0^t E\|\partial\phi_\epsilon(x_\epsilon(\tau;u_3))-\partial\phi_\epsilon(x_\epsilon(\tau;u_2))\|^2d\tau$$
$$+\int_0^t\int_Z E\|g(\tau,x_\epsilon(\tau;u_3),\eta)-g(\tau,x_\epsilon(\tau;u_2),\eta)\|^2\lambda(d\eta)d\tau$$
$$+\Big(\int_0^t\int_Z E\|g(\tau,x_\epsilon(\tau;u_3),\eta)-g(\tau,x_\epsilon(\tau;u_2),\eta)\|^4\lambda(d\eta)d\tau\Big)^{1/2}\Big\}$$
$$\leq N\{M_f+\epsilon^{-1}+M_g+\sqrt{L_g}\}\int_0^t E\|x(\tau;u_3)-x(\tau;u_2)\|^2d\tau,$$
$$\leq 4N\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{1}{2\alpha-1}\{M_f+\epsilon^{-1}+M_g+\sqrt{L_g}\}\exp\Big\{4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}$$

$$\times(M_f+\epsilon^{-1}+M_g+\sqrt{L_g})\Big\}\int_0^t \tau^{2\alpha}\|Bu_3-Bu_2\|^2_{L_2(0,t;H)}d\tau,$$

$$\leq 4N\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{1}{2\alpha-1}\{M_f+\epsilon^{-1}+M_g+\sqrt{L_g}\}\exp\Big\{4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}$$

$$\times(M_f+\epsilon^{-1}+M_g+\sqrt{L_g})\Big\}\int_0^t \tau^{2\alpha}\|Bv_2\|^2_{L_2(0,t;H)}d\tau,$$

$$\leq \Big\{4N\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{1}{2\alpha-1}\{M_f+\epsilon^{-1}+M_g+\sqrt{L_g}\}\exp\Big\{4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}$$

$$\times(M_f+\epsilon^{-1}+M_g+\sqrt{L_g})\Big\}\Big\}^2\int_0^t \tau^{2\alpha}\frac{\tau^{2\alpha+1}}{2\alpha+1}\|Bu_2-Bu_1\|^2_{L_2(0,t;H)}d\tau,$$

$$\leq \Big\{4N\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{1}{2\alpha-1}\{M_f+\epsilon^{-1}+M_g+\sqrt{L_g}\}\exp\Big\{4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}$$

$$\times(M_f+\epsilon^{-1}+M_g+\sqrt{L_g})\Big\}\Big\}^2\frac{t^{4\alpha+2}}{(2\alpha+1)(4\alpha+2)}\|Bu_2-Bu_1\|^2_{L_2(0,t;H)}.$$

By proceeding this process, a sequence $\{u_n\}_{n\geq 1}$ such that $u_{n+1}=u_n-v_n$ can be obtained and from

$$\|Bu_n-Bu_{n+1}\|^2_{L_2(0,t;H)}$$

$$\leq \|Bv_n\|^2_{L_2(0,t;H)},$$

$$\leq \Big\{4N\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{1}{2\alpha-1}\{M_f+\epsilon^{-1}+M_g+\sqrt{L_g}\}\exp\Big\{4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}$$

$$\times(M_f+\epsilon^{-1}+M_g+\sqrt{L_g})\Big\}\Big\}^{n-1}$$

$$\frac{t^{(2\alpha+1)(n-1)}}{(2\alpha+1)(4\alpha+2)\dots(2\alpha+1)(n-1)}\|Bu_2-Bu_1\|^2_{L_2(0,t;H)},$$

$$\leq \Big\{4N\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{1}{2\alpha-1}\{M_f+\epsilon^{-1}+M_g+\sqrt{L_g}\}\exp\Big\{4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\frac{b^{2\alpha}}{2\alpha-1}$$

$$\times(M_f+\epsilon^{-1}+M_g+\sqrt{L_g})\Big\}\frac{t^{2\alpha+1}}{2\alpha+1}\Big\}^{n-1}\frac{1}{(n-1)!}\|Bu_2-Bu_1\|^2_{L_2(0,t;H)},$$

it follows that

$$\sum_{n=1}^{\infty}\|Bu_{n+1}-Bu_n\|^2_{L_2(0,b;H)}$$

$$\leq \sum_{n=0}^{\infty} \Big\{4N\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2 \frac{1}{2\alpha-1}\{M_f+\epsilon^{-1}+M_g+\sqrt{L_g}\}$$
$$\exp\Big\{4\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2 \frac{b^{2\alpha}}{2\alpha-1}$$
$$\times(M_f+\epsilon^{-1}+M_g+\sqrt{L_g})\Big\}\frac{b^{2\alpha+1}}{2\alpha+1}\Big\}^n \frac{1}{(n)!}\|Bu_2-Bu_1\|^2_{L_2(0,t;H)}<\infty.$$

Therefore, there exists $u^* \in L_2(0,b;H)$ such that $\lim_{n\to\infty} Bu_n = u^*$ in $L_2(0,b;H)$. From (8) and (9) it follows that

$$E\|\xi_b-\widehat{T}_\alpha(b)x_0-\widehat{S}f(\cdot,x_\epsilon(\cdot;u_2))+\widehat{S}\partial\phi_\epsilon(x_\epsilon(\cdot;u_2))-\int_0^b\int_Z(b-s)^{\alpha-1}T_\alpha(b-s)$$
$$\times g(s,x_\epsilon(s;u_2),\eta)\tilde{N}(ds,d\eta)-\widehat{S}Bu_3\|^2 = E\|\xi_b-\widehat{T}_\alpha(b)x_0-\widehat{S}f(\cdot,x_\epsilon(\cdot;u_1))$$
$$+\widehat{S}\partial\phi_\epsilon(x_\epsilon(\cdot;u_1))-\int_0^b\int_Z(b-s)^{\alpha-1}T_\alpha(b-s)g(s,x_\epsilon(s;u_1),\eta)\tilde{N}(ds,d\eta)$$
$$-\widehat{S}Bu_2+\widehat{S}Bv_2-\widehat{S}\Big(f(\cdot,x_\epsilon(\cdot;u_2))-f(\cdot,x_\epsilon(\cdot;u_1))\Big)$$
$$+\widehat{S}\Big(\partial\phi_\epsilon(x_\epsilon(\cdot;u_2))-\partial\phi_\epsilon(x_\epsilon(\cdot;u_1))\Big)-\int_0^b\int_Z(b-s)^{\alpha-1}T_\alpha(b-s)$$
$$\times\Big(g(s,x_\epsilon(s;u_2),\eta)-g(s,x_\epsilon(s;u_1),\eta)\Big)\tilde{N}(ds,d\eta)\|^2<(\frac{1}{2^2}+\frac{1}{2^3})\epsilon.$$

By choosing $v_n \in L_2(0,b;U)$ by the Hypothesis (H_4) such that

$$E\|\widehat{S}\Big(f(\cdot,x_\epsilon(\cdot;u_n))-f(\cdot,x_\epsilon(\cdot;u_{n-1}))\Big)$$
$$+\widehat{S}\Big(\partial\phi_\epsilon(x_\epsilon(\cdot;u_n))-\partial\phi_\epsilon(x_\epsilon(\cdot;u_{n-1}))\Big)+\int_0^b\int_Z(b-s)^{\alpha-1}$$
$$\times T_\alpha(b-s)[g(s,x_\epsilon(s;u_n),\eta)-g(s,x_\epsilon(s;u_{n-1}),\eta)]\tilde{N}(ds,d\eta)-\widehat{S}Bv_n\|^2<\frac{\epsilon}{2^{n+1}},$$

putting $u_{n+1}=u_n-v_n$, we have

$$E\|\xi_b-\widehat{T}_\alpha(b)x_0-\widehat{S}f(\cdot,x_\epsilon(\cdot;u_n))+\widehat{S}\partial\phi_\epsilon(x_\epsilon(\cdot;u_n))-\int_0^b\int_Z(b-s)^{\alpha-1}T_\alpha(b-s)$$
$$\times g(s,x_\epsilon(s;u_n),\eta)\tilde{N}(ds,d\eta)-\widehat{S}Bu_{n+1}\|^2<(\frac{1}{2^2}+\cdots+\frac{1}{2^{n+1}})\epsilon, \quad n=1,2,\ldots$$

Therefore, for $\epsilon>0$, there exists an integer $\mathcal{N}$ such that

$$E\|\widehat{S}Bu_{\mathcal{N}+1}-\widehat{S}Bu_{\mathcal{N}}\|^2<\frac{\epsilon}{2^2}$$

and

$$
\begin{aligned}
& E\|\xi_b - \widehat{T}_\alpha(b)x_0 - \widehat{S}f(\cdot, x_\epsilon(\cdot; u_{\mathcal{N}})) + \widehat{S}\partial\phi_\epsilon(x_\epsilon(\cdot; u_{\mathcal{N}})) \\
& - \int_0^b \int_Z (b-s)^{\alpha-1} T_\alpha(b-s) g(s, x_\epsilon(s; u_{\mathcal{N}}), \eta)\tilde{N}(ds, d\eta) - \widehat{S}Bu_{\mathcal{N}}\|^2 \\
\leq\ & E\|\xi_b - \widehat{T}_\alpha(b)x_0 - \widehat{S}f(\cdot, x_\epsilon(\cdot; u_{\mathcal{N}})) + \widehat{S}\partial\phi_\epsilon(x_\epsilon(\cdot; u_{\mathcal{N}})) - \int_0^b \int_Z (b-s)^{\alpha-1} T_\alpha(b-s) \\
& \times g(s, x_\epsilon(s; u_{\mathcal{N}}), \eta)\tilde{N}(ds, d\eta) - \widehat{S}Bu_{\mathcal{N}+1}\|^2 + E\|\widehat{S}Bu_{\mathcal{N}+1} - \widehat{S}Bu_{\mathcal{N}}\|^2, \\
\leq\ & 2(\frac{1}{2^2} + \cdots + \frac{1}{2^{\mathcal{N}+1}})\epsilon + 2(\frac{\epsilon}{2^2}) \leq \epsilon.
\end{aligned}
$$

Thus, the system (2) is approximately controllable on $[0, b]$ as $\mathcal{N}$ tends to infinity. ■

From Theorem 2.8 and Theorem 3.2, the following result can be obtained.

Theorem 3.3. *Under the hypotheses* $(H_1) - (H_4)$, *the system (1) is approximately controllable on* $[0, b]$.

4. Example

Let $\mathbb{D}$ be a region in an n-dimensional Euclidean space $\mathbb{R}^N$ with smooth boundary Γ and closure $\overline{\mathbb{D}}$. For an integer $m \geq 0$, $C^m(\mathbb{D})$ is the set of all $m-$ times continuously differentiable function on $\mathbb{D}$. $C_0^m(\mathbb{D})$ will denote the subspace of $C^m(\mathbb{D})$ consisting of these functions which have compact support in $\mathbb{D}$. For $1 \leq p \leq \infty$, $W^{m,p}(\mathbb{D})$ is the set of all functions $f = f(x)$ whose derivative $D^\alpha f$ up to degree m in distribution sense belong to $L_p(\mathbb{D})$. As usual, the norm of $W^{m,p}(\mathbb{D})$ is given by

$$
\|f\|_{m,p} = \Big(\sum_{\alpha \leq m} \|D^\alpha f\|_p^p\Big)^{1/p}, \quad 1 \leq p < \infty, \quad \|f\|_{m,\infty} = \max_{\alpha \leq m} \|D^\alpha f\|_\infty,
$$

where $D^0 f = f$. In particular, $W^{0,p}(\mathbb{D}) = L_p(\mathbb{D})$ with the norm $\|\cdot\|_p$. $W_0^{m,p}(\mathbb{D})$ is the closure of $C_0^\infty(\mathbb{D})$ in $W^{m,p}(\mathbb{D})$. For $p = 2$ we denote $W^{m,2}(\mathbb{D}) = \mathbb{H}^m(\mathbb{D})$ (simply, $W^{1,2}(\mathbb{D}) = \mathbb{H}(\mathbb{D})$), $W_0^{m,2}(\mathbb{D}) = \mathbb{H}_0^m(\mathbb{D})$. $\mathbb{H}^{-1}(\mathbb{D})$ stands for the dual space $W_0^{1,2}(\mathbb{D})^\star$ whose norm is denoted by $\|\cdot\|_{-1}$. From now on, Gelfand triples are considered as $V = \mathbb{H}_0(\mathbb{D}), H = L_2(\mathbb{D})$ and $V^\star = \mathbb{H}^{-1}(\mathbb{D})$. Let us consider the following fractional order stochastic variational inequality driven by Poisson jumps

$$
(10) \qquad
\begin{aligned}
& {}^cD_t^\alpha y(t,x) + \mathcal{A}(x, \mathcal{D}_x)y(t,x) + \partial\phi(y(t,x)) \ni \mathcal{B}u(t,x) + \frac{e^{-t}}{k + e^t} y(t,x) \\
& \quad + \int_Z (\cos t) y(t,x)\eta\tilde{N}(dt, d\eta), \quad (t,x) \in [0,b] \times \mathbb{D}, \\
& y(t,x) = 0, \quad x \in \Gamma, \quad t \in [0,b],
\end{aligned}
$$

where ${}^cD_t^\alpha$ is a Caputo fractional partial derivative of order $0 < \alpha < 1, b > 0$. Let $\{q(t), t \in J\}$ be the Poisson point process taking values in the space $K = [0, \infty)$ with a σ- finite intensity measure $\lambda(d\eta)$ on the complete probability space $(\Omega, \mathfrak{F}, P)$. We denote by $N(ds, d\eta)$ the Poisson counting measure, which is induced by $q(\cdot)$, and the compensating martingale measure by

$$\tilde{N}(ds, d\eta) = N(ds, d\eta) - \lambda(d\eta)ds.$$

Here, $\mathcal{A}(x, \mathcal{D}_x)$ is a second order linear differential operator with smooth coefficients in $\overline{\mathbb{D}}$ and $\mathcal{A}(x, \mathcal{D}_x)$ is elliptic. If $Ay = \mathcal{A}(x, \mathcal{D}_x)y$, then it is known that $-A$ generates an analytic semigroup in $\mathbb{H}^{-1}(\mathbb{D})$ as is seen in [25].

Let us denote the realization of $\mathcal{A}$ in $L_2(\mathbb{D})$ under the Dirichlet boundary condition by $\widehat{A}$

$$D(\widehat{A}) = \mathbb{H}^2(\mathbb{D}) \cap \mathbb{H}_0(\mathbb{D}),$$

$$\widehat{A}y = \mathcal{A}y \quad \text{for} \quad y \in D(\widehat{A}).$$

The operator $-\widehat{A}$ generates analytic semigroup in $L_2(\mathbb{D})$. From now on, both A and $\widehat{A}$ are denoted simply by A. So, it may be considered that $-A$ generates an analytic semigroup in both of $H = L_2(\mathbb{D})$ and $V^* = \mathbb{H}^{-1}(\mathbb{D})$ as seen in [10].
For every $y \in \mathbb{H}_0(\mathbb{D})$, define $\phi : V \to (-\infty, \infty]$ given by

$$\phi(y) = \begin{cases} \int_{\mathbb{D}} \Big(|grad(y(x))|^2 + \varphi(y(x)) \Big) dx & \text{if} \quad \varphi(y(\cdot)) \in L_1(\mathbb{D}). \\ \infty & \text{otherwise.} \end{cases} .$$

From [2], it follows ϕ is a proper convex and lower semi continuous function. We can define the nonlinear functions $f : J \times V \to H$ and $g : J \times V \to H$ by $f(y)(x) = \frac{e^{-t}}{k+e^t} y(t,x)$ and $g(y)(x) = (\cos t) y(t,x)$ and assuming that $\int_Z \eta^2 \lambda(d\eta) < \infty$, $\int_Z \eta^4 \lambda(d\eta) < \infty$. Clearly, f and g satisfies hypotheses $(H_1) - (H_2)$. In order to verify the hypothesis (H_4), let $U = H$, $0 < \tau < b$ and the intercept control operator $\mathcal{B}$ on $L_2(0, b; H)$ (see [8]) is defined by

$$\mathcal{B}u(t) = \begin{cases} 0, & 0 \le t < \tau, \\ u(t), & \tau \le t \le b, \end{cases}$$

for $u \in L_2(0, b; H)$. For a given $q_1 \in L_2(0, b; H)$, let us choose a control function u satisfying

$$u(t) = \begin{cases} 0, & 0 \le t < \tau, \\ q_1(t) + \frac{\tau}{b-\tau} T_\alpha \Big(t - \frac{\tau}{b-\tau}(t-\tau) \Big) q_1 \Big(\frac{\tau}{b-\tau}(t-\tau) \Big), & \tau \le t \le b. \end{cases}$$

Then $u \in L_2(0, b; H)$ and $E\|\widehat{S}q_1 - \widehat{S}\mathcal{B}u\|^2 < \epsilon$. From the following

$$E\|\mathcal{B}u\|^2_{L_2(0,b;H)} = E\|u\|^2_{L_2(\tau,b;H)},$$
$$\leq 2\|q_1\|^2_{L_2(\tau,b;H)} + 2\|\frac{\tau}{b-\tau}\|^2\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\|q_1\Big(\frac{\tau}{b-\tau}(t-\tau)\Big)\|^2_{L_2(\tau,b;H)},$$
$$\leq 2\Big(1+\|\frac{\tau}{b-\tau}\|^2\Big(\frac{M\alpha}{\Gamma(1+\alpha)}\Big)^2\Big)\|q_1\|^2_{L_2(0,b;H)},$$

it follows that the controller $\mathcal{B}$ satisfies hypothesis (H_4). Then, we can rewrite (10) in the form of (1). Further, all the conditions stated in Theorem 3.2 and Theorem 3.3 are satisfied. Hence, by Theorem 3.2 and Theorem 3.3, the system (10) is approximately controllable on $[0, b]$.

5. Conclusion

This paper contains approximate controllability results for fractional order stochastic variational inequalities driven by Poisson jumps. The sufficient conditions of controllability results are obtained by using the Moreau-Yosida approximation of subdifferential operator, fractional calculus, stochastic analysis techniques and semigroup theory. An example is also included to illustrate the importance of the main results. For the future research, it is interesting to study the controllability results for fractional stochastic variational problem of order $1 < \alpha < 2$.

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