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ON THE EXISTENCE OF POSITIVE DEFINITE SOLUTIONS OF A NONLINEAR MATRIX EQUATION

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Abstract. In this paper the nonlinear matrix equation $X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = Q$ with $p_i > 0$ is investigated. Necessary and sufficient conditions for the existence of Hermitian positive definite solutions are obtained. An effective iterative method to obtain the unique solution is established. A perturbation bound and the backward error of an approximate solution to this solution is evaluated. Moreover, an explicit expression of the condition number for the positive definite solution is given. The theoretical results are illustrated by numerical examples.

1. INTRODUCTION

In this paper we consider the sensitivity analysis of the nonlinear matrix equation

$$X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = Q, \quad p_i > 0, \tag{1.1}$$

where $A_1, A_2, \cdots, A_m$ are $n \times n$ complex matrices, Q is an $n \times n$ Hermitian positive definite matrix, m is a positive integer and the Hermitian positive definite solution X is required. Here, A_i^* denotes the conjugate transpose of the matrix A_i.

This type of nonlinear matrix equations arises in many practical problems, such as ladder networks, dynamic programming, control theory, stochastic filtering, statistics and so forth [1, 2, 3, 4, 5, 6]. When $p_1 = p_2 = \cdots = p_m = 1$, the matrix equation (1.1) reduces to a special case of the nonlinear matrix equation

$$X = Q + A^*(\widehat{X} - C)^{-1} A, \tag{1.2}$$

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where Q is an $n \times n$ positive definite matrix, C is an $mn \times mn$ positive semi-definite matrix, A is an arbitrary $mn \times n$ matrix and $\widehat{X}$ is the $m \times m$ block diagonal matrix with on each diagonal entry the $n \times n$ matrix X. In [7], the matrix equation (1.2) is recognized as playing an important role in modelling certain optimal interpolation problems. Let $C = 0$, $A = (A_1^T, A_2^T, \cdots, A_m^T)^T$, where $A_i,\ i = 1, 2, \cdots, m$, are $n \times n$ matrices. Then $X = Q + A^*(\widehat{X} - C)^{-1}A$ can be written as $X - \sum\limits_{i=1}^{m} A_i^* X^{-1} A_i = Q$. The general nonlinear matrix equation $X - \sum\limits_{i=1}^{m} A_i^* X^{-p_i} A_i = Q\,(p_i > 0)$ comes from solving a system of linear equations in many physical calculations [8]. When solving the nonlinear matrix equation $X - \sum\limits_{i=1}^{m} A_i^* X^{-p_i} A_i = Q$, we often do not know A_i and Q exactly, but have only approximations $\widetilde{A}_i$ and $\widetilde{Q}$ available. Then we can solve the equation $\widetilde{X} - \sum\limits_{i=1}^{m} \widetilde{A}_i^* \widetilde{X}^{-p_i} \widetilde{A}_i = \widetilde{Q}$ exactly which gives a different solution $\widetilde{X}$. We would like to know how the errors of $\widetilde{A}_i$ and $\widetilde{Q}$ influence the error in $\widetilde{X}$. Motivated by this, we consider in this paper the sensitivity analysis of $\widetilde{X} - \sum\limits_{i=1}^{m} \widetilde{A}_i^* \widetilde{X}^{-p_i} \widetilde{A}_i = \widetilde{Q}\,(p_i > 0)$.

For the case $m > 1$, the solvability and numerical solutions to the matrix equation $\widetilde{X} - \sum\limits_{i=1}^{m} \widetilde{A}_i^* \widetilde{X}^{-p_i} \widetilde{A}_i = \widetilde{Q}$ with $0 < p_i \leq 1$ have been studied in [9, 11, 10]. Duan et al. [9] proved that the equation $X - \sum\limits_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q\ (0 < |\delta_i| < 1)$ always has a unique Hermitian positive definite solution. They also proposed an iterative method for obtaining the unique Hermitian positive definite solution. Duan and Liao [10] showed that equation $X - \sum_{i=1}^{m} A_i^* X^r A_i = Q$ with $-1 \leq r < 0$ or $0 < r < 1$ has a unique Hermitian positive definite solution. Lim [11] showed that the equation $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q\,(0 < |\delta_i| < 1)$ has a unique Hermitian positive definite solution. However, these papers have not examined the sensitivity analysis about the equation (1.1) and limited the range of p_i in $(0, 1]$. Yin and Fang [12] obtained an explicit expression of the condition number and also gave two perturbation estimates for the unique Hermitian positive definite solution of $X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q$. Duan et al. [13] gave two perturbation estimates for the Hermitian positive definite solution of the equation $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$ with $0 < |\delta_i| < 1$. Whereas, to our best knowledge, there have been no literatures paying attention to the sensitivity analysis for the equation $X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = Q$ with $p_i > 0$. The reason is that $X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = Q$ does not always have unique Hermitian positive definite solution in the case $p_i > 0$. It is hard to find sufficient conditions for the existence of a unique Hermitian positive definite solution, because the map $L(X) = Q + \sum_{i=1}^{m} A_i^* X^{-p_i} A_i$ with $p_i > 0$ is not monotonic. There are two difficulties for considering the sensitivity analysis for the equation (1.1). One is how to find some reasonable restrictions on the coefficient matrices ensuring this equation has a unique Hermitian positive definite solution. The other one is how to find a reasonable expression of the matrix X^{-p_i} in the case $p_i > 0$ which is easy to handle. By using the integral representation of matrix function, the fixed point theorem and the technique developed in [14], we derive a sufficient condition for the existence of a unique Hermitian positive definite solution to the matrix equation (1.1) and consider the sensitivity analysis of this equation.

The rest of the paper is organized as follows. In Section 2, we give some preliminary knowledge that will be used to develop this work. In Section 3, we derive necessary and sufficient conditions for the existence of Hermitian positive definite solutions to the equation (1.1). In Section 4, we give a perturbation bound for the unique solution to the equation (1.1), which is independent of the exact solution of the equation (1.1). In Section 5, the backward error estimates of an approximate solution for the unique solution to the equation (1.1) are discussed. In Section 6, applying the integral representation of matrix function, we also discuss the explicit expression of the condition number for the Hermitian positive definite solution to the equation (1.1). Finally, several numerical examples are presented in Section 7.

We denote by $\mathcal{C}^{n\times n}$ the set of $n \times n$ complex matrices, by $\mathcal{H}^{n\times n}$ the set of $n \times n$ Hermitian matrices, by I the identity matrix, by **i** the imaginary unit, by $\|\cdot\|$ the spectral norm, by $\|\cdot\|_F$ the Frobenius norm and by $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ the maximal and minimal eigenvalues of M, respectively. For $A = (a_1, \ldots, a_n) = (a_{ij}) \in \mathcal{C}^{n\times n}$ and a matrix B, $A \otimes B = (a_{ij}B)$ is a Kronecker product, and $\mathrm{vec}A$ is a vector defined by $\mathrm{vec}A = (a_1^T, \ldots, a_n^T)^T$. For $X, Y \in \mathcal{H}^{n\times n}$, we write $X \geq Y$(resp. $X > Y)$ if $X - Y$ is Hermitian positive semi-definite (resp. definite).

2. Preliminaries

In this section we quote some preliminary lemmas that we will use later.

Lemma 2.1. [15, Lemma 1]. *If $X, Y \geq \beta I > 0$ and $p > 0$, then $\|X^{-p} - Y^{-p}\| \leq p\, \beta^{-(p+1)}\|X - Y\|$.*

Lemma 2.2. [15, Lemma 2].

(i) *If $X \in \mathcal{H}^{n\times n}$, then $\|e^{-X}\| = e^{-\lambda_{\min}(X)}$.*

(ii) *If $X \in \mathcal{H}^{n\times n}$ and $r > 0$, then $X^{-r} = \frac{1}{\Gamma(r)}\int_0^\infty e^{-sX}s^{r-1}ds$.*

(iii) *If $A, B \in \mathcal{C}^{n\times n}$, then $e^{A+B} - e^A = \int_0^1 e^{(1-t)A}Be^{t(A+B)}dt$.*

3. Necessary and Sufficient Conditions

In this section, we derive the properties of the Hermitian positive definite solutions of (1.1), including uniqueness and estimates of the solutions.

Theorem 3.1. *The nonlinear matrix equation (1.1) always has Hermitian positive definite solutions.*

Proof. Let $\Omega = \{X : Q \le X \le Q + \sum_{i=1}^{m} \lambda_{\min}^{-p_i}(Q) A_i^* A_i\}$. Obviously, Ω is a nonempty bounded convex closed set. Let $F(X) = Q + \sum\limits_{i=1}^{m} A_i^* X^{-p_i} A_i$. Evidently, F is continuous. For $X \in \Omega$, we have $X \ge Q > 0$, which implies that $F(X) \ge Q$. It follows from $X \ge Q \ge \lambda_{\min}(Q) I$ that $X^{-p_i} \le \lambda_{\min}^{-p_i}(Q) I$, which implies that $F(X) \le Q + \sum\limits_{i=1}^{m} \lambda_{\min}^{-p_i}(Q) A_i^* A_i$. Therefore $F(\Omega) \subseteq \Omega$. By Brouwer fixed point theorem, there exists $X \in \Omega$ satisfies $F(X) = X$, which means X is a solution of Eq.(1.1). ■

Theorem 3.2. *If X is an Hermitian positive definite solution of (1.1), then $Q \le X \le Q + \sum\limits_{i=1}^{m} \dfrac{A_i^* A_i}{\lambda_{\min}^{p_i}(Q)}$.*

Proof. That X is an Hermitian positive definite solution of (1.1) implies $X > 0$. Then $X^{-p_i} > 0$ and $A_i^* X^{-p_i} A_i \ge 0$, $i = 1, 2, \cdots, m$. Hence $X = Q + \sum\limits_{i=1}^{m} A_i^* X^{-p_i} A_i \ge Q \ge \lambda_{\min}(Q) I$. Consequently, $X^{-p_i} \le \lambda_{\min}^{-p_i}(Q) I$ and $X \le Q + \sum\limits_{i=1}^{m} \lambda_{\min}^{-p_i}(Q) A_i^* A_i$. ■

Theorem 3.3. *If*

$$q = \sum_{i=1}^{m} p_i \|A_i\|^2 \lambda_{\min}^{-p_i-1}(Q) < 1,$$

then

(i) *The matrix equation (1.1) has a unique Hermitian positive definite solution X.*

(ii) *The iteration*

$$X_0 \in \left[Q,\ Q + \sum_{i=1}^{m} \lambda_{\min}^{-p_i}(Q) A_i^* A_i\right],\ X_n = Q + \sum_{i=1}^{m} A_i^* X_{n-1}^{-p_i} A_i, \quad n = 1, 2, \cdots \tag{3.1}$$

converges to X. Moreover,

$$\|X_n - X\| \le \frac{q^n}{1-q} \|X_1 - X_0\|.$$

Proof. Let $\Omega = \{X : Q \le X \le Q + \sum_{i=1}^{m} \lambda_{\min}^{-p_i}(Q) A_i^* A_i\}$. Obviously, Ω is a nonempty bounded convex closed set. Let $F(X) = Q + \sum\limits_{i=1}^{m} A_i^* X^{-p_i} A_i$. Evidently, F

is continuous. The proof of Theorem 3.1 implies that $F(\Omega) \subseteq \Omega$. According to Lemma 2.1, we obtain $\forall X_1, X_2 \in \Omega$,

$$\begin{aligned}\|F(X_1) - F(X_2)\| &= \left\|\sum_{i=1}^{m} A_i^*(X_1^{-p_i} - X_2^{-p_i})A_i\right\| \leq \sum_{i=1}^{m} \left\|A_i^*(X_1^{-p_i} - X_2^{-p_i})A_i\right\| \\ &\leq \sum_{i=1}^{m} p_i \lambda_{\min}^{-(p_i+1)}(Q)\|A_i^* A_i\|\|X_1 - X_2\| \\ &= q\|X_1 - X_2\|.\end{aligned}$$

The last equality is due to the fact that $||A_i^* A_i|| = ||A_i||^2$ (refer to [19, Problem 11. Page 312]). From $q < 1$, it follows that F is a contractive mapping. By Banach contraction mapping principle, the theorem is proved. ■

4. Perturbation Bounds

In this section we develop a relative perturbation bound for the unique solution of (1.1), which does not need any knowledge of the exact solution X of (1.1) and is easy to calculate.

Li and Zhang [16] proved the existence of a unique positive definite solution to the equation $X - A^* X^{-p} A = Q$ $(0 < p < 1)$ and also obtained a perturbation bound for the unique solution. However, their approach becomes invalid for $X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = Q$ $(p_i > 0)$. Since the latter equation does not always have a unique positive definite solution, there are two difficulties for a perturbation analysis of the equation $X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = Q$ $(p_i > 0)$. One is how to find some reasonable restrictions on the coefficient matrices of perturbed equation ensuring this perturbed equation has a unique positive definite solution. The other one is how to find an expression of $X^{-p_i} (p_i > 0)$ which is easy to handle.

Consider the perturbed matrix equation

$$\widetilde{X} - \sum_{i=1}^{m} \widetilde{A_i}^* \widetilde{X}^{-p_i} \widetilde{A_i} = \widetilde{Q}, \quad p_i > 0, \tag{4.1}$$

where $\widetilde{Q}$ is an Hermitian positive definite matrix.

If $\sum_{i=1}^{m} p_i \|A_i\|^2 \lambda_{\min}^{-p_i-1}(Q) < 1$ and $\sum_{i=1}^{m} p_i \|\widetilde{A}_i\|^2 \lambda_{\min}^{-p_i-1}(\widetilde{Q}) < 1$. According to Theorem 3.3, equations (1.1) and (4.1) have unique Hermitian positive definite solutions X and $\widetilde{X}$, respectively. Let $\Delta A_i = \widetilde{A_i} - A_i$, $i = 1, 2, \cdots, m$, $\Delta Q = \widetilde{Q} - Q$ and $\Delta X = \widetilde{X} - X$, then we have the following theorem.

Theorem 4.1. *If*

$$\sum_{i=1}^{m} p_i \|A_i\|^2 \Lambda^{-p_i-1} < 1 \; and \; \sum_{i=1}^{m} p_i \|\widetilde{A}_i\|^2 \Lambda^{-p_i-1} < 1 \tag{4.2}$$
$$with \;\; \Lambda = \min\{\lambda_{\min}(Q), \lambda_{\min}(\widetilde{Q})\},$$

then

$$X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = Q \ \ and \ \ \widetilde{X} - \sum_{i=1}^{m} \widetilde{A_i}^* \widetilde{X}^{-p_i} \widetilde{A_i} = \widetilde{Q}$$

have unique Hermitian positive definite solutions X and $\widetilde{X}$, respectively. Furthermore,

$$\frac{\|\widetilde{X} - X\|}{\|X\|} \leq \frac{\sum_{i=1}^{m}(2\|A_i\| + \|\Delta A_i\|)\|\Delta A_i\|\Lambda^{-p_i} + ||\Delta Q||}{\Lambda - \sum_{i=1}^{m} p_i\|A_i\|^2\Lambda^{-p_i}} \triangleq \xi.$$

Proof. According to Theorem 3.3, the condition (4.2) ensures that (1.1) and (4.1) have unique positive definite solution X and $\widetilde{X}$, respectively. Furthermore, we obtain that

$$X \geq \lambda_{\min}(Q)I \geq \Lambda I, \quad \widetilde{X} \geq \lambda_{\min}(\widetilde{Q})I \geq \Lambda I. \tag{4.3}$$

Subtracting (4.1) from (1.1) gives

$$\begin{aligned} \Delta X &= \sum_{i=1}^{m} \left(\widetilde{A_i}^* \widetilde{X}^{-p_i} \widetilde{A_i} - A_i^* X^{-p_i} A_i\right) + \Delta Q \\ &= \sum_{i=1}^{m} \left[A_i^*(\widetilde{X}^{-p_i} - X^{-p_i})A_i + \Delta A_i^* \widetilde{X}^{-p_i} A_i + \widetilde{A_i}^* \widetilde{X}^{-p_i} \Delta A_i\right] + \Delta Q. \end{aligned} \tag{4.4}$$

By Lemma 2.2 and the inequalities in (4.3), we have

$$\begin{aligned} &\left\|\Delta X + \sum_{i=1}^{m} \left(A_i^* X^{-p_i} A_i - A_i{}^* \widetilde{X}^{-p_i} A_i\right)\right\| \\ = &\left\|\Delta X + \sum_{i=1}^{m} A_i^* \frac{1}{\Gamma(p_i)} \int_0^\infty (e^{-sX} - e^{-s\widetilde{X}}) s^{p_i - 1} ds A_i\right\| \\ = &\left\|\Delta X + \sum_{i=1}^{m} A_i^* \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-(1-t)s\widetilde{X}} \Delta X e^{-tsX} dt s^{p_i} ds A_i\right\| \\ \geq &\ \|\Delta X\| - \sum_{i=1}^{m} \frac{\|A_i\|^2 \|\Delta X\|}{\Gamma(p_i)} \int_0^\infty \int_0^1 \left\|e^{-(1-t)s\widetilde{X}}\right\| \left\|e^{-tsX}\right\| dt s^{p_i} ds \\ = &\ \|\Delta X\| - \sum_{i=1}^{m} \frac{\|A_i\|^2 \|\Delta X\|}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-(1-t)s\lambda_{\min}(\widetilde{X})} e^{-ts\lambda_{\min}(X)} dt s^{p_i} ds \\ \geq &\ \|\Delta X\| - \sum_{i=1}^{m} \frac{\|A_i\|^2 \|\Delta X\|}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-(1-t)s\Lambda} e^{-ts\Lambda} dt s^{p_i} ds \end{aligned} \tag{4.5}$$

$$
\begin{aligned}
&= \|\Delta X\| - \sum_{i=1}^{m} \frac{\|A_i\|^2\|\Delta X\|}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-s\Lambda} dt s^{p_i} ds \\
&= \|\Delta X\| - \sum_{i=1}^{m} \frac{\Gamma(p_i+1)}{\Gamma(p_i)} \cdot \frac{\|A_i\|^2\|\Delta X\|}{\Lambda^{p_i+1}} = \left(1 - \sum_{i=1}^{m} \frac{p_i\|A_i\|^2}{\Lambda^{p_i+1}}\right) \|\Delta X\|.
\end{aligned}
$$

Noticing the conditions in (4.2), we have

$$
1 - \sum_{i=1}^{m} \frac{p_i\|A_i\|^2}{\Lambda^{p_i+1}} > 0.
$$

Combining (4.4) with (4.5), one sees that

$$
\begin{aligned}
&\left(1 - \sum_{i=1}^{m} \frac{p_i\|A_i\|^2}{\Lambda^{p_i+1}}\right) \|\Delta X\| \le \left\| \sum_{i=1}^{m} (\Delta A_i^* \widetilde{X}^{-p_i} A_i + \widetilde{A_i}^* \widetilde{X}^{-p_i} \Delta A_i) + \Delta Q \right\| \\
&\le \sum_{i=1}^{m} (\|\Delta A_i\| + 2\|A_i\|) \|\Delta A_i\| \|\widetilde{X}^{-p_i}\| + ||\Delta Q|| \\
&\le \sum_{i=1}^{m} (\|\Delta A_i\| + 2\|A_i\|) \|\Delta A_i\| \Lambda^{-p_i} + ||\Delta Q||,
\end{aligned}
$$

which implies that

$$
\frac{\|\Delta X\|}{\|X\|} \le \frac{\displaystyle\sum_{i=1}^{m} (2\|A_i\| + \|\Delta A_i\|) \|\Delta A_i\| \Lambda^{-p_i} + ||\Delta Q||}{\displaystyle\Lambda - \sum_{i=1}^{m} p_i \|A_i\|^2 \Lambda^{-p_i}}. \qquad \blacksquare
$$

5. Backward Error

In this section, we obtain some estimates for the backward error of the approximate solution of (1.1).

Let $\widetilde{X} > 0$ be an approximation to the unique solution X to the equation (1.1), and let $\Delta A_i \in \mathcal{C}^{n\times n}\,(i = 1, 2, \cdots, m)$ and $\Delta Q \in \mathcal{H}^{n\times n}$ be the corresponding perturbations of the coefficient matrices $A_i\,(i = 1, 2, \cdots, m)$ and Q in the equation (1.1). A backward error of the approximate solution $\widetilde{X}$ can be defined by

$$
\begin{aligned}
\eta(\widetilde{X}) = \min \Bigg\{ &\left\| \left(\frac{\Delta A_1}{\alpha_1}, \frac{\Delta A_2}{\alpha_2}, \cdots, \frac{\Delta A_m}{\alpha_m}, \frac{\Delta Q}{\rho} \right) \right\|_F : \\
&\widetilde{X} - \sum_{i=1}^{m} (A_i + \Delta A_i)^* \widetilde{X}^{-p_i} (A_i + \Delta A_i) = Q + \Delta Q \Bigg\},
\end{aligned} \tag{5.1}
$$

where $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_m$ and ρ are positive parameters. Taking $\alpha_i = ||A_i||_F,\ i = 1, 2, \cdots, m$ and $\rho = ||Q||_F$ in (5.1) gives the relative backward error $\eta_{rel}(\widetilde{X})$, and taking $\alpha_i = 1,\ i = 1, 2, \cdots, m$ and $\rho = 1$ in (5.1) gives the absolute backward error $\eta_{abs}(\widetilde{X})$.

Let

$$R = Q - \widetilde{X} + \sum_{i=1}^{m} A_i^* \widetilde{X}^{-p_i} A_i. \tag{5.2}$$

Note that

$$Q = \widetilde{X} - \sum_{i=1}^{m} (A_i + \Delta A_i)^* \widetilde{X}^{-p_i} (A_i + \Delta A_i) - \Delta Q.$$

It follows from (5.2) that

$$-\sum_{i=1}^{m} (\Delta A_i^* \widetilde{X}^{-p_i} A_i + A_i^* \widetilde{X}^{-p_i} \Delta A_i) - \Delta Q = R + \sum_{i=1}^{m} \Delta A_i^* \widetilde{X}^{-p_i} \Delta A_i. \tag{5.3}$$

Let

$$\begin{aligned}
&\left(I \otimes (\widetilde{X}^{-p_i} A_i)^*\right) = U_{i1} + \mathbf{i}\Omega_{i1}, \quad \left((\widetilde{X}^{-p_i} A_i)^T \otimes I\right)\Pi = U_{i2} + \mathbf{i}\Omega_{i2},\\
&\mathrm{vec}\Delta A_i = x_i + \mathbf{i} y_i, \quad \mathrm{vec}\Delta Q = q_1 + \mathbf{i} q_2, \quad \mathrm{vec} R = r_1 + \mathbf{i} r_2,\\
&\mathrm{vec}(\Delta A_i^* \widetilde{X}^{-p_i} \Delta A_i) = a_i + \mathbf{i} b_i, \quad i = 1, 2, \cdots, m,\\
&g = \left(\frac{x_1^T}{\alpha_1}, \frac{y_1^T}{\alpha_1}, \cdots, \frac{x_m^T}{\alpha_m}, \frac{y_m^T}{\alpha_m}, \frac{q_1^T}{\rho}, \frac{q_2^T}{\rho}\right)^T,\\
&U_i = \begin{pmatrix} U_{i1} + U_{i2} & \Omega_{i2} - \Omega_{i1} \\ \Omega_{i1} + \Omega_{i2} & U_{i1} - U_{i2} \end{pmatrix},\\
&T = [-\alpha_1 U_1, -\alpha_2 U_2, \cdots, -\alpha_m U_m, -\rho I_{2n^2}],
\end{aligned}$$

where Π is the vec-permutation. Then (5.3) can be rewritten as

$$Tg = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \sum_{i=1}^{m} \begin{pmatrix} a_i \\ b_i \end{pmatrix}. \tag{5.4}$$

It follows from $\rho > 0$ that $2n^2 \times 2(m+1)n^2$ matrix T is full row rank. Hence, $TT^\dagger = I_{2n^2}$, which implies that every solution to the equation

$$g = T^\dagger \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + T^\dagger \left(\sum_{i=1}^{m} \begin{pmatrix} a_i \\ b_i \end{pmatrix}\right) \tag{5.5}$$

must be a solution to the equation (5.4). Consequently, for any solution g to (5.5) we have

$$\eta(\widetilde{X}) \leq ||g||. \tag{5.6}$$

Then we can state the estimates of the backward error as follows.

Theorem 5.1. *Let $A_1, A_2, \cdots, A_m$, Q and $\widetilde{X}$ be given matrices. Where $A_i \in \mathcal{C}^{n\times n}$, $\widetilde{X}$ and Q are Hermitian positive definite matrices. Let $\eta(\widetilde{X})$ be the backward error defined by (5.1). If*

$$r < \frac{s}{4\Big(\sum\limits_{i=1}^{m} t_i\alpha_i^2\Big)}, \tag{5.7}$$

then we have that

$$U(r) \leq \eta(\widetilde{X}) \leq B(r), \tag{5.8}$$

where

$$r = \left\| T^{\dagger}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \right\|, \quad s = \|T^{\dagger}\|^{-1}, \quad t_i = \left\| \widetilde{X}^{-p_i} \right\|, \tag{5.9}$$

$$B(r) = \frac{2rs}{s + \sqrt{s^2 - 4rs(\sum_{i=1}^{m} t_i\alpha_i^2)}}, \quad U(r) = \frac{2r\sqrt{s^2 - 4rs(\sum_{i=1}^{m} t_i\alpha_i^2)}}{s + \sqrt{s^2 - 4rs(\sum_{i=1}^{m} t_i\alpha_i^2)}}. \tag{5.10}$$

Proof. Let

$$L(g) = T^{\dagger}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + T^{\dagger}\left(\sum_{i=1}^{m}\begin{pmatrix} a_i \\ b_i \end{pmatrix}\right).$$

Obviously, $L : \mathcal{C}^{2(m+1)n^2\times 1} \to \mathcal{C}^{2(m+1)n^2\times 1}$ is continuous. The condition (5.7) ensures that the quadratic equation

$$x = r + \frac{1}{s}(\sum_{i=1}^{m} t_i\alpha_i^2)x^2 \tag{5.11}$$

in x has two positive real roots. The smaller one is

$$B(r) = \frac{2rs}{s + \sqrt{s^2 - 4rs(\sum_{i=1}^{m} t_i\alpha_i^2)}}.$$

Define $\Omega = \left\{ g \in \mathcal{C}^{2(m+1)n^2 \times 1} : ||g|| \leq B(r) \right\}$. Then for any $g \in \Omega$, we have

$$\begin{aligned}
&||L(g)|| \\
&\leq r + \frac{1}{s} \sum_{i=1}^{m} \left\| \begin{pmatrix} a_i \\ b_i \end{pmatrix} \right\| = r + \frac{1}{s} \sum_{i=1}^{m} \left\| \Delta A_i^* \widetilde{X}^{-p_i} \Delta A_i \right\|_F \leq r + \frac{1}{s} \sum_{i=1}^{m} t_i \|\Delta A_i\|_F^2 \\
&\leq r + \frac{1}{s} \left(\sum_{i=1}^{m} t_i \alpha_i^2 \right) \left\| \left(\frac{\Delta A_1}{\alpha_1}, \frac{\Delta A_2}{\alpha_2}, \cdots, \frac{\Delta A_m}{\alpha_m} \right) \right\|_F^2 \leq r + \frac{1}{s} \left(\sum_{i=1}^{m} t_i \alpha_i^2 \right) \|g\|^2 \\
&\leq r + \frac{1}{s} \left(\sum_{i=1}^{m} t_i \alpha_i^2 \right) B^2(r) = B(r).
\end{aligned}$$

The last equality is due to the fact that $B(r)$ is a solution to the quadratic equation (5.11). Thus we have proved that $L(\Omega) \subset \Omega$. By the Schauder fixed-point theorem, there exists a $g_* \in \Omega$ such that $L(g_*) = g_*$, which means that g_* is a solution to (5.5), and hence it follows from (5.6) that

$$\eta(\widetilde{X}) \leq ||g_*|| \leq B(r).$$

Next we derive a lower bound for $\eta(\widetilde{X})$. Suppose that $(\frac{\Delta A_{1\,\min}}{\alpha_1}, \cdots, \frac{\Delta A_{m\,\min}}{\alpha_m}, \frac{\Delta Q_{\min}}{\rho})$ satisfies

$$\eta(\widetilde{X}) = \left\| \left(\frac{\Delta A_{1\,\min}}{\alpha_1}, \cdots, \frac{\Delta A_{m\,\min}}{\alpha_m}, \frac{\Delta Q_{\min}}{\rho} \right) \right\|_F. \tag{5.12}$$

Then we have

$$T g_{\min} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \sum_{i=1}^{m} \begin{pmatrix} a_{i*} \\ b_{i*} \end{pmatrix}, \tag{5.13}$$

where

$$\begin{aligned}
&\mathrm{vec}(\Delta A_{i\,\min}^T \widetilde{X}^{-p_i} \Delta A_{i\,\min}) = a_{i*} + \mathbf{i} b_{i*}, \\
&\mathrm{vec}(\Delta A_{i\,\min}) = x_{i*} + \mathbf{i} y_{i*}, \\
&\mathrm{vec}(\Delta Q_{\min}) = q_{1*} + \mathbf{i} q_{2*}, \\
&g_{\min} = \left(\frac{x_{1*}^T}{\alpha_1}, \frac{y_{1*}^T}{\alpha_1}, \cdots, \frac{x_{m*}^T}{\alpha_m}, \frac{y_{m*}^T}{\alpha_m}, \frac{q_{1*}^T}{\rho}, \frac{q_{2*}^T}{\rho} \right)^T.
\end{aligned}$$

Let a singular value decomposition of T be $T = W(E, 0)Z^T$, where W and Z are orthogonal matrices, $E = diag(e_1, e_2, \cdots, e_{2n^2})$ with $e_1 \geq \cdots \geq e_{2n^2} > 0$. Substituting this decomposition into (5.13) and letting

$$Z^T g_{\min} = \begin{pmatrix} v \\ * \end{pmatrix}, \; v \in \mathbb{R}^{2n^2},$$

we get

$$v = E^{-1}W^T \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + E^{-1}W^T \sum_{i=1}^{m} \begin{pmatrix} a_{i*} \\ b_{i*} \end{pmatrix}.$$

It follows from (5.12) that

$$\begin{aligned}
&\eta(\widetilde{X}) = \|g_{\min}\| = \left\| \begin{pmatrix} v \\ * \end{pmatrix} \right\| \geq \|v\| \\
&\geq \left\| E^{-1}W^T \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \right\| - \left\| E^{-1}W^T \sum_{i=1}^{m} \begin{pmatrix} a_{i*} \\ b_{i*} \end{pmatrix} \right\| \\
&\geq \left\| T^{\dagger} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \right\| - \left\| T^{\dagger} \right\| \cdot \sum_{i=1}^{m} \left\| \begin{pmatrix} a_{i*} \\ b_{i*} \end{pmatrix} \right\| \\
&\geq r - \frac{1}{s} \sum_{i=1}^{m} \left\| \Delta A_{i\min}^{*} \widetilde{X}^{-p_i} \Delta A_{i\min} \right\|_F \geq r - \frac{1}{s} \sum_{i=1}^{m} t_i \|\Delta A_{i\min}\|_F^2 \\
&\geq r - \frac{1}{s} \left(\sum_{i=1}^{m} t_i \alpha_i^2 \right) \left\| \left(\frac{\Delta A_{1\min}}{\alpha_1}, \cdots, \frac{\Delta A_{m\min}}{\alpha_m} \right) \right\|_F^2 \\
&\geq r - \frac{1}{s} \left(\sum_{i=1}^{m} t_i \alpha_i^2 \right) B^2(r).
\end{aligned}$$

Here we have used the fact that

$$\begin{aligned}
&\left\| \left(\frac{\Delta A_{1\min}}{\alpha_1}, \cdots, \frac{\Delta A_{m\min}}{\alpha_m} \right) \right\|_F \\
&\leq \left\| \left(\frac{\Delta A_{1\min}}{\alpha_1}, \cdots, \frac{\Delta A_{m\min}}{\alpha_m}, \frac{\Delta Q_{\min}}{\rho} \right) \right\|_F = \eta(\widetilde{X}) \leq B(r).
\end{aligned}$$

Let

$$U(r) = r - \frac{1}{s} \left(\sum_{i=1}^{m} t_i \alpha_i^2 \right) B^2(r).$$

Since $B(r)$ is a solution to the equation (5.11), we have

$$B(r) = r + \frac{1}{s} (\sum_{i=1}^{m} t_i \alpha_i^2) B^2(r),$$

which implies that

$$U(r) = r - \frac{1}{s} \left(\sum_{i=1}^{m} t_i \alpha_i^2 \right) B^2(r) = 2r - B(r) = \frac{2r\sqrt{s^2 - 4rs(\sum_{i=1}^{m} t_i \alpha_i^2)}}{s + \sqrt{s^2 - 4rs(\sum_{i=1}^{m} t_i \alpha_i^2)}} > 0.$$

Then $\eta(\widetilde{X}) \geq U(r)$. ■

6. Condition Number

A condition number is a measurement of the sensitivity of the Hermitian positive definite stabilizing solutions to small changes in the coefficient matrices. In this section, we apply the theory of condition number developed by Rice [17] to study condition numbers of the unique solution to (1.1). The difficulty for obtaining explicit expressions of the condition number for (1.1) is how to find expressions of ΔX and $X^{-p_i}(p_i > 0)$ which are easy to handle. Here we consider the perturbed equation

$$\widetilde{X} - \sum_{i=1}^{m} \widetilde{A}_i^* \widetilde{X}^{-p_i} \widetilde{A}_i = \widetilde{Q}. \tag{6.1}$$

Suppose that $\sum_{i=1}^{m} p_i \|A_i\|^2 \lambda_{\min}^{-p_i-1}(Q) < 1$ and $\sum_{i=1}^{m} p_i \|\widetilde{A}_i\|^2 \lambda_{\min}^{-p_i-1}(\widetilde{Q}) < 1$. According to Theorem 3.3, equations (1.1) and (6.1) have unique Hermitian positive definite solutions X and $\widetilde{X}$, respectively. Let $\Delta A_i = \widetilde{A_i} - A_i$, $\Delta Q = \widetilde{Q} - Q$ and $\Delta X = \widetilde{X} - X$. Subtracting (6.1) from (1.1) gives

$$\begin{aligned}
\Delta X &= \sum_{i=1}^{m} \left(\widetilde{A}_i^* \widetilde{X}^{-p_i} \widetilde{A}_i - A_i^* X^{-p_i} A_i \right) + \Delta Q \\
&= \sum_{i=1}^{m} \left(A_i^* (\widetilde{X}^{-p_i} - X^{-p_i}) A_i + \Delta A_i^* \widetilde{X}^{-p_i} A_i + \widetilde{A_i}^* \widetilde{X}^{-p_i} \Delta A_i \right) + \Delta Q \\
&= -\sum_{i=1}^{m} \frac{A_i^*}{\Gamma(p_i)} \int_0^{\infty} \left(e^{-sX} - e^{-s\widetilde{X}} \right) s^{p_i-1} ds A_i + \sum_{i=1}^{m} \left(\Delta A_i^* \widetilde{X}^{-p_i} A_i + \widetilde{A}_i^* \widetilde{X}^{-p_i} \Delta A_i \right) + \Delta Q \\
&= -\sum_{i=1}^{m} \frac{A_i^*}{\Gamma(p_i)} \int_0^{\infty} \int_0^{1} e^{-(1-t)s\widetilde{X}} \left(\widetilde{X} - X \right) e^{-tsX} dt s^{p_i} ds A_i \\
&\quad + \sum_{i=1}^{m} \left(\Delta A_i^* \widetilde{X}^{-p_i} A_i + \widetilde{A}_i^* \widetilde{X}^{-p_i} \Delta A_i \right) + \Delta Q \\
&= -\sum_{i=1}^{m} \frac{A_i^*}{\Gamma(p_i)} \int_0^{\infty} \int_0^{1} (e^{-(1-t)s\widetilde{X}} - e^{-(1-t)sX}) \Delta X e^{-tsX} dt s^{p_i} ds A_i \\
&\quad + \sum_{i=1}^{m} \widetilde{A}_i^* (X + \Delta X)^{-p_i} \Delta A_i \\
&\quad - \sum_{i=1}^{m} \frac{A_i^*}{\Gamma(p_i)} \int_0^{\infty} \int_0^{1} e^{-(1-t)sX} \Delta X e^{-tsX} dt s^{p_i} ds A_i - \sum_{i=1}^{m} \widetilde{A}_i^* X^{-p_i} \Delta A_i
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\widetilde{A_i}^{*}X^{-p_i}\Delta A_i-\sum_{i=1}^{m}\left(\Delta A_i^{*}X^{-p_i}A_i-\Delta A_i^{*}(X+\Delta X)^{-p_i}A_i\right)\\
&+\sum_{i=1}^{m}\Delta A_i^{*}X^{-p_i}A_i+\Delta Q\\
=&\sum_{i=1}^{m}\frac{A_i^{*}}{\Gamma(p_i)}\int_0^{\infty}\int_0^{1}\int_0^{1}e^{-(1-m)(1-t)sX}\\
&\qquad\Delta X e^{-m(1-t)s\widetilde{X}}\Delta X e^{-tsX}dm(1-t)dts^{p_i+1}dsA_i+\Delta Q\\
&-\sum_{i=1}^{m}\frac{A_i^{*}}{\Gamma(p_i)}\int_0^{\infty}\int_0^{1}e^{-(1-t)sX}\Delta X e^{-tsX}dts^{p_i}dsA_i+\sum_{i=1}^{m}\Delta A_i^{*}X^{-p_i}\Delta A_i\\
&-\sum_{i=1}^{m}\frac{\widetilde{A_i}^{*}}{\Gamma(p_i)}\int_0^{\infty}\int_0^{1}e^{-(1-t)s(X+\Delta X)}\Delta X e^{-tsX}dts^{p_i}ds\Delta A_i\\
&+\sum_{i=1}^{m}(A_i^{*}X^{-p_i}\Delta A_i+\Delta A_i^{*}X^{-p_i}A_i)\\
&-\sum_{i=1}^{m}\frac{\Delta A_i^{*}}{\Gamma(p_i)}\int_0^{\infty}\int_0^{1}e^{-(1-t)s(X+\Delta X)}\Delta X e^{-tsX}dts^{p_i}dsA_i.
\end{aligned}
$$

Therefore

$$
\Delta X+\sum_{i=1}^{m}\frac{A_i^{*}}{\Gamma(p_i)}\int_0^{\infty}\int_0^{1}e^{-(1-t)sX}\Delta X e^{-tsX}dts^{p_i}dsA_i=E+h(\Delta X), \tag{6.2}
$$

where

$$
\begin{aligned}
B_i\;&=\;X^{-p_i}A_i,\\
E\;&=\;\Delta Q+\sum_{i=1}^{m}(B_i^{*}\Delta A_i+\Delta A_i^{*}B_i)+\sum_{i=1}^{m}\Delta A_i^{*}X^{-p_i}\Delta A_i,\\
h(\Delta X)\;&=\;\sum_{i=1}^{m}\frac{A_i^{*}}{\Gamma(p_i)}\int_0^{\infty}\int_0^{1}\int_0^{1}e^{-(1-m)(1-t)sX}\\
&\qquad\Delta X e^{-m(1-t)s\widetilde{X}}\Delta X e^{-tsX}dm(1-t)dts^{p_i+1}dsA_i\\
&\quad-\sum_{i=1}^{m}\frac{\widetilde{A_i}^{*}}{\Gamma(p_i)}\int_0^{\infty}\int_0^{1}e^{-(1-t)s(X+\Delta X)}\Delta X e^{-tsX}dts^{p_i}ds\Delta A_i\\
&\quad-\sum_{i=1}^{m}\frac{\Delta A_i^{*}}{\Gamma(p_i)}\int_0^{\infty}\int_0^{1}e^{-(1-t)s(X+\Delta X)}\Delta X e^{-tsX}dts^{p_i}dsA_i.
\end{aligned}
$$

Lemma 6.3. *If*

$$\sum_{i=1}^{m} p_i\|A_i\|^2\lambda_{\min}^{-p_i-1}(Q) < 1, \tag{6.3}$$

then the linear operator $\mathbf{V} : \mathcal{H}^{n\times n} \to \mathcal{H}^{n\times n}$ *defined by*

$$\mathbf{V}W = W + \sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 A_i^* e^{-(1-t)sX} W e^{-tsX} A_i dts^{p_i} ds, \quad W \in \mathcal{H}^{n\times n} \tag{6.4}$$

is invertible.

Proof. Defining the operator $\mathbf{R} : \mathcal{H}^{n\times n} \to \mathcal{H}^{n\times n}$ by

$$\mathbf{R}Z = \sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 A_i^* e^{-(1-t)sX} Z e^{-tsX} A_i dts^{p_i} ds,\ Z \in \mathcal{H}^{n\times n},$$

it follows that

$$\mathbf{V}W = W + \mathbf{R}W.$$

Then $\mathbf{V}$ is invertible if and only if $I + \mathbf{R}$ is invertible.

According to Lemma 2.2 and the condition (6.3), we have

$$\begin{aligned}
||\mathbf{R}W|| &\le \sum_{i=1}^{m} ||A_i||^2||W||\frac{1}{\Gamma(p_i)}\int_0^\infty \int_0^1 \left\|e^{-(1-t)sX}\right\| \left\|e^{-tsX}\right\| dts^{p_i} ds\\
&= \sum_{i=1}^{m} ||A_i||^2||W||\frac{1}{\Gamma(p_i)}\int_0^\infty \int_0^1 e^{-(1-t)s\lambda_{min}(X)} e^{-ts\lambda_{min}(X)} dts^{p_i} ds\\
&\le \sum_{i=1}^{m} ||A_i||^2||W||\frac{1}{\Gamma(p_i)}\int_0^\infty \int_0^1 e^{-(1-t)s\lambda_{min}(Q)} e^{-ts\lambda_{min}(Q)} dts^{p_i} ds\\
&= \sum_{i=1}^{m} ||A_i||^2||W||\frac{1}{\Gamma(p_i)}\int_0^\infty e^{-s\lambda_{min}(Q)} s^{p_i} ds\\
&= \sum_{i=1}^{m} \frac{p_i||A_i||^2}{\lambda_{min}^{p_i+1}(Q)}||W|| < ||W||,
\end{aligned}$$

which implies that $||\mathbf{R}|| < 1$ and $I + \mathbf{R}$ is invertible. Therefore, the operator $\mathbf{V}$ is invertible. ∎

Based on the arguments above, we can rewrite (6.2) as

$$\Delta X = \mathbf{V}^{-1}\Delta Q + \mathbf{V}^{-1}\sum_{i=1}^{m}(B_i^*\Delta A_i + \Delta A_i^* B_i) + \mathbf{V}^{-1}\sum_{i=1}^{m}(\Delta A_i^* X^{-p_i}\Delta A_i) + \mathbf{V}^{-1}(h(\Delta X)).$$

Obviously,

$$\begin{aligned}\Delta X = & \mathbf{V}^{-1}\Delta Q + \mathbf{V}^{-1}\sum_{i=1}^{m}(B_i^*\Delta A_i + \Delta A_i^* B_i) \\ & + O(||(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q)||_F^2)\end{aligned} \tag{6.5}$$

as $(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q) \to 0$.

By the condition number theory developed by Rice [17], we define the condition number of the Hermitian positive definite solution X to (6.5) by

$$c(X) = \lim_{\delta\to 0} \sup_{||(\frac{\Delta A_1}{\eta_1}, \frac{\Delta A_2}{\eta_2}, \cdots, \frac{\Delta A_m}{\eta_m}, \frac{\Delta Q}{\rho})||_F \le \delta} \frac{||\Delta X||_F}{\xi\delta}, \tag{6.6}$$

where ξ, ρ and η_i, $i = 1, 2, \cdots, m$ are positive parameters. Taking $\xi = \eta_i = \rho = 1$ in (6.6) gives the absolute condition number $c_{abs}(X)$, and taking $\xi = ||X||_F$, $\eta_i = ||A_i||_F$ and $\rho = ||Q||_F$ in (6.6) gives the relative condition number $c_{rel}(X)$.

Substituting (6.5) into (6.6), we get

$$\begin{aligned} c(X) &= \frac{1}{\xi} \max_{\substack{\left(\frac{\Delta A_1}{\eta_1}, \frac{\Delta A_2}{\eta_2}, \cdots, \frac{\Delta A_m}{\eta_m}, \frac{\Delta Q}{\rho}\right) \neq 0 \\ \Delta A_i \in \mathcal{C}^{n\times n},\ \Delta Q \in \mathcal{H}^{n\times n}}} \frac{\left\|\mathbf{V}^{-1}\left(\sum\limits_{i=1}^{m}(B_i^*\Delta A_i + \Delta A_i^* B_i) + \Delta Q\right)\right\|_F}{\left\|\left(\frac{\Delta A_1}{\eta_1}, \frac{\Delta A_2}{\eta_2}, \cdots, \frac{\Delta A_m}{\eta_m}, \frac{\Delta Q}{\rho}\right)\right\|_F} \\ &= \frac{1}{\xi} \max_{\substack{(E_1, E_2, \cdots, E_m, H) \neq 0 \\ E_i \in \mathcal{C}^{n\times n},\ H \in \mathcal{H}^{n\times n}}} \frac{\left\|\mathbf{V}^{-1}\left(\sum\limits_{i=1}^{m}\eta_i(B_i^* E_i + E_i^* B_i + \rho H)\right)\right\|_F}{\|(E_1, E_2, \cdots, E_m, H)\|_F}. \end{aligned}$$

Let V be the matrix representation of the linear operator $\mathbf{V}$. It follows from Lemma 4.3.2 in [18] that

$$\mathrm{vec}(\mathbf{V}W) = V \cdot \mathrm{vec}W.$$

By Lemma 4.3.1 in [18], we have

$$\mathrm{vec}(\mathbf{V}W) = \left(I \otimes I + \sum_{i=1}^{m}\frac{1}{\Gamma(p_i)}\int_0^\infty\int_0^1 (e^{-tsX}A_i)^T \otimes (A_i^* e^{-(1-t)sX})dt s^{p_i} ds\right)\cdot \mathrm{vec}W.$$

Then

$$V = I \otimes I + \sum_{i=1}^{m}\frac{1}{\Gamma(p_i)}\int_0^\infty\int_0^1 (e^{-tsX}A_i)^T \otimes (A_i^* e^{-(1-t)sX})dt s^{p_i} ds. \tag{6.7}$$

Let

$$V^{-1}=S+i\Sigma,\ V^{-1}(I\otimes B_i^*)=U_{i1}+\mathbf{i}\Omega_{i1},\ V^{-1}(B_i^T\otimes I)\Pi=U_{i2}+\mathbf{i}\Omega_{i2}, \tag{6.8}$$

$$S_c=\begin{bmatrix} S & -\Sigma \\ \Sigma & S \end{bmatrix},\quad U_i=\begin{bmatrix} U_{i1}+U_{i2} & \Omega_{i2}-\Omega_{i1} \\ \Omega_{i1}+\Omega_{i2} & U_{i1}-U_{i2} \end{bmatrix},\ \ i=1,2,\cdots,m, \tag{6.9}$$

$$\begin{aligned}&\mathrm{vec}H=x+\mathbf{i}y,\ \mathrm{vec}E_i=a_i+\mathbf{i}b_i,\ \ g=(a_1^T,b_1^T,\cdots,a_m^T,b_m^T,x^T,y^T)^T,\\ &M=(E_1,E_2,\cdots,E_m,H),\end{aligned}$$

where $x,y,a_i,b_i\in\mathcal{R}^{n^2},\ \ S,\Sigma,U_{i1},U_{i2},\Omega_{i1},\Omega_{i2}\in\mathcal{R}^{n^2\times n^2},\ \ i=1,2,\cdots,m,\ \mathbf{i}=\sqrt{-1}$, Π is the vec-permutation matrix, such that

$$\mathrm{vec}\ A^T=\Pi\ \mathrm{vec}\ A.$$

Then we obtain that

$$\begin{aligned}
&c(X)\\
&=\frac{1}{\xi}\max_{M\neq 0}\frac{\left\|\mathbf{V}^{-1}\left(\sum\limits_{i=1}^m\eta_i(\rho H+B_i^*E_i+E_i^*B_i)\right)\right\|_F}{||(E_1,E_2,\cdots,E_m,H)||_F}\\
&=\frac{1}{\xi}\max_{M\neq 0}\frac{\left\|\rho V^{-1}\mathrm{vec}H+\sum\limits_{i=1}^m\eta_iV^{-1}\left((I\otimes B_i^*)\mathrm{vec}E_i+(B_i^T\otimes I)\mathrm{vec}E_i^*\right)\right\|}{\|\mathrm{vec}\,(E_1,E_2,\cdots,E_m,H)\|}\\
&=\frac{1}{\xi}\max_{M\neq 0}\frac{\left\|\rho(S+\mathbf{i}\Sigma)(x+\mathbf{i}y)+\sum\limits_{i=1}^m\eta_i[(U_{i1}+\mathbf{i}\Omega_{i1})(a_i+\mathbf{i}b_i)+(U_{i2}+\mathbf{i}\Omega_{i2})(a_i-\mathbf{i}b_i)]\right\|}{\|\mathrm{vec}\,(E_1,E_2,\cdots,E_m,H)\|}\\
&=\frac{1}{\xi}\max_{g\neq 0}\frac{||(\rho S_c,\eta_1U_1,\eta_2U_2,\cdots,\eta_mU_m)g||}{\|g\|}\\
&=\frac{1}{\xi}\ ||\ (\rho S_c,\eta_1U_1,\ \eta_2U_2,\cdots,\eta_mU_m)||,\ \ E_i\in\mathcal{C}^{n\times n}.
\end{aligned}$$

Theorem 6.1. *If* $\sum\limits_{i=1}^m p_i\|A_i\|^2\lambda_{\min}^{-p_i-1}(Q)<1$ *and* $\sum\limits_{i=1}^m p_i\|A_i+\Delta A_i\|^2\lambda_{\min}^{-p_i-1}(Q+\Delta Q)<1$, *the condition number* $c(X)$ *defined by (6.6) has the explicit expression*

$$c(X)=\frac{1}{\xi}\ ||\ (\rho S_c,\ \eta_1U_1,\ \eta_2U_2,\cdots,\eta_mU_m)||, \tag{6.10}$$

where the matrices S_c,U_i *are defined as in (6.8).*

Remark 6.1. From (6.10) we have the relative condition number

$$c_{rel}(X)=\frac{||\ (||Q||_FS_c,\ ||A_1||_FU_1,\ ||A_2||_FU_2,\ \cdots,||A_m||_FU_m)||}{||X||_F}.$$

7. Numerical Examples

To illustrate the theoretical results of the previous sections, in this section several simple examples are given, which were carried out using MATLAB 7.1. For the stopping criterion we take $\varepsilon_{k+1}(X) = \|X_k - \sum_{i=1}^{m} A_i^* X_k^{-p_i} A_i - Q\| < 1.0e-10$.

Example 7.1. In this example, we study the matrix equation

$$X - A_1^* X^{-2} A_1 - A_2^* X^{-3} A_2 = I,$$

with

$$A_k = \frac{\frac{1}{k+2} + 2 \times 10^{-2}}{||A||} A, \quad k = 1, 2, \quad A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

By computation, $q = 2\|A_1^* A_1\| + 3\|A_2^* A_2\| = 0.4673 < 1$. Let $X_0 = 1.1I \in [I, I + A_1^* A_1 + A_2^* A_2]$. The conditions in Theorem 3.3 are satisfied. Algorithm (3.1) needs 17 iterations to obtain the unique positive definite solution

$$X = \begin{pmatrix} 1.0578 & 0.0426 & 0.0073 & -0.0013 & 0.0001 \\ 0.0426 & 1.0651 & 0.0413 & 0.0074 & -0.0013 \\ 0.0073 & 0.0413 & 1.0652 & 0.0413 & 0.0073 \\ -0.0013 & 0.0074 & 0.0413 & 1.0651 & 0.0426 \\ 0.0001 & -0.0013 & 0.0073 & 0.0426 & 1.0578 \end{pmatrix}$$

with the residual $\|X - A_1^* X^{-2} A_1 - A_2^* X^{-3} A_2 - I\| = 5.6312e-011$.

Example 7.2. In this example, we consider the corresponding perturbation bound for the solution X in Theorem 4.1.

We consider the matrix equation

$$X - A_1^* X^{-2} A_1 - A_2^* X^{-3/2} A_2 = I,$$

with

$$A_1 = \frac{\frac{1}{3} + 2 \times 10^{-2}}{||A||} A, \quad A_2 = \frac{\frac{1}{6} + 3 \times 10^{-2}}{||A||} A, \quad A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

In this case, the matrix equation (1.1) has a unique positive definite solution X. Suppose that the coefficient matrices A_1, A_2 and Q are respectively perturbed to $\widetilde{A_i} = A_i + \Delta A_i, i = 1, 2$ and $\widetilde{Q} = I + \Delta Q$, where

$$\Delta A_1 = \frac{10^{-j}}{\|C^T + C\|}(C^T + C), \quad \Delta A_2 = \frac{3 \times 10^{-j-1}}{\|C^T + C\|}(C^T + C), \quad \Delta Q = 10^{-j} \times A$$

and C is a random matrix generated by MATLAB function **randn**.

By Theorem 4.1, we can compute the relative perturbation bound ξ. The results averaged as the geometric mean of 20 randomly perturbed runs. Some results are listed in Table 1.

Table 1: Perturbation bounds for Example 7.2 with different values of j

j	4	5	6	7
$\frac{\|\widetilde{X}-X\|}{\|X\|}$	2.8216×10^{-4}	2.6097×10^{-5}	2.8774×10^{-6}	2.7732×10^{-7}
ξ	7.4788×10^{-4}	6.9630×10^{-5}	7.4725×10^{-6}	7.5760×10^{-7}

The results listed in Table 1 show that the perturbation bound ξ given by Theorem 4.1 is sharp.

Example 7.3. In this example, we consider the backward error of an approximate solution for the unique solution X to the equation (1.1) in Theorem 5.1. We consider

$$X - A_1^* X^{-1/2} A_1 - A_2^* X^{-3/2} A_2 = Q,$$

with the coefficient matrices

$$A_1 = \frac{1}{5}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \quad A_2 = \frac{2\sqrt{3}}{45} A_1, \quad Q = X - A_1^* X^{-1/2} A_1 - A_2^* X^{-3/2} A_2,$$

where $X = diag(1, 2, 3)$, which ensures that there exists a unique positive solution in equation (1.1).

Let

$$\widetilde{X} = X + \begin{pmatrix} 0.5 & -0.1 & 0.2 \\ -0.1 & 0.3 & 0.6 \\ 0.2 & 0.6 & -0.4 \end{pmatrix} \times 10^{-j}$$

be an approximate solution to (1.1). Take $\alpha_1 = ||A_1||_F$, $\alpha_2 = ||A_2||_F$ and $\rho = ||Q||_F$ in Theorem 5.1. Some results on lower and upper bounds for the backward error $\eta(\widetilde{X})$ are displayed in Table 2.

The results listed in Table 2 show that the backward error of $\widetilde{X}$ decreases as the error $||\widetilde{X} - X||_F$ decreases.

Table 2: Backward error for Example 7.3 with different values of j

j	$\|\|\widetilde{X}-X\|\|_F$	r	$U(r)$	$B(r)$
1	0.2298	0.0636	0.0632	0.0639
3	2.3×10^{-3}	6.3587×10^{-4}	6.3583×10^{-4}	6.3591×10^{-4}
5	2.2978×10^{-5}	6.3587×10^{-6}	6.3587×10^{-6}	6.3587×10^{-6}
7	2.2978×10^{-7}	6.3587×10^{-8}	6.3587×10^{-8}	6.3587×10^{-8}
9	2.2978×10^{-9}	6.3587×10^{-10}	6.3587×10^{-10}	6.3587×10^{-10}

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