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EXTINCTION FOR A QUASILINEAR PARABOLIC EQUATION WITH A NONLINEAR GRADIENT SOURCE

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Abstract. We investigate the extinction, non-extinction and decay estimates of the non-negative nontrivial weak solutions of the initial-boundary value problem for the quasilinear parabolic equation with nonlinear gradient source.

1. Introduction

In this paper, we consider the following degenerate singular equation with nonlinear gradient source

$$(1.1)\quad \begin{cases} u_t = \operatorname{div}\left(u^\alpha \left|\nabla u\right|^{m-1} \nabla u\right) + \lambda \left|\nabla u\right|^q, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$, $N \geq m+1$, is an open bounded domain with smooth boundary $\partial\Omega$, m, q and λ are positive parameters, $0 < m + \alpha < 1$, and $u_0 \in L^\infty(\Omega) \cap W_0^{1,m+1}(\Omega)$ is a nonzero nonnegative function.

Model (1.1) is encountered in a variety of physical phenomena and applications. For instance, when $\alpha = 0$, $m = 1$, the equation in problem (1.1) can be viewed as the viscosity approximation of Hamilton-Jacobi type equation from stochastic control theory [20]. In particular, when $\alpha = 0$, $m = 1$ and $q = 2$, the equation in problem (1.1) appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation [14]. It is obvious that the equation in problem (1.1) is degenerate and singular at the points where $u = 0$ or $\nabla u = 0$, and hence there is no classical solution in general. We introduce the definition of the weak solution for problem (1.1) as follows.

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Definition 1.1. We say that a nonnegative measurable function $u(x,t)$ defined in $\Omega\times(0,T)$ is a weak solution of problem (1.1) if $u^{\alpha}|\nabla u|^{m+1}\in L^1\left(0,T;L^1(\Omega)\right)$, $u_t\in L^2\left(0,T;L^2(\Omega)\right)$, $u\in C\left(0,T;L^{\infty}(\Omega)\right)\cap L^q\left(0,T;W^{1,q}(\Omega)\right)$, and the integral identity

$$\begin{aligned}&\int_{\Omega}u(x,t_2)\zeta(x,t_2)\,dx+\int_{t_1}^{t_2}\int_{\Omega}\left[-u\zeta_t+u^{\alpha}|\nabla u|^{m-1}\nabla u\cdot\nabla\zeta\right]dxdt\\&=\int_{\Omega}u(x,t_1)\zeta(x,t_1)\,dx+\lambda\int_{t_1}^{t_2}\int_{\Omega}|\nabla u|^q\zeta dxdt\end{aligned}\tag{1.2}$$

holds for any $\zeta\in C_0^{\infty}(\Omega\times(0,T))$ and $0<t_1<t_2<T$. Furthermore,

$$u(x,0)=u_0(x)\ \text{ a.e. } x\in\Omega.\tag{1.3}$$

Remark 1.1. The weak subsolution (resp. supersolution) of problem (1.1) can be defined in the similar way. In fact, one needs only to change “ $=$ ” in (1.2) and (1.3) into “ $\le$ ” (resp. “ $\ge$ ”) for every nonnegative $\zeta\in C_0^{\infty}(\Omega\times(0,T))$.

Recently, based on a priori estimates, the local existence of the solution for fast diffusion equations with gradient source terms was studied in [25, 26, 27] through an approximation and regularization process. Following a slight modification of those arguments previously used in [25, 26, 27], we can prove the local existence result of the weak solution for problem (1.1). Furthermore, from Theorem 3.9 in [24] and Subsection 1.1 in [12], we know that comparison principle is granted for problem (1.1). The main goal of this article is to investigate the extinction property of the nonnegative weak solution $u(x,t)$ of problem (1.1), namely, whether there exists a finite time $T>0$ such that $u(x,t)$ is nontrivial for $t\in(0,T)$ but $u(x,t)\equiv 0$ for any $(x,t)\in\Omega\times[T,+\infty)$. As one of the most remarkable properties that distinguish nonlinear parabolic problems from the linear ones, the study of extinction and non-extinction of nonnegative solutions to parabolic equations attracted extensive attentions of mathematicians in the past few decades (see [3, 4, 5, 6, 10, 11, 13, 17, 21, 28, 31, 32] and the references therein). For example, Gu [9] considered the following semilinear heat equation with a cool source

$$\begin{cases}u_t=\Delta u-u^p, & (x,t)\in\Omega\times(0,+\infty),\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,+\infty),\\ u(x,0)=u_0(x), & x\in\overline{\Omega},\end{cases}\tag{1.4}$$

where $p>0$, and proved that the nontrivial solution of problem (1.1) vanishes in finite time if and only if $p\in(0,1)$. In other words, the strong absorption in problem (1.4) causes extinction in finite time. Meanwhile, in [9], Gu also gave the necessary and sufficient conditions on the occurrence of extinction phenomenon of the weak solution to the following p-Laplace equation

$$\text{(1.5)} \quad \begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \lambda u^q, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases}$$

where $p > 1$, λ and q are positive parameters.

Tian and Mu [29] studied the extinction behavior of the weak solution for a p-Laplace equation with a hot source as the form

$$\text{(1.6)} \quad \begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda u^q, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases}$$

where λ, $q > 0$ and $p \in (1,2)$, and showed that $q = p-1$ is the critical extinction exponent of the weak solution of problem (1.6). Moreover, for the critical case $q = p-1$, the authors pointed out that whether the weak solution vanishes in finite time or not depends strongly on the first eigenvalue of p-Laplace equation with zero boundary condition.

Jin et al. [15], Zhou and Mu [34] dealt with the following fast diffusive polytropic filtration equation with source term

$$\text{(1.7)} \quad \begin{cases} u_t = \operatorname{div}\left(|\nabla u^m|^{p-2}\nabla u^m\right) + \lambda u^q, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases}$$

where $p > 1$, m, λ, $q > 0$, $m(p-1) < 1$. The authors concluded that the critical extinction exponent of the weak solution to problem (1.7) is $q = m(p-1)$. Furthermore, in the critical case $q = m(p-1)$, they pointed out that the value of parameter λ plays an important role in determining the extinction property of the weak solution for problem (1.7).

Let us point out that extinction behaviors for parabolic equations with both hot source and cool source have been studied extensively by many researchers, such as [7, 19, 20, 22, 30, 33], and so on.

However, to our best knowledge, there is little literature on the study of the extinction and non-extinction properties for parabolic equations with nonlinear gradient terms. In [1], Benachour et al. discussed the following Cauchy problem for a viscous Hamilton-Jacobi equation

$$\text{(1.8)} \quad \begin{cases} u_t = \Delta u - |\nabla u|^p, & x \in \mathbf{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases}$$

where $p > 0$, $u_0(x) \in \mathcal{BC}(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$ is nonnegative, here $\mathcal{BC}(\mathbf{R}^N)$ denotes the space of bounded and continuous functions in $\mathbf{R}^N$. The authors showed that extinction phenomenon takes place for any nonnegative and integrable solution to problem (1.8) if $p \in \left(0, \frac{N}{N+1}\right)$. Moreover, they established some temporal decay estimates for the $L^\infty-$norm of the nonnegative solutions in the case $p \geq \frac{N}{N+1}$. Later, Benachour et al. [2] investigated problem (1.8) with $p \in (0,1)$ and $u_0 \in \mathcal{BC}(\mathbf{R}^N)$. They pointed out that the occurrence of the extinction phenomenon depends on the asymptotic behavior of u_0 as $|x|$ tends to infinity. Roughly speaking, they proved that if the decay of initial data $u_0(x)$ is faster than that of $|x|^{-\frac{p}{1-p}}$ as $|x| \to \infty$, then extinction occurs. Otherwise, the solution of (1.8) is strictly positive for any positive initial data. In addition, they also claimed that the critical extinction exponent $p = \frac{N}{N+1}$ introduced in [1] is optimal.

Iagar and Laurençot [12] concerned with the following Cauchy problem

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) - |\nabla u|^q, & x \in \mathbf{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.9}$$

where $q > 0$ and $p \in (1,2)$. Based on comparison principle and gradient estimates of the solutions, they classify the behavior of the solutions for large time, obtaining either positivity as $t \to \infty$ for $q > p - \frac{N}{N+1}$, optimal decay estimates as $t \to \infty$ for $q \in \left[\frac{p}{2}, p - \frac{N}{N+1}\right]$, or extinction in finite time for $q \in \left(0, \frac{p}{2}\right)$. In addition, the authors showed that how the diffusion prevents extinction in finite time in some ranges of exponents where extinction occurs for the non-diffusive Hamilton-Jacobi equation.

Recently, Mu et al. [23] considered the following fast diffusion equation

$$\begin{cases} u_t = \Delta u^m + \lambda |\nabla u|^p, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.10}$$

where $0 < m < 1$, p, $\lambda > 0$, and proved that $p = m$ is the critical extinction exponent of the nonnegative weak solution for problem (1.10).

Motivated by those works above, we consider the extinction property of the weak solutions for problem (1.1) by using energy estimates approach and constructing suitable subsolutions. The main results of this article are stated as follows.

Theorem 1.1. *Assume that $0 < m + \alpha < 1$ and $m + \alpha < q < \frac{m+1}{2-\alpha}$, then the nonnegative weak solution of problem (1.1) vanishes in finite time provided that u_0 is sufficiently small. Furthermore, we have*

$$\begin{cases} \|u\|_{\frac{2m+\alpha}{m}} \leq \|u_0\|_{\frac{2m+\alpha}{m}} \left[1 - C_6 t \|u_0\|^{m+\alpha-1}_{\frac{2m+\alpha}{m}}\right]^{\frac{1}{1-m-\alpha}}, & t \in [0, T_1), \\ \|u\|_{\frac{2m+\alpha}{m}} \equiv 0, & t \in [T_1, +\infty), \end{cases}$$

for $m\left(\frac{N-m-1}{Nm+m+1}-1\right)\leq\alpha<1$, *and*

$$\begin{cases} \|u\|_{\frac{N(1-m-\alpha)}{m+1}} \leq \|u_0\|_{\frac{N(1-m-\alpha)}{m+1}} \left[1-C_{10}t\,\|u_0\|^{m+\alpha-1}_{\frac{N(1-m-\alpha)}{m+1}}\right]^{\frac{1}{1-(m+\alpha)}}, & t\in[0,T_2),\\ \|u\|_{\frac{N(1-m-\alpha)}{m+1}}\equiv 0, & t\in[T_2,+\infty), \end{cases}$$

for $-m<\alpha<m\left(\frac{N-m-1}{Nm+m+1}-1\right)$, *where* C_6, T_1, T_2 *and* C_{10} *are positive constants, given by* (*2.10*), (*2.11*), (*2.18*) *and* (*2.19*), *respectively.*

Theorem 1.2. *Assume that* $0<m+\alpha<1$ *and* $q<m+\alpha$, *then for any nonzero nonnegative initial data* u_0, *the nonnegative weak solution* u *of problem (1.1) cannot possess extinction phenomenon provided that* λ *is sufficiently large.*

Theorem 1.3. *Assume that* $0<m+\alpha<1$ *and* $q=m+\alpha$.

(*1*). *The nonnegative weak solution of problem (1.1) vanishes in finite time provided that* λ *is sufficiently small. Furthermore, we have*

$$\begin{cases} \|u\|_{\frac{2m+\alpha}{m}} \leq \|u_0\|_{\frac{2m+\alpha}{m}} \left[1-C_{12}t\,\|u_0\|^{m+\alpha-1}_{\frac{2m+\alpha}{m}}\right]^{\frac{1}{1-m-\alpha}}, & t\in[0,T_3),\\ \|u\|_{\frac{2m+\alpha}{m}}\equiv 0, & t\in[T_3,+\infty), \end{cases}$$

for $m\left(\frac{N-m-1}{Nm+m+1}-1\right)\leq\alpha<1$, *and*

$$\begin{cases} \|u\|_{\frac{N(1-m-\alpha)}{m+1}} \leq \|u_0\|_{\frac{N(1-m-\alpha)}{m+1}} \left[1-C_{13}t\,\|u_0\|^{m+\alpha-1}_{\frac{N(1-m-\alpha)}{m+1}}\right]^{\frac{1}{1-m-\alpha}}, & t\in[0,T_4),\\ \|u\|_{\frac{N(1-m-\alpha)}{m+1}}\equiv 0, & t\in[T_4,+\infty), \end{cases}$$

for $-m<\alpha<m\left(\frac{N-m-1}{Nm+m+1}-1\right)$, *where* T_3, C_{12}, T_4 *and* C_{13} *are positive constants, given by* (*4.2*), (*4.3*), (*4.4*) *and* (*4.5*), *respectively.*

(*2*). *The nonnegative weak solution of problem (1.1) cannot vanish in finite time provided that* λ *is sufficiently large.*

Remark 1.2. From theorems 1.1, 1.2 and 1.3, we know that $q=m+\alpha$ is the critical extinction exponent of the weak solution of problem (1.1).

The rest of this article is organized as follows. In Section 2, we will discuss the extinction behaviour and decay estimate of the weak solution for problem (1.1) in the case $q\in\left(m+\alpha,\frac{m+1}{2-\alpha}\right)$, and give the proof of Theorem 1.1. Section 3 is mainly about the non-extinction property of problem (1.1) in the case $q\in(0,m+\alpha)$ and the proof of Theorem 1.2. Finally, the critical case $q=m+\alpha$ and the proof of Theorem 1.3 are the main subject of Section 4.

2. Extinction of the Solution

The main goal of this section is to discuss the extinction behavior of the weak solution for problem (1.1) in the case $q \in \left(m+\alpha, \frac{m+1}{2-\alpha}\right)$. By establishing appropriate L^r-norm estimate of the weak solution, here $r > 1$, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Multiplying the first equation in (1.1) by u^s with $s > 0$, and integrating over Ω, one has

$$\begin{aligned} &\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1}dx + s\left(\frac{m+1}{m+\alpha+s}\right)^{m+1}\int_\Omega \left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx \\ &= \lambda\left(\frac{m+1}{m+\alpha+s}\right)^q \int_\Omega u^{\frac{s(m+1)-q(\alpha+s-1)}{m+1}} \left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^q dx. \end{aligned} \tag{2.1}$$

Now, we will divide the proof of Theorem 1.1 into two cases according to the different values of α.

Case 1. $m\left[\frac{N-(m+1)}{Nm+m+1} - 1\right] \le \alpha < 1$. For this case, taking $s = \frac{m+\alpha}{m}$ in (2.1), then (2.1) becomes

$$\begin{aligned} &\frac{m}{2m+\alpha}\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}dx + \left(\frac{m}{m+\alpha}\right)^m \int_\Omega \left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1}dx \\ &= \lambda\left(\frac{m}{m+\alpha}\right)^q \int_\Omega u^{\frac{m+\alpha(1-q)}{m}} \left|\nabla u^{\frac{m+\alpha}{m}}\right|^q dx. \end{aligned} \tag{2.2}$$

Since $q \in \left(m+\alpha, \frac{m+1}{2-\alpha}\right)$, Young's inequality can be used to obtain

$$\begin{aligned} &\int_\Omega u^{\frac{m+\alpha(1-q)}{m}} \left|\nabla u^{\frac{m+\alpha}{m}}\right|^q dx \\ &\le \epsilon_1 \int_\Omega \left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1} dx + C(\epsilon_1)\int_\Omega u^{\frac{(m+1)[m+\alpha(1-q)]}{m(m+1-q)}}dx. \end{aligned} \tag{2.3}$$

Furthermore, Hölder's inequality tells us

$$\int_\Omega u^{\frac{(m+1)[m+\alpha(1-q)]}{m(m+1-q)}}dx \le |\Omega|^{1-\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}} \left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}}. \tag{2.4}$$

Inserting this estimate into (2.3), we get

$$\begin{aligned} &\int_\Omega u^{\frac{m+\alpha(1-q)}{m}} \left|\nabla u^{\frac{m+\alpha}{m}}\right|^q dx \\ &\le \epsilon_1 \int_\Omega \left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1} dx + C(\epsilon_1)|\Omega|^{1-\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}} \\ &\quad \times \left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}}. \end{aligned} \tag{2.5}$$

On the other hand, by using Hölder's inequality and Sobolev embedding inequality, we can easily arrive at the following estimate

$$\begin{aligned}\int_\Omega u^{\frac{2m+\alpha}{m}}dx &\le |\Omega|^{1-\frac{(2m+\alpha)[N-(m+1)]}{N(m+1)(m+\alpha)}}\left(\int_\Omega u^{\frac{N(m+1)(m+\alpha)}{m[N-(m+1)]}}dx\right)^{\frac{(2m+\alpha)[N-(m+1)]}{N(m+1)(m+\alpha)}}\\ &\le \kappa_1|\Omega|^{1-\frac{(2m+\alpha)[N-(m+1)]}{N(m+1)(m+\alpha)}}\left(\int_\Omega\left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1}dx\right)^{\frac{2m+\alpha}{(m+1)(m+\alpha)}},\end{aligned}\tag{2.6}$$

which implies that

$$C_1\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}} \le \int_\Omega\left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1}dx,\tag{2.7}$$

where

$$C_1=\kappa_1^{-\frac{(m+1)(m+\alpha)}{2m+\alpha}}|\Omega|^{1-\frac{m+1}{N}-\frac{(m+1)(m+\alpha)}{2m+\alpha}},$$

and κ_1 is the embedding constant, depending only on m, α and N. Choosing ϵ_1 sufficiently small such that

$$C_2=\left(\frac{m}{m+\alpha}\right)^m-\lambda\epsilon_1\left(\frac{m}{m+\alpha}\right)^q>0,$$

then from (2.2), (2.5), and (2.7), it follows that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}dx+C_3\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}}\\ \le{}& C_4\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}},\end{aligned}\tag{2.8}$$

where

$$C_3=\frac{C_1C_2(2m+\alpha)}{m},$$

and

$$C_4=\frac{\lambda(2m+\alpha)C(\epsilon_1)}{m}\left(\frac{m}{m+\alpha}\right)^q|\Omega|^{1-\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}}.$$

Next, let $u_0(x)$ be sufficiently small such that

$$\left(\int_\Omega u_0^{\frac{2m+\alpha}{m}}dx\right)^{\frac{m(m+1)[q-(m+\alpha)]}{(2m+\alpha)(m+1-q)}}\le C_3C_4^{-1},$$

then we have

$$\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}dx\le C_5\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}},\tag{2.9}$$

where

$$C_5 = C_4 \left(\int_\Omega u_0^{\frac{2m+\alpha}{m}} dx \right)^{\frac{m(m+1)[q-(m+\alpha)]}{(2m+\alpha)(m+1-q)}} - C_3 < 0.$$

Integrating (2.9), we arrive at the following inequality

$$\left(\int_\Omega u^{\frac{2m+\alpha}{m}} dx \right)^{\frac{m[1-(m+\alpha)]}{2m+\alpha}} \le \left(\int_\Omega u_0^{\frac{2m+\alpha}{m}} dx \right)^{\frac{m[1-(m+\alpha)]}{2m+\alpha}} - C_6 t,$$

where

$$C_6 = \frac{m\,(m+\alpha-1)}{2m+\alpha} C_5 > 0, \tag{2.10}$$

as long as the right side is nonnegative. From this, one has

$$\int_\Omega u^{\frac{2m+\alpha}{m}} dx \le \int_\Omega u_0^{\frac{2m+\alpha}{m}} dx \left[1 - \frac{C_6 t}{\left(\int_\Omega u_0^{\frac{2m+\alpha}{m}} dx \right)^{\frac{m[1-(m+\alpha)]}{2m+\alpha}}} \right]_+^{\frac{2m+\alpha}{m[1-(m+\alpha)]}},$$

that is

$$\|u\|_{\frac{2m+\alpha}{m}} \le \|u_0\|_{\frac{2m+\alpha}{m}} \left[1 - C_6 t \, \|u_0\|_{\frac{2m+\alpha}{m}}^{m+\alpha-1} \right]_+^{\frac{1}{1-(m+\alpha)}},$$

which implies that $u\,(x,t)$ vanishes in finite time

$$T_1 = C_6^{-1} \, \|u_0\|_{\frac{2m+\alpha}{m}}^{1-(m+\alpha)}. \tag{2.11}$$

Case 2. $-m < \alpha < m \left[\frac{N-(m+1)}{Nm+m+1} - 1 \right]$. For this case, choosing

$$s = \frac{N\,[1-(m+\alpha)] - m - 1}{m+1} > \frac{m+\alpha}{m}$$

in (2.1). By Sobolev embedding inequlity and the choice of s, we find

$$\begin{aligned} \left(\int_\Omega u^{s+1} dx \right)^{\frac{m+\alpha+s}{(m+1)(s+1)}} &= \left(\int_\Omega u^{\frac{N(\alpha+m+s)}{N-(m+1)}} dx \right)^{\frac{N-(m+1)}{N(m+1)}} \\ &\le \kappa_2 \left(\int_\Omega \left| \nabla u^{\frac{\alpha+m+s}{m+1}} \right|^{m+1} dx \right)^{\frac{1}{m+1}}, \end{aligned} \tag{2.12}$$

where κ_2 is the embedding constant, depending only on α, m and N. On the other hand, since $q \in \left(m+\alpha, \frac{m+1}{2-\alpha}\right)$, according to Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_\Omega u^{\frac{s(m+1)-q(\alpha+s-1)}{m+1}} \left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^q dx \\
&\le \epsilon_2 \int_\Omega \left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1} dx + C(\epsilon_2) \int_\Omega u^{\frac{s(m+1)-q(\alpha+s-1)}{m+1-q}} dx \\
&\le C(\epsilon_2) |\Omega|^{1-\frac{s(m+1)-q(\alpha+s-1)}{(s+1)(m+1-q)}} \left(\int_\Omega u^{s+1} dx\right)^{\frac{s(m+1)-q(\alpha+s-1)}{(s+1)(m+1-q)}} \\
&\quad + \epsilon_2 \int_\Omega \left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1} dx.
\end{aligned} \tag{2.13}$$

Choosing

$$\epsilon_2 < \frac{s}{\lambda}\left(\frac{m+1}{m+\alpha+s}\right)^{m+1-q},$$

then from (2.1), (2.12) and (2.13), one has

$$\frac{d}{dt}\int_\Omega u^{s+1}dx + C_7\left(\int_\Omega u^{s+1}dx\right)^{\frac{m+\alpha+s}{s+1}} \le C_8\left(\int_\Omega u^{s+1}dx\right)^{\frac{s(m+1)-q(\alpha+s-1)}{(s+1)(m+1-q)}}, \tag{2.14}$$

where

$$C_7 = \frac{s+1}{\kappa_2}\left[s\left(\frac{m+1}{m+\alpha+s}\right)^{m+1} - \lambda\epsilon_2\left(\frac{m+1}{m+\alpha+s}\right)^q\right] > 0,$$

and

$$C_8 = \lambda C(\epsilon_2) |\Omega|^{1-\frac{s(m+1)-q(\alpha+s-1)}{(s+1)(m+1-q)}} \left(\frac{m+1}{m+\alpha+s}\right)^q.$$

Choosing u_0 sufficiently small such that

$$\left(\int_\Omega u_0^{s+1}dx\right)^{\frac{(m+1)[q-(m+\alpha)]}{(s+1)(m+1-q)}} \le C_7 C_8^{-1}. \tag{2.15}$$

It follows easily from (2.14) and (2.15) that

$$\frac{d}{dt}\int_\Omega u^{s+1}dx \le C_9\left(\int_\Omega u^{s+1}dx\right)^{\frac{m+\alpha+s}{s+1}}, \tag{2.16}$$

here

$$C_9 = C_8\left(\int_\Omega u_0^{s+1}dx\right)^{\frac{(m+1)[q-(m+\alpha)]}{(s+1)(m+1-q)}} - C_7 < 0.$$

Integrating (2.16), we get

$$\int_\Omega u^{s+1}dx \le \int_\Omega u_0^{s+1}dx \left[1-\frac{C_{10}t}{\left(\int_\Omega u_0^{s+1}dx\right)^{\frac{1-(m+\alpha)}{s+1}}}\right]_+^{\frac{s+1}{1-(m+\alpha)}}, \tag{2.17}$$

that is

$$\|u\|_{\frac{N[1-(m+\alpha)]}{m+1}} \le \|u_0\|_{\frac{N[1-(m+\alpha)]}{m+1}} \left[1-C_{10}t\,\|u_0\|^{m+\alpha-1}_{\frac{N[1-(m+\alpha)]}{m+1}}\right]_+^{\frac{1}{1-(m+\alpha)}},$$

which means that $u(x,t)$ vanishes in finite time

$$T_2 = C_{10}^{-1}\,\|u_0\|^{1-(m+\alpha)}_{\frac{N[1-(m+\alpha)]}{m+1}}, \tag{2.18}$$

where

$$C_{10} = \frac{m(m+\alpha-1)}{2m+\alpha}C_9 > 0. \tag{2.19}$$

The proof of Theorem 1.1 is complete. ■

3. Non-extinction of the Solution

In this section, we will deal with the non-extinction of the weak solution for problem (1.1) in the case $q \in (0, m+\alpha)$ and give the proof of Theorem 1.2. Our main tool is the combination of constructing suitable positive weak subsolution and comparison principle.

Proof of Theorem 1.2. Let λ_1 be the first eigenvalue and $\psi(x)$ be the corresponding eigenfunction of the following problem

$$\begin{cases} -\operatorname{div}\left(\mathcal{U}^\alpha\,|\nabla\mathcal{U}|^{m-1}\,\nabla\mathcal{U}\right) = \lambda\mathcal{U}^{\alpha+1}\,|\mathcal{U}|^{m-1}, & x\in\Omega,\\ \mathcal{U}(x) = 0, & x\in\partial\Omega. \end{cases} \tag{3.1}$$

In what follows, we assume that $\psi(x) > 0$ and $\max\limits_{x\in\Omega}\psi(x) = 1$. Define a function $f(t)$ as follows

$$f(t) = d^{\frac{1}{m+\alpha-q}}\left(1-e^{-ct}\right)^{\frac{1}{1-q}},$$

where $d\in(0,1)$, and $c>0$ will be determined later. Then it is easy to check that

$$f(0)=0 \text{ and } f(t)\in(0,1) \text{ for } t>0. \tag{3.2}$$

In addition, for convenience, we denote $d=\frac{1}{a}$, where $a\in(1,+\infty)$ is a constant. Taking

$$0<c<(m+\alpha-q)\,a^{\frac{1-q}{m+\alpha-q}},$$

then using the inequality

$$(1-x)^a + ax < 1 \text{ for } x, a \in (0,1),$$

we have

$$f'(t) + af^{m+\alpha} - f^q < 0. \tag{3.3}$$

Letting

$$\mathcal{V}_1(x,t) = f(t)\,\psi(x).$$

Our next goal is to show that $\mathcal{V}_1(x,t)$ is a weak subsolution of problem (1.1). By a straightforward computation, using (3.3) and the definition of $\psi(x)$, we arrive at

$$\begin{aligned}
I :=& \int_0^t \int_\Omega \mathcal{V}_{1s}\zeta dxds + \int_0^t \int_\Omega \mathcal{V}_1^\alpha |\nabla \mathcal{V}_1|^{m-1} \nabla \mathcal{V}_1 \cdot \nabla \zeta dxds - \lambda \int_0^t \int_\Omega |\nabla \mathcal{V}_1|^q \zeta dxds \\
=& \int_0^t \int_\Omega f_s(s)\,\psi(x)\,\zeta(x,s)\,dxds - \lambda \int_0^t \int_\Omega f^q(s)\,|\nabla\psi|^q\,\zeta(x,s)\,dxds \\
&+ \int_0^t \int_\Omega f^{\alpha+m}(s)\,\psi^\alpha(x)\,|\nabla\psi(x)|^{m-1}\,\nabla\psi(x)\cdot\nabla\zeta(x,s)\,dxds \\
<& \int_0^t \int_\Omega \left[f^q(s) - af^{m+\alpha}(s)\right]\psi(x)\,\zeta(x,s)\,dxds \\
&- \lambda \int_0^t \int_\Omega f^q(s)\,|\nabla\psi(x)|^q\,\zeta(x,s)\,dxds \\
&+ \lambda_1 \int_0^t \int_\Omega f^{\alpha+m}(s)\,\psi^{m+\alpha}(x)\,\zeta(x,s)\,dxds \\
<& \int_0^t \int_\Omega f^q(s)\left[\psi(x) + \lambda_1 f^{\alpha+m-q}(s)\,\psi^{m+\alpha}(x) - \lambda\,|\nabla\psi(x)|^q\right]\zeta(x,s)\,dxds.
\end{aligned}$$

Recalling that f, $\psi \in (0,1)$, then $0 < q < m + \alpha < 1$ tells us that

$$\psi(x) + \lambda_1 f^{\alpha+m-q}(s)\,\psi^{m+\alpha}(x) < (\lambda_1 + 1)\,\psi^{m+\alpha}(x).$$

If

$$\lambda > \frac{(\lambda_1+1)\,\|\psi\|_{m+\alpha}^{m+\alpha}}{\|\nabla\psi\|_q^q},$$

then we can immediately get $I < 0$, which implies that $\mathcal{V}_1(x,t)$ is a strict weak subsolution of problem (1.1). Then by comparison principle, we know that $u(x,t) > \mathcal{V}_1(x,t) > 0$ for all $(x,t) \in \Omega \times (0,+\infty)$, which means that, for any nonzero nonnegative initial data u_0, the weak solution of problem (1.1) cannot vanish in finite time provided that λ is sufficiently large. The proof of Theorem 1.2 is complete. ■

4. Critical Case

This section is devoted to consider the critical case $q = m + \alpha$ and prove Theorem 1.3.

Proof of Theorem 1.3. **(1).** By the similar arguments in the proof of Theorem 1.1, for $m\left[\frac{N-(m+1)}{Nm+m+1}-1\right] \le \alpha < 1$, we have

$$\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}dx \le C_{11}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}}, \tag{4.1}$$

where

$$C_{11} = \frac{2m+\alpha}{m}\left(\frac{m}{m+\alpha}\right)^m\left\{\lambda\left(\frac{m}{m+\alpha}\right)^\alpha\left[\epsilon_1 C_1 + C(\epsilon_1)\left|\Omega\right|^{\frac{m[1-(m+\alpha)]}{2m+\alpha}}\right] - C_1\right\}.$$

If

$$\lambda < C_1\left(\frac{m}{m+\alpha}\right)^{-\alpha}\left[\epsilon_1 C_1 + C(\epsilon_1)\left|\Omega\right|^{\frac{m[1-(m+\alpha)]}{2m+\alpha}}\right]^{-1},$$

then we have $C_{11} < 0$. From (4.1), it follows that

$$\|u\|_{\frac{2m+\alpha}{m}} \le \|u_0\|_{\frac{2m+\alpha}{m}}\left[1 - C_{12}t\,\|u_0\|^{m+\alpha-1}_{\frac{2m+\alpha}{m}}\right]_+^{\frac{1}{1-(m+\alpha)}},$$

which implies that $u(x,t)$ vanishes in finite time

$$T_3 = C_{12}^{-1}\,\|u_0\|^{1-(m+\alpha)}_{\frac{2m+\alpha}{m}}, \tag{4.2}$$

where

$$C_{12} = \frac{m(m+\alpha-1)}{2m+\alpha}C_{11} > 0. \tag{4.3}$$

Likewise, for $-m < \alpha < m\left[\frac{N-(m+1)}{Nm+m+1}-1\right]$, we have

$$\|u\|_{\frac{N[1-(m+\alpha)]}{m+1}} \le \|u_0\|_{\frac{N[1-(m+\alpha)]}{m+1}}\left[1 - C_{13}t\,\|u_0\|^{m+\alpha-1}_{\frac{N[1-(m+\alpha)]}{m+1}}\right]_+^{\frac{1}{1-(m+\alpha)}}$$

holds for

$$\lambda < \frac{s(s+1)}{\kappa_2}\left(\frac{m+1}{m+\alpha+s}\right)^{1-\alpha}\left[C(\epsilon_2)\left|\Omega\right|^{\frac{1-m-\alpha}{s+1}} + \frac{\epsilon_2(s+1)}{\kappa_2}\right]^{-1},$$

which leads to $u(x,t)$ vanishes in finite time

$$T_4 = C_{13}^{-1} \|u_0\|^{1-(m+\alpha)}_{\frac{N[1-(m+\alpha)]}{m+1}}, \tag{4.4}$$

where

$$\begin{aligned} C_{13} = (m+\alpha-1)\left(\frac{m+1}{m+\alpha+s}\right)^{m+\alpha} &\left\{\lambda\left[C(\epsilon_2)|\Omega|^{\frac{1-m-\alpha}{s+1}} + \frac{\epsilon_2(s+1)}{\kappa_2}\right]\right. \\ &\left. - \frac{s(s+1)}{\kappa_2}\left(\frac{m+1}{m+\alpha+s}\right)^{1-\alpha}\right\} \\ > 0. \end{aligned} \tag{4.5}$$

(2). Letting

$$\mathcal{V}_2(x,t) = [(1-m-\alpha)t]^{\frac{1}{1-m-\alpha}}\psi(x).$$

Similarly as in the proof of Theorem 1.2, we can easily verify that $\mathcal{V}_2(x,t)$ is a weak subsolution of problem (1.1) if

$$\lambda > \frac{(\lambda_1+1)\|\psi\|_q^q}{\|\nabla\psi\|_q^q},$$

therefore extinction phenomenon in finite time cannot occur for sufficiently large λ. The proof of Theorem 1.3 is complete. ∎

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