TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 3, pp. 871-893, June 2014
DOI: 10.11650/tjm.18.2014.3860
This paper is available online at http://journal.taiwanmathsoc.org.tw

FIXED POINTS AND SOLUTIONS OF OPERATOR EQUATIONS FOR THE WEAK TOPOLOGY IN BANACH ALGEBRAS

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Abstract. In this paper we prove some fixed point theorems for operators acting in Banach algebras and satisfying conditions expressed mainly with help of weak topology and measures of weak noncompactness. The existence of solutions to some class of operator equations in Banach algebras is also discussed. Some examples are presented to illustrate our results.

1. Introduction

Operator equations of various kind create the base of numerous considerations conducted in nonlinear analysis and in the theories of differential and integral equations. The existence of solutions of those operator equations is mostly proved with aid of miscellaneous fixed point theorems.

In the present paper we prove some fixed point theorems for operators acting on Banach algebras. We also discuss the solvability of some operator equations in Banach algebras. Our analysis uses the concept of measures of noncompactness. In what follows we formulate introductory facts allowing us to present concepts, tools and auxiliary results needed further on.

Assume that X is a given Banach space with norm $\|.\|$ and the zero element θ. We denote by X^* the dual space of X. We use standard notation $M + M', \lambda M$ to denote algebraic operations on subsets of X. The symbol $B(x, r)$ denotes the closed ball centered at x with radius r. We write B_r to denote $B(\theta, r)$. We write $\overline{M}$ and $ConvM$ to denote the closure and the closed convex hull of a set M, respectively. The symbol $\overline{M^w}$ stands for the weak closure of M. Moreover, we write $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong convergence (with respect to the norm of X) and the weak convergence (with respect to the weak topology of X) of a sequence (x_n) to x.

Received October 2, 2013, accepted November 20, 2013.
Communicated by Biagio Ricceri.
2010 *Mathematics Subject Classification*: 47H08, 47H09.
Key words and phrases: Banach algebra, Weakly compact set, Weakly sequentially continuous operator, Measure of weak noncompactness, Fixed point theorem, Operator equation, Dunford-Pettis property.

Further, denote by $\mathcal{B}(X)$ the family of all nonempty and bounded subsets of X. The symbol $\mathcal{W}(X)$ stands for the family of all nonempty and relatively weakly compact subsets of X. The set of all real numbers will be denoted by $\mathbb{R}$ while $\mathbb{R}_+ = [0, \infty)$.

In the sequel we will use the following definition of the concept of a measure of weak noncompactness [5].

Definition 1.1. A function $\psi\colon \mathcal{B}(X) \to \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions :

(1) The family $\ker\psi = \{M \in \mathcal{B}(X) : \psi(M) = 0\}$ is nonempty and $\ker\psi \subset \mathcal{W}(X)$.

(2) $M_1 \subset M_2 \Rightarrow \psi(M_1) \leq \psi(M_2)$.

(3) $\psi(ConvM) = \psi(M)$.

(4) $\psi(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\psi(M_1) + (1-\lambda)\psi(M_2)$ for $\lambda \in [0,1]$.

(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty, weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$ and such that $\lim_{n\to\infty}\psi(M_n) = 0$, then the set $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

The family $\ker\psi$ described in (1) is said to be *the kernel of the measure of weak noncompactness* ψ. Notice that the intersection set M_∞ from (5) belongs to $\ker\psi$ since $\psi(M_\infty) \leq \psi(M_n)$ for every n and $\lim_{n\to\infty}\psi(M_n) = 0$. Also, it can be easily verified that the measure ψ satisfies

$$\psi(\overline{M^w}) = \psi(M), \tag{1.1}$$

for all $M \in \mathcal{B}(X)$.

In applications, there are measures of noncompactness satisfying some additional handy conditions. Thus, if a measure of weak noncompactness ψ satisfies the condition

$$\psi(M) = 0 \Leftrightarrow M \in \mathcal{W}(X), \tag{1.2}$$

it is called a *regular measure of weak noncompactness*. Moreover, if ψ is such that

$$\psi(M_1 + M_2) \leq \psi(M_1) + \psi(M_2), \tag{1.3}$$

it is called *subadditive,* and if

$$\psi(\lambda M) = |\lambda|\psi(M), \tag{1.4}$$

for $\lambda \in \mathbb{R}$, then ψ is called *homogeneous.* We say that ψ has the *maximum property* if

$$\psi(M_1 \cup M_2) = max(\psi(M_1), \psi(M_2)). \tag{1.5}$$

The first important example of a measure of weak noncompactness was defined by De Blasi [16] as follows :

$$w(M) = \inf\{r > 0 : \text{ there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\}, \tag{1.6}$$

for each $M \in \mathcal{B}(X)$.

Notice that the De Blasi measure of weak noncompactness w is regular, homogeneous, subadditive and has the maximum property [16].

By a *measure of noncompactness* on a Banach space X we mean a map $\psi\colon \mathcal{B}(X) \to \mathbb{R}_+$ which satisfies conditions (1)–(5) of Definition 1.1 relative to the strong topology instead of the weak topology. Let us recall that the concept of a measure of noncompactness was initiated by fundamental papers of Kuratowski [33] and Darbo [15]. It turns out that measures of noncompactness create very useful tools in nonlinear analysis. An important example of a measure of noncompactness is the so-called Hausdorff measure of noncompactness χ [7], which is defined in the following way:

$$\chi(M) := \inf\{r > 0 : \text{ there exists a finite set } F \text{ with } M \subset F + B_r\}.$$

Observe that since any finite subset F of the space X is weakly compact in X then we have

$$w(M) \leq \chi(M), \tag{1.7}$$

for any $M \in \mathcal{B}(X)$.

Now we formulate the definition of *Darbo condition* [7] being parallel to the classical definition of Lipschitz condition.

Definition 1.2. Let X be a Banach space and let ψ be a measure of (weak) noncompactness on X. Let $T\colon D(T) \subset X \to X$ be an operator. We say that T satisfies the Darbo condition with respect to ψ if there exists k, $k \geq 0$, such that for each bounded subset M of $D(T)$ the set $T(M)$ is bounded and $\psi(T(M)) \leq k\psi(M)$.

Next, we provide definitions distinguishing important classes of operators acting in Banach spaces.

Definition 1.3. [29]. A mapping $T\colon D(T) \to X$ is said to be ws- compact if it is continuous and for any weakly convergent sequence (x_n) in $D(T)$ the sequence (Tx_n) has a strongly convergent subsequence in X.

Definition 1.4. A mapping $T\colon D(T) \to X$ is said to be ww- compact if it is continuous and for any weakly convergent sequence (x_n) in $D(T)$ the sequence (Tx_n) has a weakly convergent subsequence in X.

Notice that the concepts of ww-compact and ws-compact mappings arise naturally in the study of integral and partial differential equations (see [29, 25, 26, 27, 34, 35, 41, 1, 42, 43, 20]).

2. Fixed Point Theorems

At the beginning we recall the definition of the concept of $\mathcal{D}$-Lipschizian mapping playing an important role in fixed point theory [19].

Definition 2.1. Let X be a Banach space. A mapping $T : X \to X$ is called $\mathcal{D}$-Lipschitzian if there exists a continuous nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|Tx - Ty\| \leq \phi(\|x - y\|),$$

for all $x, y \in X$. The function ϕ is called a $\mathcal{D}$-function of T. If, moreover, ϕ satisfies $\phi(r) < r$ for $r > 0$, then T is called a nonlinear contraction with a contraction function ϕ.

Remark 2.1. Obviously, every Lipschitzian mapping is $\mathcal{D}$-Lipschitzian. The converse may not be true. For example, take $f(x) = \sqrt{|x|}$, $x \in \mathbb{R}$ and consider $\phi(r) = \sqrt{r}$, $r \geq 0$. Clearly, ϕ is continuous and nondecreasing. First notice that f is subadditive. To see this, let $x, y \in \mathbb{R}$. Then,

$$\begin{aligned} (f(x+y))^2 &= |x+y| \leq |x| + |y| \\ &\leq \left(\sqrt{|x|} + \sqrt{|y|}\right)^2 \\ &= (f(x) + f(y))^2 . \end{aligned}$$

Thus, for all $x, y \in \mathbb{R}$ we have :

$$f(x+y) \leq f(x) + f(y).$$

Using the subadditivity of f we get

$$|f(x) - f(y)| \leq f(x-y) = \phi(|x-y|), \tag{2.1}$$

for all $x, y \in \mathbb{R}$. Thus, f is $\mathcal{D}$- Lipschitzian with $\mathcal{D}-$ function ϕ. Now, suppose that f is Lipschitzian with constant k. Then, for all $x \in \mathbb{R}$ we have $f(x) \leq k|x|$. Hence, for all $x \neq 0$ we have $k \geq \frac{1}{\sqrt{|x|}}$. Letting x go to zero we obtain a contradiction. Consequently, f is not Lipschitzian.

Lemma 2.1. *Let T be a $\mathcal{D}$-Lipschitzian mapping defined on a Banach space X with a $\mathcal{D}$-function ϕ. Then,*

(i) *for each bounded subset M of X we have $\chi(TM) \leq \phi(\chi(M))$. Here, χ is the Hausdorff measure of noncompactness.*

(ii) *if T is ww-compact then for each bounded subset M of X we have $w(TM) \leq \phi(w(M))$, where w stands for the De Blasi measure of weak noncompactness.*

Proof. Let M be a bounded subset of X and $r > \chi(M)$. Then there exists a finite subset F of X such that $M \subseteq F + B_r$. Let $x \in M$. Then there exist $f \in F$ and $b \in B_r$ such that $x = f + b$. Since T is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function ϕ, then

$$\|Tx - Tf\| \leq \phi(\|x - f\|) \leq \phi(\|b\|) \leq \phi(r). \tag{2.2}$$

Hence,

$$TM \subseteq TF + B_{\phi(r)}. \tag{2.3}$$

Accordingly,

$$\chi(TM) \leq \phi(r). \tag{2.4}$$

Letting $r \to \chi(M)$ and using the continuity of ϕ we deduce that

$$\chi(TM) \leq \phi(\chi(M)). \tag{2.5}$$

This proves the first assertion. Now we prove the second assertion. To this end, let M be a bounded subset of X and $r > w(M)$. Then there exists a weakly compact subset W of X such that $M \subseteq W + B_r$. Since T is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function ϕ, then

$$TM \subseteq TW + B_{\phi(r)} \subseteq \overline{TW}^w + B_{\phi(r)}. \tag{2.6}$$

Note that $\overline{TW}^w$ is weakly compact since W is weakly compact and T is ww-compact. Thus,

$$w(TM) \leq \phi(r). \tag{2.7}$$

Letting $r \to w(M)$ and using the continuity of ϕ we deduce that

$$w(TM) \leq \phi(w(M)). \tag{2.8}$$

This achieves the proof.

Remark 2.2. Note that Lemma 2.1 is a sharpening of Lemma 2.8 in [2].

In what follows we will often consider a Banach Algebra X with the norm $\|.\|$ satisfying the condition $\|xy\| \leq \|x\|.\|y\|$ for all $x, y \in X$.

Lemma 2.2. *Let* (x_n) *be a sequence in a Banach algebra* X *such that* $x_n \rightharpoonup x$ *then* $ax_n \rightharpoonup ax$ *and* $x_n a \rightharpoonup xa$ *for any fixed* $a \in X$.

Proof. Fix arbitrarily $a \in X$. Consider the left hand multiplication operator $L_a(x) = ax$ and the right hand multiplication operator $R_b(x) = xb$. Clearly L_a and R_a are continuous linear operators. Taking into account the fact that a linear operator between normed spaces is continuous if and only if it is weakly continuous, we deduce the desired assertions.

Lemma 2.3. *Let* X *be a Banach algebra. Then*

(i) *If* K *and* K' *are compact then* KK' *is also compact.*

(ii) *If* K *is weakly compact and* K' *is compact then* KK' *is weakly compact.*

Proof. The first assertion (i) is a consequence of the fact that $KK' = \psi(K \times K')$ where ψ is the continuous multiplication $\psi : (x, y) \to x.y$. Now we prove the second assertion (ii). To this end, take a sequence (x_n) in K and a sequence (y_n) in K'. Keeping in mind the compactness of K and the weak compactness of K' and by extracting a subsequence if necessary, we may assume that (x_n) converges strongly to some $x \in K$ and (y_n) converges weakly to some $y \in K'$. In view of Lemma 2.2 we infer that

$$x(y_n - y) \rightharpoonup \theta. \tag{2.9}$$

Moreover, taking into account the fact that a weakly convergent sequence is norm bounded [21] we get

$$\|(x_n - x)y_n\| \le \|x_n - x\|\|y_n\| \to 0. \tag{2.10}$$

Combining (2.9), (2.10) and the following equality

$$x_n y_n - xy = (x_n - x)y_n + x(y_n - y) \tag{2.11}$$

we conclude that $x_n y_n \rightharpoonup xy$. The proof is complete.

In the sequel the concept defined below will play a crucial role in our considerations.

Definition 2.2. Let X be a Banach algebra. We say that X is a WC-Banach algebra if the product KK' of arbitrary weakly compact subsets K, K' of X is weakly compact.

Example 2.1. Assume that S is a Hausdorff compact space and E is a Banach space. The following characterization of weak sequential convergence in the space $C(S, E)$ is well known (cf. [14, Theorem 9]: A bounded sequence $(f_n) \subset C(S, E)$ converges weakly to $f \in C(S, E)$ if and only if the sequence $(f_n(x))$ converges weakly (in E) to $f(x)$ for each $x \in S$.

Thus, if we take E to be finite dimensional Banach space then $C(S,E)$ is a WC-Banach algebra. To prove this fact, let us take two arbirary weakly compact subsets K and K' of $C(S,E)$. Let (f_n) and (g_n) be two sequences in K and K', respectively. By extracting subsequences, if necessary, we may assume that (f_n) converges weakly to some element $f \in K$ and (g_n) converges weakly to some element $g \in K'$. Hence, for an arbitrary $x \in S$, the sequence $(f_n(x))$ converges weakly in E to $f(x)$ while the sequence $(g_n(x))$ converges weakly (in E) to $g(x)$. Since E is finite dimensional, the weak convergence in E is equivalent to the strong convergence. This yields that

$$(f_n g_n)(x) = f_n(x)g_n(x) \to f(x)g(x).$$

The arbitrariness of x implies that $f_n g_n \rightharpoonup fg$. Thus the set KK' is weakly compact and our assertion follows. By similar reasoning we may prove that if E is a WC-Banach algebra then $C(S,E)$ is also a WC-Banach algebra.
The following lemma will play a key role in our further study. In order to present this lemma assume that X is a Banach algebra. Moreover, let χ be the Hausdorff measure of noncompactness on X and let w be the De Blasi measure of weak noncompactness on X.

Lemma 2.4. *Let M and M' be bounded subsets of a Banach algebra X. Then we have the following assertions:*

(i) $w(MM') \leq \|M'\|w(M) + \|M\|\chi(M') + w(M)\chi(M')$, *where the symbol $\|H\|$ denotes the norm of a set H $(H \subset X)$ i.e. $\|H\| = \sup\{\|x\| : x \in H\}$.*

(ii) *If X is a WC-Banach algebra then*

$$w(MM') \leq \|M'\|w(M) + \|M\|w(M') + w(M)w(M').$$

(iii) $\chi(MM') \leq \|M'\|\chi(M) + \|M\|\chi(M') + \chi(M)\chi(M')$.

Proof. For the proof of (i) let us take two bounded subsets M and M' of X. Next, fix arbitrarily numbers r, t such $r > w(M)$ and $t > \chi(M')$. Then there exist a weakly compact set W and a finite set F in X such that

$$M \subseteq W + B_r \tag{2.12}$$

and

$$M' \subseteq F + B_t. \tag{2.13}$$

Further, take $z \in MM'$. then we can find $x \in M$ and $y \in M'$ such that $z = xy$. Keeping in mind (2.12) and (2.13) we infer that there are $w \in W$, $f \in F$, $u \in B_r$ and $v \in B_t$ such that $x = w + u$ and $y = f + v$. Hence we get

$$\begin{aligned} z = xy &= (w+u)(f+v) = wf + wv + uf + uv \\ &= wf + (x-u)v + u(y-v) + uv \\ &= wf + xv + uy - uv. \end{aligned}$$

This yields the inclusion

$$MM' \subset WF + MB_t + B_rM' + B_rB_t \subset WF + B_{\|M\|t+\|M'\|r+rt}. \tag{2.14}$$

Now, taking into account Lemma 2.3 (ii) and the definition of the De Blasi measure of weak noncompactness w, we obtain

$$w(MM') \leq \|M\|t + \|M'\|r + rt. \tag{2.15}$$

Next, letting $r \to w(M)$ and $t \to \chi(S')$ we get

$$w(MM') \leq \|M\|\chi(M') + \|M'\|w(M) + w(M)\chi(M'). \tag{2.16}$$

To prove (ii) assume that M and M' are arbitrary bounded subsets of a WC-Banach algebra X. Let r, t be fixed numbers with $r > w(M)$ and $t > w(M')$. Then we can find two weakly compact subsets W_1 and W_2 of X such that

$$M \subseteq W_1 + B_r, \tag{2.17}$$

and

$$M' \subseteq W_2 + B_t. \tag{2.18}$$

Now, take $z \in MM'$. Then z can be represented in the form $z = xy$ with $x \in M$ and $y \in M'$. In view of (2.17) and (2.18) there exist $w_1 \in W_1$, $w_2 \in W_2$, $u \in B_r$, and $v \in B_t$ such that $x = w_1 + u$, $y = w_2 + v$. Hence, similarly as in the proof of part (i), we get

$$z = xy = (w_1 + u)(w_2 + v) = w_1w_2 + xv + uy - uv.$$

The above equality implies the following inclusion

$$MM' \subset W_1W_2 + MB_t + B_rM' + B_tB_r \subset W_1W_2 + B_{\|M\|t+\|M'\|r+rt}.$$

Thus, keeping in mind the fact that X is a WC-Banach algebra (cf. Definition 2.2), in view of the definition of the De Blasi measure of weak noncompactness w, we obtain

$$w(MM') \leq \|M\|t + \|M'\|r + rt.$$

Letting $r \to w(M)$ and $t \to w(M')$, we get

$$w(MM') \leq \|M\|w(M') + \|M'\|w(M) + w(M)w(M').$$

The proof of the assertion (iii) is similar to (ii), so it is omitted.

On the basis of Lemma 2.4 we can now derive the following result concerning the existence of fixed points for operators acting in a WC-Banach algebra and satisfying some conditions expressed in terms of weak sequential continuity and the measure of weak noncompactness w.

Theorem 2.1. *Assume that Ω is a nonempty, closed and convex subset of a WC-Banach algebra X. Further, assume that P and T are operators acting weakly sequentially continuously from Ω into X in such a way that $P\Omega$ and $T\Omega$ are bounded. Apart from this we require that the operator $S = PT$ (the product of P and T) transforms Ω into itself and is weakly sequentially continuous. If the operators P and T satisfy the Darbo condition with respect to the De Blasi measure of weak noncompactness w, with constants k_1 and k_2, respectively, then the operator S satisfies on Ω the Darbo condition (with respect to w) with the constant $k_1\|T\Omega\| + k_2\|P\Omega\| + k_1k_2w(\Omega)$. Particularly, if $k_1\|T\Omega\| + k_2\|P\Omega\| + k_1k_2w(\Omega) < 1$, then S is a contraction with respect to w and has at least one fixed point in the set Ω.*

Proof. Take an arbitrary nonempty bounded subset M of Ω. Then, in view of our assumptions and Lemma 2.4 (ii), we obtain

$$\begin{aligned} w(SM) &\le w((PM)(TM)) \\ &\le \|PM\|w(TM) + \|TM\|w(PM) + w(PM)w(TM) \\ &\le \|PM\|k_2w(M) + \|TM\|k_1w(M) + k_1k_2w(M)^2 \\ &\le [k_1\|T\Omega\| + k_2\|P\Omega\| + k_1k_2w(\Omega)]\, w(M) \\ &= kw(M), \end{aligned}$$

where $k = k_1\|T\Omega\| + k_2\|P\Omega\| + k_1k_2w(\Omega)$.

Since $k < 1$, we have that S is a contraction with respect to the measure of weak noncompactness w. On the other hand the operator S transforms Ω into itself and is weakly sequentially continuous on Ω. Thus, keeping in mind a fixed point theorem of Arino, Gautier and Penot [3] and its Darbo type generalization for the measure of weak noncompactness w [4] we conclude that the operator S has at least one fixed point in the set Ω. The proof is complete.

Observe that the assumption requiring that the operator $S = PT$ is weakly sequentially continuous on Ω can be omitted if we assume that the operation of multiplication $(x, y) \to xy$ is weakly sequentially continuous in a Banach algebra X, i.e. the following condition is satisfied :

$(\mathcal{P})$ if $\{x_n\}, \{y_n\}$ are sequences in a Banach algebra X such that $x_n \rightharpoonup x, y_n \rightharpoonup y$, for some $x, y \in X$, then $x_ny_n \rightharpoonup xy$.

As an example of the Banach algebra which satisfies condition $(\mathcal{P})$ may serve the Banach algebra $C[a, b]$ consisting of real functions being continuous on the interval $[a, b]$, with the standard maximum norm. Note also that condition $(\mathcal{P})$ was used in [9].

Further, let us observe that condition $(\mathcal{P})$ defined above implies the WC-Banach algebra structure. Indeed, we have the following:

Lemma 2.5. *If X is a Banach algebra satisfying condition $(\mathcal{P})$ then X is a WC-Banach algebra.*

Proof. Let K, K' be arbitrarily weakly compact subsets of X. Take an arbitrary sequence $(z_n) \subset KK'$. Then, for any fixed n we can find $x_n \in K$ and $y_n \in K'$ such that $z_n = x_n y_n$. Consider the sequences $(x_n) \subset K$ and $(y_n) \subset K'$. Since K is weakly compact we can extract a subsequence (x_{k_n}) of the sequence (x_n) which is weakly convergent to some $x \in K$. Further, keeping in mind the weak compactness of the set K' we can extract a subsequence (y_{l_n}) of the sequence (y_{k_n}) which is weakly convergent to some $y \in K'$. Obviously, $x_{l_n} \rightharpoonup x$. In view of our assumption we deduce that $z_{l_n} = x_{l_n} y_{l_n} \rightharpoonup xy$. This shows that X is a WC-Banach algebra and completes the proof.

In what follows we indicate a wide class of Banach algebras satisfying condition $(\mathcal{P})$.

Definition 2.3. We say that a Banach space X has the Dunford-Pettis Property (DPP, in short) if for each Banach space Y every weakly compact linear operator $T\colon X \to Y$ maps weakly convergent sequences into strongly convergent sequences.

Since every Banach algebra is a Banach space then we can also consider Banach algebras with DPP. It can be shown that $L^1(\mu)$ and $C(K)$ have the DPP. We refer to [17] for an excellent survey of DPP.

In order to formulate our result let us first recall some relevant definitions and results. Namely, a homogeneous continuous polynomial on a Banach space X is a mapping P having the form $P(x) = T(x, x, \ldots, x)$, where $T\colon X \times X \times \ldots \times X \to \mathbb{R}$ is a multilinear continuous map on X. Notice that continuous polynomials are usually not continuous with respect to the weak topology. It was proved by Ryan [40] that in spaces with DPP all multilinear forms are weakly sequentially continuous.

Now we formulate the above announced result.

Theorem 2.2. *Let X be a commutative Banach algebra with DPP. Then X satisfies condition $(\mathcal{P})$. Particularly, X is a WC-Banach algebra.*

Proof. Take an arbitrary functional $\varphi \in X^*$, where X^* denotes the dual space of X. Consider the mapping $m_\varphi\colon X \times X \to \mathbb{R}$ defined by the formula $m_\varphi(x, y) = \varphi(xy)$. Obviously, m_φ is a continuous bilinear map. Put $P_\varphi(x) = m_\varphi(x, x) = \varphi(x^2)$. In view of the above mentioned Ryan's result we have that P_φ is weakly sequentially continuous. Further, let $(x_n)_n$ and $(y_n)_n$ be sequences in X such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$. Then, from the equality

$$x_n y_n = \frac{1}{4}((x_n + y_n)^2 - (x_n - y_n)^2) \tag{2.19}$$

we infer that

$$\varphi(x_n y_n) = \frac{1}{4}(P_\varphi(x_n + y_n) + P_\varphi(x_n - y_n)) \tag{2.20}$$

Hence, taking into account the weak sequential continuity of P_φ we obtain

$$\varphi(x_n y_n) \to \frac{1}{4}(P_\varphi(x+y) + P_\varphi(x-y)) = \varphi(xy) \tag{2.21}$$

Obviously this means that

$$x_n y_n \rightharpoonup xy. \tag{2.22}$$

Thus we proved that X satisfies condition $(\mathcal{P})$. Finally, applying Lemma 2.5 we complete the proof.

On the basis of Theorem 2.1 together with Lemma 2.1 we obtain the following weak version of [36, Theorem 1 and Theorem 2].

Theorem 2.3. *Let Ω be a bounded subset of a Banach algebra X with property $(\mathcal{P})$, and suppose $T\colon \Omega \to X$ is of the form $Tx = x_0 + (Ax)(Bx)$ where*

- (i) $x_0 \in \Omega$;
- (ii) *$A : \Omega \to X$ is weakly sequentially continuous and $\mathcal{D}$-lipschitzian with $\mathcal{D}$-function ϕ_A; and*
- (iii) *$B\colon \Omega \to X$ is weakly sequentially continuous and maps bounded sets onto relatively weakly compact sets.*

Suppose $\beta \equiv \sup_{x\in\Omega} \|Bx\| < \infty$. If $\beta\phi_A(r) < r$, whenever $r > 0$, then T is contraction with respect to w.

Theorem 2.4. *Suppose that B is a weakly sequentially continuous, weakly compact operator on a Banach algebra X with property $(\mathcal{P})$ and suppose $x_0 \in X$. If there exists a closed, convex set Ω in X such that $\sup_{x\in\Omega} \|Bx\| < 1$ and such that $x_0 + xBx \in \Omega$ for each $x \in \Omega$, then the equation*

$$x = x_0 + xBx$$

has a solution $x \in \Omega$.

3. Operator Equations in Banach Algebras

In this section we are going to prove the existence of solutions of some operator equations considered in Banach algebras. We will use the ideas developed in the previous sections. The results which we will prove generalize those obtained in [9], among others.

We start with the following key lemma.

Lemma 3.6. *Let Ω be a nonempty bounded closed subset of a Banach algebra X and let $A, C\colon X \to X$ be $\mathcal{D}$-Lipschitzian mappings with $\mathcal{D}$- functions ϕ_A and ϕ_C respectively. Assume that for each $r > 0$ we have $\|\Omega\|\phi_A(r) + \phi_C(r) < r$. Then $(\frac{I-C}{A})^{-1}\colon \Omega \to X$ exists and is continuous.*

Proof. Let $y \in \Omega$ be fixed. The map τ_y which assigns to each $x \in X$ the value $A(x).y + C(x)$ defines a nonlinear contraction with a contraction function $\psi(r) = \|\Omega\|\phi_A(r) + \phi_C(r)$, $r > 0$. Indeed, for all $x_1, x_2 \in X$ we have:

$$\begin{aligned}\|\tau_y(x_1) - \tau_y(x_2)\| &\leq \|Ax_1 - Ax_2\|\|y\| + \|Cx_1 - Cx_2\| \\ &\leq \|\Omega\|\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|).\end{aligned}$$

Now the Boyd and Wong fixed point theorem [10] guarantees that there exists a unique point $x^* \in X$ such that $\tau_y(x^*) = x^*$, i.e. $y = (\frac{I-C}{A})x^*$. Thus, the operator $N := \left(\frac{I-C}{A}\right)^{-1} : \Omega \to X$ is well defined. Now we show that $N \colon \Omega \to X$ is continuous. To see this, let (x_n) be a sequence in Ω converging to a point x. Since Ω is closed, then $x \in \Omega$. First notice that for each $z \in \Omega$ we have

$$Nz = CNz + (ANz)z. \tag{3.1}$$

Hence,

$$\begin{aligned}\|Nx_n - Nx\| &\leq \|CNx_n - CNx\| + \|(ANx_n)x_n - (ANx)x\| \\ &\leq \|CNx_n - CNx\| + \|ANx\|\|x_n - x\| + \|ANx_n - ANx\|\|x_n\| \\ &\leq \phi_C(\|Nx_n - Nx\|) + \phi_A(\|Nx_n - Nx\|)\|\Omega\| + \|ANx\|\|x_n - x\|.\end{aligned}$$

Thus,

$$\begin{aligned}&\limsup_n \|Nx_n - Nx\| \\ &\leq \phi_C(\limsup_n \|Nx_n - Nx\|) + \phi_A(\limsup_n \|Nx_n - Nx\|)\|\Omega\|.\end{aligned} \tag{3.2}$$

This shows that $\lim_n \|Nx_n - Nx\| = 0$ and consequently N is continuous on Ω. This completes the proof.

Now we are in position to state the following result.

Theorem 3.1. *Let X be a Banach algebra and let ψ be a measure of weak noncompactness on X. Let Ω be a nonempty closed and convex subset of X and let $A, C \colon X \to X$ and $B : \Omega \to X$ be weakly sequentially continuous operators satisfying the following conditions:*

- (i) *The operators A and C are $\mathcal{D}$-Lipschitzian mappings with $\mathcal{D}$- functions ϕ_A and ϕ_C respectively.*
- (ii) *The set $B(\Omega)$ is bounded and the operator $\left(\frac{I-C}{A}\right)^{-1} B$ is ψ-condensing on Ω.*
- (iii) *The equality $(x = AxBy + Cx)$ with $y \in \Omega$ implies $x \in \Omega$.*

Then the operator equation $x = AxBx + Cx$ *has a solution in the set* Ω *provided that* $Q\phi_A(r) + \phi_C(r) < r$ *for* $r > 0$, *where* $Q = \|B(\Omega)\|$.

Remark 3.3. In Theorem 3.1, the set Ω need not be bounded.

Proof. In view of Lemma 3.6, the operator $\tau := \left(\frac{I-C}{A}\right)^{-1} B\colon \Omega \to X$ is well-defined. Notice also by assumption (iii) we have $\tau(\Omega) \subset \Omega$. Now, let $x_0 \in \Omega$ and

$$\mathcal{F} = \{M : x_0 \in M \subset \Omega, M \text{ is a closed convex set and } \tau(M) \subset M\}.$$

Clearly, $\mathcal{F} \neq \emptyset$ since $\Omega \in \mathcal{F}$. Put $\Lambda = \cap_{M\in\mathcal{F}} M$. Then $x_0 \in \Lambda \subset \Omega$, Λ is a closed convex set and $\tau(\Lambda) \subset \Lambda$. Notice that $Conv\{\tau(\Lambda), x_0\} \subset \Lambda$, where $Conv(M)$ denotes the closed convex hull of the set M. We thus have

$$\tau(Conv\{\tau(\Lambda), x_0\}) \subset \tau(\Lambda) \subset Conv\{\tau(\Lambda), x_0\},$$

which shows that $Conv\{\tau(\Lambda), x_0\} \in \mathcal{F}$. It then follows that $Conv\{\tau(\Lambda), x_0\} = \Lambda$. Using the properties of measures of weak noncompactness we get:

$$\psi(\Lambda) = \psi(Conv\{\tau(\Lambda), x_0\}) = \psi(\{\tau(\Lambda), x_0\}) = \psi(\tau(\Lambda)).$$

From our assumptions we get $\psi(\Lambda) = 0$, and therefore, Λ is a nonempty weakly compact convex set. Now we show that $\tau\colon \Lambda \to \Lambda$ is weakly sequentially continuous. To see this, let (x_n) be a sequence of Λ which converges weakly to some $x \in \Omega$. Since, $\tau(\Lambda)$ is relatively weakly compact then there is a subsequence (x_{n_k}) of (x_n) such that

$$\tau x_{n_k} \rightharpoonup z. \tag{3.3}$$

Taking into account that $\tau(x_{n_k}) = A\tau(x_{n_k})Bx_{n_k} + C\tau(x_{n_k})$, the weak sequential continuity of A, B and C yields

$$z = AzBx + Cz \tag{3.4}$$

Thus, $z = \tau(x)$.

Consequently,

$$\tau x_{n_k} \rightharpoonup z = \tau x \tag{3.5}$$

Now we show that

$$\tau x_n \rightharpoonup \tau x. \tag{3.6}$$

Suppose the contrary, then there exists a weak neighborhood N^w of $\tau(x)$ and a subsequence (x_{n_j}) of (x_n) such that $\tau x_{n_j} \notin N^w$ for all $j \geq 1$. Since (x_{n_j}) converges weakly to x, then arguing as before we may extract a subsequence $(x_{n_{j_k}})$ of (x_{n_j}) such that $\tau x_{n_{j_k}} \rightharpoonup \tau(x)$. This is not possible, since $\tau x_{n_{j_k}} \notin N^w$ for all $k \geq 1$. As a result, τ is weakly sequentially continuous. Now by the Arino-Gautier-Penot fixed point theorem we infer that there exists $x \in \Lambda$ such that $x = \tau(x) = A(\tau(x))B(x) + C(\tau(x)) = AxBx + Cx$. This completes the proof.

Taking $C \equiv 0$ in Theorem 3.1 we obtain the following result.

Theorem 3.2. *Let X be a Banach algebra and let ψ be a measure of weak noncompactness on X. Let Ω be a nonempty closed and convex subset of X and let $A\colon X \to X$ and $B : \Omega \to X$ be weakly sequentially continuous operators satisfying the following conditions:*

(i) *The operator A is a $\mathcal{D}$-Lipschitzian mapping with $\mathcal{D}$- functions ϕ_A.*

(ii) *The set $B(\Omega)$ is bounded and the operator $\left(\frac{I}{A}\right)^{-1} B$ is ψ-condensing on Ω.*

(iii) *The equality $(x = AxBy)$ with $y \in \Omega$ implies $x \in \Omega$.*

Then the operator equation $x = AxBx$ has a solution in the set Ω provided $Q\phi_A(r) < r$ for $r > 0$, where $Q = \|B(\Omega)\|$.

Now, we prove the following useful result.

Theorem 3.3. *Let Ω be a nonempty, closed and convex subset of a WC- Banach algebra X and let $A, C\colon X \to X$ and $B : \Omega \to X$ be weakly sequentially continuous operators satisfying the following conditions:*

(i) *The operators A and C are $\mathcal{D}$-Lipschitzian mappings with $\mathcal{D}$- functions ϕ_A and ϕ_C respectively.*

(ii) *The set $B(\Omega)$ is relatively weakly compact.*

(iii) *The equality $(x = AxBy + Cx)$ with $y \in \Omega$ implies $x \in \Omega$.*

Then the operator equation $x = AxBx + Cx$ has a solution in the set Ω provided $Q\phi_A(r) + \phi_C(r) < r$ for $r > 0$, where $Q = \|B(\Omega)\|$.

Proof. According to Theorem 3.1 it suffices to show that $\tau := \left(\frac{I-C}{A}\right)^{-1} B$ maps bounded sets into weakly compact sets. To see this, let M be a bounded subset of Ω. It is easily seen that

$$\tau(M) \subset C(\tau(M)) + A(\tau(M))B(M). \tag{3.7}$$

Using Lemmas 2.1 and 2.4 together with the compactness of $B(\Omega)$ we infer that

$$\begin{aligned} w(\tau(M)) &\leq w(C(\tau(M)) + A(\tau(M))B(M)) \\ &\leq w(C(\tau(M))) + w(A(\tau(M))B(M)) \\ &\leq \phi_C(w(\tau(M)) + Q\phi_A(\tau(M)). \end{aligned}$$

This shows that $w(\tau(M)) = 0$. Thus, $\tau(M)$ is relatively weakly compact. The result follows from Theorem 3.1.

The below given result presents a variant of Theorem 3.3 but, in fact, it is a corollary of that theorem.

Corollary 3.1. *Let Ω be a nonempty, closed and convex subset of a WC- Banach algebra X and let $A, B, C\colon X \to X$ be weakly sequentially continuous operators such that:*

(i) *A is nonexpansive.*

(ii) *C is a $\mathcal{D}$-Lipschitzian mapping with a $\mathcal{D}$-function ϕ_C.*

(iii) *The set $B(\Omega)$ is relatively weakly compact.*

(iv) *The equality $x = AxBy + Cx$ with $y \in \Omega$ implies $x \in \Omega$.*

Then the operator equation $x = AxBx+Cx$ has a solution, whenever $Qr+\phi_C(r) < r$ for $r > 0$, where $Q = \|B(\Omega)\|$.

As a spacial case of Corollary 3.1 we derive the following result.

Corollary 3.2. *Let Ω be a nonempty, closed and convex subset of a WC- Banach algebra X and let $B, C\colon X \to X$ be weakly sequentially continuous operators such that:*

(i) *The set $B(\Omega)$ is relatively weakly compact.*

(ii) *C is a $\mathcal{D}$-Lipschitzian mapping with a $\mathcal{D}$-function ϕ_C.*

(iii) *The equality $x = xBy + Cx$ with $y \in \Omega$ implies $x \in \Omega$.*

Then the operator equation $x = xBx + Cx$ has a solution, whenever $Qr + \phi_C(r) < r$ for $r > 0$, where $Q = \|B(\Omega)\|$.

As a special case of Corollary 3.2 we obtain the following weak version of [36, Theorem 2].

Corollary 3.3. *Suppose that B is a weakly sequentially continuous, weakly compact operator on the WC-Banach algebra X, and suppose $x_0 \in X$. If there exists a closed, convex set Ω in X such that $\sup_{x\in\Omega} \|Bx\| < 1$ and such that the equality $(x = x_0 + xBy)$ with $y \in \Omega$ implies $x \in \Omega$, then the equation*

$$x = x_0 + xBx,$$

has a solution $x \in \Omega$.

Our next result has a similar form as those given above but assumptions imposed in it are different.

Theorem 3.4. *Let X be a Banach algebra and let ψ be a measure of weak noncompactness on X. Let Ω be a nonempty closed and convex subset of X and let $A, C\colon X \to X$ and $B : \Omega \to X$ be continuous operators satisfying the following conditions:*

(i) *The operators A and C are ww-compact $\mathcal{D}$-Lipschitzian mappings with $\mathcal{D}$- functions ϕ_A and ϕ_C respectively.*

(*ii*) B *is ws-compact.*

(*iii*) *The set* $B(\Omega)$ *is bounded and the operator* $\left(\frac{I-C}{A}\right)^{-1}B$ *is* ψ*-condensing on* Ω*.*

(*iv*) *The equality* $(x = AxBy + Cx)$ *with* $y \in \Omega$ *implies* $x \in \Omega$*.*

Then the operator equation $x = AxBx + Cx$ *has a solution in the set* Ω *provided* $Q\phi_A(r) + \phi_C(r) < r$ *for* $r > 0$*, where* $Q = \|B(\Omega)\|$*.*

Proof. In view of Lemma 3.6, the operator $\tau := \left(\frac{I-C}{A}\right)^{-1}B\colon \Omega \to X$ is well-defined and continuous. Notice also since B is ws-compact then τ is ws-compact. Next, by assumption (iii) we know that $\tau(\Omega) \subset \Omega$. Now, let $x_0 \in \Omega$ and

$$\mathcal{F} = \{M : x_0 \in M \subset \Omega, M \text{ is a closed convex set and } \tau(M) \subset M\}.$$

Clearly, $\mathcal{F} \neq \emptyset$ since $\Omega \in \mathcal{F}$. Put $\Lambda = \cap_{M\in\mathcal{F}} M$. Then $x_0 \in \Lambda \subset \Omega$, Λ is a closed convex set and $\tau(\Lambda) \subset \Lambda$. Notice that $Conv\{\tau(\Lambda), x_0\} \subset \Lambda$, where $Conv(M)$ denotes the closed convex hull of the set M. We thus have

$$\tau(Conv\{\tau(\Lambda), x_0\}) \subset \tau(\Lambda) \subset Conv\{\tau(\Lambda), x_0\},$$

which shows that $Conv\{\tau(\Lambda), x_0\} \in \mathcal{F}$. It then follows that $Conv\{\tau(\Lambda), x_0\} = \Lambda$. Using the properties of measures of weak noncompactness we get:

$$\psi(\Lambda) = \psi(Conv\{\tau(\Lambda), x_0\}) = \psi(\{\tau(\Lambda), x_0\}) = \psi(\tau(\Lambda)).$$

In view of assumption (ii) , we obtain $\psi(\Lambda) = 0$, and therefore, Λ is a nonempty weakly compact convex set. Since B is ws-compact, then $B(\Lambda)$ is relatively compact. Now Lemma 3.6 guarantees that $\tau(\Lambda)$ is compact. Invoking the Schauder fixed point theorem we conclude immediately that there exists $x \in \Lambda$ such that $x = \tau(x) = A(\tau(x))B(x) + C(\tau(x)) = AxBx + Cx$. The proof is complete.

Now we state the following result.

Theorem 3.5. *Let* Ω *be a nonempty, closed and convex subset of a WC- Banach algebra* X *and let* $A, B, C\colon X \to X$ *be continuous operators satisfying the following conditions:*

(*i*) *The operators* A *and* C *are ww-compact* $\mathcal{D}$*-Lipschitzian mappings with* $\mathcal{D}$*- functions* ϕ_A *and* ϕ_C *respectively.*

(*ii*) B *is ws-compact and* $B(\Omega)$ *is relatively weakly compact.*

(*iii*) *The equality* $x = AxBy + Cx$ *with* $y \in \Omega$ *implies* $x \in \Omega$*.*

Then the operator equation $x = AxBx + Cx$ *has a solution provided* $Qr + \phi_C(r) < r$ *for* $r > 0$*, where* $Q = \|B(\Omega)\|$*.*

Proof. The reasoning in Theorem 3.3 shows that $\tau := \left(\frac{I-C}{A}\right)^{-1} B$ maps bounded sets into weakly compact sets. The result follows from Theorem 3.4.

4. Application

In this section we discuss the solvability of the quadratic integral equation

$$x(t) = u(x(t)) + f(x(t)) \int_0^1 k(t,s)g(s,x(s))ds, \quad t \in [0,1]. \tag{4.1}$$

The integral in (4.1) is understood to be the Pettis integral and solutions to (4.1) will be sought in $E := C([0,1], X)$, where X is a (real) reflexive Banach algebra and E is endowed with its standard norm $\|x\| = \sup_{t\in[0,1]} \|x(t)\|$.

This equation is a general form of many integral equations, such as the Chandrasekhar integral equation arising in radiative transfer [12] and the Hammerstein integral equation [13, 22, 23].

We consider equation (4.1) under the following assumptions :

(i) $u\colon X \to X$ is weakly sequentially continuous and there exists a constant C_u such that $\|u(r) - u(s)\| \le C_u \|r - s\|$ for all $r, s \in X$,

(ii) $f\colon X \to X$ is weakly sequentially continuous and there exists a constant C_f such that $\|f(r) - f(s)\| \le C_f \|r - s\|$ for all $r, s \in X$,

(iii) for each continuous $x\colon [0,1] \to X$, $g(.,x(.))$ is weakly measurable on $[0,1]$ and for each $t \in [0,1]$, $g(t,.)$ is weakly sequentially continuous,

(iv) for any $r > 0$, there exists a nonnegative constant h_r with $\|g(s,x(s))\| \le h_r$ for all $t \in [0,1]$ and all $x \in E$ with $\|x\| \le r$,

(v) $k\colon [0,1] \times [0,1] \to \mathbb{R}$ is continuous with respect to the first variable, integrable with respect to the second variable and there is a constant k^* with $\int_0^1 |k(t,s)|ds \le k^*$ for all $t \in [0,1]$,

(vi) there is a $r_0 > 0$ such that

$$C_u + C_f k^* h_{r_0} < 1,$$

and

$$\frac{\|u(0)\| + \|f(0)\| k^* h_{r_0}}{1 - C_u - C_f k^* h_{r_0}} \le r_0.$$

Remark 4.1. It should be noted that if X is a WC-Banach algebra then $E := C([0,1], X)$ is also a WC-Banach algebra. In our considerations, X need not be a WC-Banach algebra.

Observe that the equation (4.1) may be written in the form

$$x(t) = (Ux)(t) + (Fx(t))(Kx(t)), \tag{4.2}$$

where the operators F, G and K are defined on $E := C([0,1], X)$ by

$$(Kx)(t) = \int_0^1 k(t,s)g(s,x(s))ds, \tag{4.3}$$

$$(Ux)(t) = u(x(t)), \quad (Fx)(t) = f(x(t)). \tag{4.4}$$

We emphasize that for all $t \in [0,1]$ and for all $y \in E$ with $\|y\| \leq r_0$ (the constant r_0 is defined in assumption (vi)) we have :

$$\begin{aligned} \|(Ky)(t)\| &\leq \int_0^1 |k(t,s)| \|g(s,y(s))\| ds \\ &\leq h_{r_0} \int_0^1 |k(t,s)| ds \\ &\leq k^* h_{r_0} \end{aligned}$$

Thus,

$$\|(Ky)(t)\| \leq k^* h_{r_0}, \tag{4.5}$$

for all $y \in E$ with $\|y\| \leq r_0$ and for all $t \in [0,1]$.
Consequently,

$$\|K(B_{r_0})\| = \sup_{\|y\| \leq r_0} \|Ky\| \leq k^* h_{r_0}. \tag{4.6}$$

We are now ready to state the main result of this section.

Theorem 4.1. *Let X be a reflexive Banach algebra and suppose (i)-(vi) hold. Then (4.1) has a solution in $E = C([0,1], X)$.*

Proof. We shall point out that the assumptions of Theorem 3.1 are fulfilled. The proof is divided into several steps.

Step 1. By assumptions (i) and (ii) we know that U and F maps E into itself. Now, we show that K maps B_{r_0} into E. First notice that for $x \in B_{r_0}$ we have $g(.,x(.))$ is weakly measurable on $[0,1]$ and $\|g(t,x(t))\| \leq h_{r_0}$ for all $t \in [0,1]$. Hence, $\phi(g(.,x(.))$ is Lebesgue integrable on $[0,1]$ for each $\phi \in E^*$. Furthermore, since, $s \to k(t,s)$ is Lebesgue integrable, then $\phi(k(t,.)g(.,x(.))) = k(t,.)\phi(g(.,x(.))$ is Lebesgue integrable for each $\phi \in E^*$. Thus, $k(t,.)g(.,x(.))$ is Pettis integrable on $[0,1]$. This implies that K is well defined. Now, suppose $x\colon [0,T] \to X$ is continuous with $\|x\| \leq r_0$. Then there exists a constant h_{r_0} with $\|g(t, x(t))\| \leq h_{r_0}$ for all $t \in [0,1]$. Let $t, s \in [0,1]$ with $t > s$. Without loss of generality we assume that $Kx(t) - Kx(s) \neq 0$. Then, from the Hahn-Banach theorem it follows that there exists $\phi \in E^*$ with $\|\phi\| = 1$ and $\|Kx(t) - Kx(s)\| = \phi(Kx(t) - Kx(s))$. Hence,

$$\begin{aligned}\|Kx(t) - Kx(s)\| &= \phi\left(\int_0^1 (k(t,\rho) - k(s,\rho))g(\rho, x(\rho))d\rho\right)\\ &= \int_0^1 (k(t,\rho) - k(s,\rho))\phi(g(\rho, x(\rho)))d\rho\\ &\leq h_{r_0}\int_0^1 (k(t,\rho) - k(s,\rho))d\rho.\end{aligned}$$

By assumption (v) we infer that Kx is continuous.

Step 2. We show that the operators U, F and K are weakly sequentially continuous. By assumptions (i) and (ii) we know that U and F are weakly sequentially continuous on E. Now we show that K is weakly sequentially continuous on E. To see this, let (x_n) be a sequence in E which converges weakly to $x \in E$. Clearly, the sequence (x_n) is norm bounded. From our assumptions it follows that for all $t \in [0,1]$, $g(t, x_n(t))$ converges weakly to $g(t, x(t))$. Thus, $\phi(g(t, x_n(t)))$ converges strongly to $\phi(g(t, x(t)))$ for all $\phi \in E^*$. Applying the Lebesgue dominated convergence theorem for Pettis integral [28], we get

$$\begin{aligned}\phi\left(\int_0^1 k(t,s)g(s, x_n(s))ds\right) &= \int_0^1 k(t,s)\phi(g(s, x_n(s)))ds\\ &\to \int_0^1 k(t,s)\phi(g(s, x(s)))ds.\end{aligned}$$

This implies that $(Kx_n)(t)$ converges weakly to $(Kx)(t)$ for all $t \in [0,1]$. Thus, Kx_n converges weakly to Kx.

Step 3. We illuminate that the equality $(x = FxKy + Ux)$ with $y \in B_{r_0}$ implies $x \in B_{r_0}$. Let $x \in E$ with $x = Ux + (Fx)(Ky)$ and $\|y\| \leq r_0$. Then for all $t \in [0,1]$ we have

$$\begin{aligned}\|x(t)\| &\leq \|u(x(t))\| + \|f(x(t))\|\|Ky(t)\|\\ &\leq \|u(0)\| + C_u\|x(t)\| + (\|f(0)\| + C_f\|x(t)\|)(\|Ky(t)\|)\\ &\leq \|u(0)\| + C_u\|x(t)\| + k^*h_{r_0}\|f(0)\| + k^*h_{r_0}C_f\|x(t)\|\end{aligned}$$

In view of assumption (vi) we obtain the estimate :

$$\|x\| \leq \frac{\|u(0)\| + k^*h_{r_0}\|f(0)\|}{1 - C_u - k^*h_{r_0}c_f} \leq r_0.$$

Step 4. We prove that $K(B_{r_0})$ is equicontinuous. To see this, let $x \in B_{r_0}$ and $t, s \in [0,1]$. Then,

$$\begin{aligned} \|Kx(t) - Kx(s)\| &\leq \int_0^1 |k(t,\rho) - k(s,\rho)| \|g(\rho, x(\rho))\| d\rho \\ &\leq h_{r_0} \int_0^1 |k(t,\rho) - k(s,\rho)| d\rho \end{aligned}$$

This together with assumption (v) guarantees that $K(B_{r_0})$ is equicontinuous.

Step 5. We show that $\tau(B_{r_0}) := \left(\frac{I-U}{F}\right)^{-1} K(B_{r_0})$ is relatively weakly compact. To see this, notice first that for all $x \in B_{r_0}$ we have

$$\tau(x) = U\tau(x) + (F\tau(x))(Kx). \tag{4.7}$$

From the equicontinuity of $K(B_{r_0})$, it follows that for any $\epsilon > 0$, there is a $\delta > 0$ such that $|t - s| < \delta$ implies

$$\|Kx(t) - Kx(s)\| \leq \left(\frac{1 - C_u - k^* h_{r_0} C_f}{\|f(0)\| + C_f r_0}\right) \epsilon, \ x \in B_{r_0}.$$

Hence for $t, s \in [0,1]$ with $|t - s| < \delta$ we have

$$\begin{aligned} \|\tau x(t) - \tau x(s)\| &= \|U\tau x(t) + (F\tau x(t))(Kx(t)) - U\tau x(s) - (F\tau x(s))(Kx(s))\| \\ &\leq \|U\tau x(t) - U\tau x(s)\| + \|F\tau x(t)\| \|Kx(t) - Kx(s)\| \\ &\quad + \|F\tau x(t) - F\tau x(s)\| \|Kx(s)\| \\ &\leq C_u \|\tau x(t) - \tau x(s)\| + (\|f(0)\| + C_f r_0) \|Kx(t) - Kx(s)\| \\ &\quad + k^* h_{r_0} C_f \|\tau x(t) - \tau x(s)\| \end{aligned}$$

This implies

$$\|\tau x(t) - \tau x(s)\| \leq \left(\frac{\|f(0)\| + C_f r_0}{1 - C_u - k^* h_{r_0} C_f}\right) \|Kx(t) - Kx(s)\| \leq \epsilon.$$

Thus, $\tau(B_{r_0})$ is equicontinuous. Let (τx_n) be any sequence in $\tau(B_{r_0})$. By reflexivity, for each $t \in [0,1]$ the set $\{\tau x_n(t) \ : n \in \mathbb{N}\}$ is relatively weakly compact. The use of the weak version of the Arzela-Ascoli theorem [30] guarantees that $\tau(B_{r_0})$ is relatively weakly compact. This accomplishes the proof of step 5.

Now, invoking Theorem 3.1 we infer that there is a $x \in E$ with $x = Ux + (Fx)(Kx)$ i.e., x is a solution to (4.1) in E. The proof is complete.

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