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ENTIRE FUNCTIONS SHARING ZERO CM WITH THEIR HIGH ORDER DIFFERENCE OPERATORS

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Abstract. In this paper, we investigate uniqueness of entire functions of order less than 2 sharing the value 0 with their difference operators and obtain a result as follows:
Let f be a transcendental entire function such that $\sigma(f) < 2$ and $\lambda(f) < \sigma(f)$. If f and $\Delta^n f$ share the value 0 CM, then f must be form of $f(z) = Ae^{\alpha z}$, where A and α are two nonzero constants. This result confirms a conjecture posed earlier on the topic.

1. Introduction and Main Results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the following standard notations in value distribution theory (see [4, 8, 9]):

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \cdots .$$

And we denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o\{T(r,f)\}, \text{ as } r \to \infty,$$

possibly outside of a set E with finite linear or logarithmic measure, not necessarily the same at each occurrence. A meromorphic function $a(z)$ is said to be a small function with respect to $f(z)$ if and only if $T(r,a) = S(r,f)$. We use $\lambda(f)$ and $\sigma(f)$ to denote the exponent of convergence of zeros of f and the order of f respectively. We say that

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two meromorphic functions $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicities) if $f(z)-a$ and $g(z)-a$ have the same zeros. If $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities, then we say that they share the value a CM (counting multiplicities). We define the difference operators $\Delta f = f(z+1)-f(z)$ and $\Delta^n f = \Delta^{n-1}(\Delta f)$. Moreover, $\Delta^n f = \sum\limits_{j=0}^{n} C_n^j (-1)^{n-j} f(z+j)$.

In 1996, R. Brück [1] studied the uniqueness theory about some entire functions sharing one value with their derivatives and posed the following interesting and famous conjecture.

Conjecture 1. Let $f(z)$ be non-constant entire function satisfying

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}$$

is neither infinity nor a positive integer. If $f(z)$ and $f'(z)$ share one finite value a CM, then

$$f'(z)-a = c(f(z)-a)$$

holds for some constant $c \neq 0$.

He also proved that the conjecture is true provided $a \neq 0$ and $N(r, \frac{1}{f'}) = S(r,f)$. But the conjecture is still open by now. It is well known that Δf can be considered as the difference counterpart of f'. Recently, the difference analogue of the lemma on the logarithmic derivative and Nevanlinna theory for the difference operator have been founded, which bring about a number of papers focusing on the uniqueness study of meromorphic functions sharing a small function with their difference operators. Furthermore, people obtained lots of results expressly for the meromorphic function whose order is less than 1 because if $\sigma(g) < 1$, then we have $g(z+\eta) = g(z)(1+o(1))$ as $z \to \infty$, possibly outside of a small set (see Lemma 3). For example, the authors in [7] obtained the following result.

Theorem A. [7]. *Let f be a transcendental entire function such that $\sigma(f) < 1$. If f and $\Delta^n f$ share a finite value a CM, then*

$$\Delta^n f - a = c(f-a)$$

holds for some nonzero complex number c.

But we find that such probability $\Delta^n f - a = c(f-a)$ in the conclusion of Theorem A does not exist. That is to say if transcendental entire function f and $\Delta^n f$ share a finite value a CM, then $\sigma(f) \geq 1$. As a matter of fact, the authors in [10] obtained the following results.

Theorem B. [10]. *Let f be a transcendental entire function such that $\sigma(f) < 1$. Then f and $\Delta^n f$ can not share any finite value a CM.*

Theorem C. [10]. *Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 2$, and $\alpha(z) \not\equiv 0$ be an entire function such that $\sigma(\alpha) < \sigma(f)$ and $\lambda(f - \alpha) < \sigma(f)$. If $\Delta^n f - \alpha(z)$ and $f(z) - \alpha(z)$ share the value 0 CM, then $\alpha(z)$ is a polynomial with degree at most $n - 1$ and $f(z)$ must be form of*

$$f(z) = \alpha(z) + H(z)e^{dz},$$

where $H(z)$ is a polynomial such that $cH(z) = -\alpha(z)$ and c, d are nonzero constants such that $e^d = 1$.

At the same time, they conjectured that the condition $\alpha(z) \not\equiv 0$ is not necessary in Theorem C and posed one conjecture as follows.

Conjecture 2. Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 2$ and $\lambda(f) < \sigma(f)$. If $f(z)$ and $\Delta^n f$ share the value 0 CM, then $f(z)$ must be form of

$$f(z) = He^{dz},$$

where H and d are two nonzero constants.

The hypothesis $\sigma(f) < 1$ plays an important role in the proof of Theorem A. In this paper, we continue to consider the case of the meromorphic function whose order is not less than 1. Here we prove conjecture 2 is true and obtain our main theorem as follows.

Theorem 1. *Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 2$ and $\lambda(f) < \sigma(f)$. If $f(z)$ and $\Delta^n f$ share the value 0 CM, then $f(z)$ must be form of*

$$f(z) = Ae^{\alpha z},$$

where A and α are two nonzero constants.

2. Some Lemmas

To prove our results, we need some lemmas as follows.

Lemma 1. (see [3]). *Let $f(z)$ be a transcendental meromorphic function with finite order σ. Then for each $\varepsilon > 0$, we have*

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 2. (see [3]). *Let $f(z)$ be a transcendental meromorphic function with finite order σ and η be a nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r),$$

$$i.e., \ T(r, f(z+\eta)) = T(r, f) + S(r, f).$$

Lemma 3. (see [2]). *Let g be a transcendental function of order less than 1 and h be a positive constant. Then there exists an ε set E such that*

$$\frac{g'(z+\eta)}{g(z+\eta)} \to 0, \ \frac{g(z+\eta)}{g(z)} \to 1, \textit{as } z \to \infty \textit{ in } C \backslash E$$

uniformly in η for $|\eta| \le h$. Further, the set E may be chosen so that for large $|z| \notin E$, the function g has no zeroes or poles in $|\zeta - z| \le h$.

Remark. According to Hayman [5], an ε set is defined to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. Suppose E is an ε set, then the set of $r \ge 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

3. The Proof of Theorem

Proof. On the one hand, from our assumption $\lambda(f) < \sigma(f) < 2$, there exist an entire function $a(z)$ which is from the canonical product of the zeros of $f(z)$ and a nonconstant polynomial $Q(z)$ such that

$$f(z) = a(z)e^{Q(z)},$$

where $\sigma(a) = \lambda(a) = \lambda(f) < \sigma(f)$. From the equation above, we can obtain the following inequality, namely

$$0 < \sigma(e^{Q(z)}) = \deg Q(z) = \sigma(f) < 2,$$

which leads to

$$\deg Q(z) = \sigma(f) = 1$$

and then $\sigma(a) < 1$. Therefore, we can rewrite the equation $f(z) = a(z)e^{Q(z)}$ as the form as follows.

$$f(z) = A(z)e^{\alpha z}, \tag{3.1}$$

where α is a nonzero constant and $A(z)$ is an entire function satisfying $\lambda(A) = \sigma(A) < 1$. In addition, we can assume that $A(z)$ has one zero at least, otherwise it is a non zero constant which implies our conclusion has holden already;

On the other hand, since $\Delta^n f$ and $f(z)$ share the value 0 CM, then there exists an entire function said $P(z)$ such that

$$\Delta^n f(z) = f(z)e^{P(z)}.$$

From the equation above, we see

$$e^{P(z)} = \frac{\Delta^n f}{f} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{f(z+j)}{f(z)}.$$

By applying Lemma 1 to the equation above, we can apparently obtain that

$$m(r, e^{P(z)}) = O(r^{\sigma(f)-1+\varepsilon})$$

holds for any $\varepsilon > 0$. That is to say $P(z)$ is a polynomial with degree

$$\deg P(z) \le \sigma(f) - 1 + \varepsilon = \varepsilon,$$

which means $P(z)$ is a constant because ε can be set small enough. So we can rewrite the equation $\Delta^n f(z) = f(z)e^{P(z)}$ as the following form.

$$\Delta^n f = \eta f(z), \tag{3.2}$$

where η is a nonzero constant. Set $H_0(z) = A(z)$ and

$$H_j(z) = kH_{j-1}(z+1) - H_{j-1}(z)$$

for $j = 1, 2 \ldots, n, \ldots,$ where $k = e^{\alpha} (\neq 0)$. Then from Equation (3.1) and the conformation of H_j, we can see

$$\Delta^j f = H_j(z)e^{\alpha z}, \quad j = 1, 2 \ldots, n \ldots. \tag{3.3}$$

Next we need to show that H_n can be indicated as the following form which plays an important role in our proof. That is

$$H_n(z) = \sum_{j=0}^{n} k^j C_n^j (-1)^{n-j} A(z+j). \tag{3.4}$$

We prove it by mathematical induction. First of all, we suppose that Equation (3.4) has holden for $s = n$, then from the definition of H_j, we see

$$\begin{aligned}
H_{n+1}(z) &= kH_n(z+1) - H_n(z) \\
&= k\sum_{j=0}^{n} k^j C_n^j (-1)^{n-j} A(z+j+1) - \sum_{j=0}^{n} k^j C_n^j (-1)^{n-j} A(z+j) \\
&= \sum_{j=1}^{n+1} k^j C_n^{j-1} (-1)^{n+1-j} A(z+j) - \sum_{j=0}^{n} k^j C_n^j (-1)^{n-j} A(z+j) \\
&= \sum_{j=1}^{n} k^j (C_n^{j-1} + C_n^j)(-1)^{n+1-j} A(z+j) + k^{n+1}A(z+n+1) - A(z)(-1)^n \\
&= \sum_{j=1}^{n} k^j C_{n+1}^j (-1)^{n+1-j} A(z+j) + k^{n+1}A(z+n+1) - A(z)(-1)^n \\
&= \sum_{j=0}^{n+1} k^j C_{n+1}^j (-1)^{n+1-j} A(z+j).
\end{aligned}$$

It means Equation (3.4) can still hold for $s = n+1$. Thus, Equation (3.4) holds for all $s \in N$ by mathematical induction. By Equations (3.1)-(3.3), we see

$$H_n(z) = \eta A(z). \tag{3.5}$$

Combining Equation (3.4) and Equation (3.5), we see

$$\sum_{j=0}^{n} k^j C_n^j (-1)^{n-j} A(z+j) = \eta A(z). \tag{3.6}$$

By applying Lemma 3 to Equation (3.6), we see

$$\eta = \sum_{j=0}^{n} k^j C_n^j (-1)^{n-j} \frac{A(z+j)}{A(z)} \to \sum_{j=0}^{n} k^j C_n^j (-1)^{n-j} \tag{3.7}$$

as $z \to \infty$ in $C \backslash E$, where E is an ε set. Then from Equation (3.7), we can obtain

$$\eta = \sum_{j=0}^{n} k^j C_n^j (-1)^{n-j}. \tag{3.8}$$

By substituting the equation above into Equation (3.6), we can obtain the following equation.

$$\sum_{j=0}^{n} k^j C_n^j (-1)^{n-j} (A(z+j) - A(z)) = 0. \tag{3.9}$$

Set $B(z) = \Delta A(z) = A(z+1) - A(z)$, then from Lemma 2, it is easy for us to see

$$T(r, B) \le 2T(r, A) + S(r, A),$$

which means $\sigma(B) \le \sigma(A) < 1$. From the definition of $B(z)$, we can obtain

$$\begin{aligned}
&A(z+1) - A(z) = B,\\
&A(z+2) - A(z) = \Delta B + 2B,\\
&A(z+3) - A(z) = \Delta^2 B + 3\Delta B + 3B,\\
&\quad\vdots\\
&A(z+j) - A(z) = \Delta^{j-1} B + \cdots + jB,\\
&\quad\vdots
\end{aligned}$$

Here we just need to show that the last term in $A(z+j) - A(z)$ is jB, and we prove it by mathematical induction also. Firstly, suppose

$$A(z+j) - A(z) = \Delta^{j-1} B + \cdots + jB \tag{3.10}$$

has holden for $s = j$, then take difference operator of both sides of Equation (3.10) and we see

$$\begin{aligned}
&\Delta^j B + \cdots + j\Delta B \\
&= \Delta(A(z+j) - A(z)) \\
&= (A(z+j+1) - A(z+1)) - (A(z+j) - A(z)) \\
&= (A(z+j+1) - A(z)) - (A(z+1) - A(z)) - (A(z+j) - A(z)) \\
&= (A(z+j+1) - A(z)) - B - (\Delta^{j-1}B + \cdots + jB).
\end{aligned}$$

Thus

$$A(z+j+1) - A(z) = \Delta^j B + \cdots + (j+1)B$$

holds which means Equation (3.10) still holds for $s = j+1$. Therefore, we can obtain the last term in $A(z+j) - A(z)$ is jB by mathematical induction. By substituting Equation (3.10) into Equation (3.9), we see

$$\sum_{j=1}^{n} k^j C_n^j (-1)^{n-j} (\Delta^{j-1}B + \cdots + jB) = 0. \tag{3.11}$$

From Equation (3.11), we can get

$$\sum_{t=1}^{s} a_t \Delta^t B + \sum_{j=1}^{n} k^j C_n^j (-1)^{n-j} jB = 0, \tag{3.12}$$

where $a_t (t = 1, 2, \ldots, s)$ are some constants. If $B(z) \not\equiv 0$, then from Equation (3.12), we can see

$$\sum_{t=1}^{s} a_t \frac{\Delta^t B}{B} + \sum_{j=1}^{n} k^j C_n^j (-1)^{n-j} j = 0. \tag{3.13}$$

Since $\sigma(B) < 1$, then by applying Lemma 3 to $\frac{\Delta^t B}{B}$ described in Equation (3.13), we can obtain

$$\frac{\Delta^t B}{B} = \sum_{j=0}^{t} C_t^j (-1)^{t-j} \frac{B(z+j)}{B} \to \sum_{j=0}^{t} C_t^j (-1)^{t-j} = (1-1)^t = 0 \tag{3.14}$$

as $z \to \infty$ in $C \backslash E$, where E is an ε set. Thus from Equations (3.13)- (3.14), we see

$$\sum_{j=1}^{n} k^j C_n^j (-1)^{n-j} j = -\sum_{t=1}^{s} a_t \frac{\Delta^t B}{B} \to 0. \tag{3.15}$$

as $z \to \infty$ in $C \backslash E$. Thus, from Equation (3.15), we see

$$\sum_{j=1}^{n} k^j C_n^j (-1)^{n-j} j = 0.$$

That is

$$\sum_{j=1}^{n} k^j n C_{n-1}^{j-1}(-1)^{n-j} = 0.$$

Thus

$$\sum_{j=1}^{n} k^j C_{n-1}^{j-1}(-1)^{n-j} = 0,$$

which implies

$$k\sum_{s=0}^{n-1} k^s C_{n-1}^{s}(-1)^{n-s-1} = (-1)^{n-1}(1-k)^{n-1}k = 0.$$

Therefore, we get $k = 1$. From Equation (3.8), we see

$$\eta = \sum_{j=0}^{n} C_n^j(-1)^{n-j} = 0,$$

which is a contradiction.

Therefore, $B(z) \equiv 0$, and then $A(z+1) = A(z)$. If $A(z)$ is not a constant, then from our assumption that $A(z)$ has a zero at least, we see

$$n(r, \frac{1}{A(z)}) \geq r(1 + o(1)),$$

which implies $\sigma(A) \geq 1$. This is a contradiction. So $A(z)$ is a nonzero constant.

The proof of Theorem 1 is completed. ∎

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References

1. R. Brück, On entire functions which share one value CM with their first derivative, *Results Math.*, **30** (1996), 21-24.
2. W. Bergweiler and J. K. Langley, Zeros of differences of meromorphic functions, *Math. Proc. Cambridge Philos. Soc.*, **142** (2007), 133-147.
3. Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, *Ramanujian. J.*, **16** (2008), 105-129.
4. W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
5. W. K. Hayman, Slowly growing integral and subharmonic functions, *Comment Math. Helv.*, **34** (1960), 75-84.

6. I. Laine and C. C. Yang, Value distribution of difference polynomials, *Pro Japan Acad Ser A.*, **83** (2007), 148-151.

7. K. Liu and L. Z. Yang, Value distribution of the difference operator, *Arch. Math.*, **92** (2009), 270-278.

8. C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Science Press, Beijing, Second Printed in 2006.

9. L. Yang, *Value Distribution Theory*, Springer-Verlag & Science Press, Berlin, 1993.

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