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ON ENTIRE SOLUTIONS OF CERTAIN TYPE OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. In this paper, we deal with differential-difference equations of the form

$$f(z)^2 + p(z)f(z+c) + h(z)f'(z) + g(z) = d_1e^{\lambda z} + d_2e^{-\lambda z}$$

where $p(z),\ h(z),\ g(z)$ are polynomials, and $c,\ d_1,\ d_2,\ \lambda \in \mathbb{C}$ are constants with $d_1d_2\lambda \neq 0$. By utilizing Nevanlinna's value distribution theory, some sufficient conditions on the nonexistence of entire solutions regarding the equations are provided.

1. Introduction and Results

Let f denote a nonconstant meromorphic function. We assume the readers are familiar with the basic Nevanlinna's value distribution theory and its standard notations such as $m(r,\ f),\ N(r,\ f),\ T(r,\ f),\ S(r,\ f)$ and etc., see e.g. [4, 5]. Also we shall use the notation $\sigma(f)$ to denote the order of f. Moreover, we shall use $P_d(f)$ to denote a differential polynomial in f and its derivatives $f',\ f'',\ \cdots$, with a total degree d, which has rational functions as the coefficients. However, without confusion, we also use $P_d(f)$ to denote a differential-difference polynomial in f, namely a polynomial in $f,\ f',\ f'',\ \cdots$, and its shifts $f(z+c_j)$, (where $c_j(j=1,\ 2,\ \cdots)$ are constants), with a total degree d.

Recently, several papers [6–8, 9] have been published regarding entire solutions of nonlinear differential equations of the form:

$$f(z)^n + P_d(f) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}, \tag{1.1}$$

where $d,\ n$ are integers, $n > d$, $P_d(f)$ a differential polynomial in $f(z)$, and $p_1,\ p_2$ nonzero polynomials and $\alpha_1,\ \alpha_2$ nonzero constants. More specifically, we recall the following Theorems A, B, C and D.

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Theorem A. ([8]). *Let $n \geq 4$ be an integer and $P_d(f)$ denote an algebraic differential polynomial in f of degree $d \leq n-3$. Let $p_1,\ p_2$ be two nonzero polynomials, α_1 and α_2 be two nonzero constants with $\alpha_1/\alpha_2 \neq rational$. Then the differential equation (1.1) has no transcendental entire solutions.*

Theorem B. ([7]). *Let $n \geq 2$ be an integer and $P_d(f)$ denote a differential polynomial in f of degree $d \leq n-1$. Let $p_1,\ p_2$ be small functions of e^z, and α_1 and α_2 be two positive number satisfying $(n-1)\alpha_2 \geq n\alpha_1 > 0$. If α_1/α_2 is irrational, then the differential equation (1.1) has no entire solutions.*

Theorem C. ([6]). *Let $n \geq 3$ be an integer and $P_d(f)$ denote a differential polynomial in f of degree $d \leq n-2$ with polynomial coefficients such that $P_d(0) \neq 0$. Provided that $p_1,\ p_2$ are non-vanishing polynomials and α_1 and α_2 are distinct nonzero complex constants, then the differential equation (1.1) has no entire solutions.*

Remark 1.1. The condition $P_d(0) \neq 0$ is a necessary one.

Theorem D. ([9]). *Let $p_1,\ p_2$ and λ be nonzero constants. For the difference equation*

$$f(z)^3 + a(z)f(z+1) = p_1e^{\lambda z} + p_2e^{-\lambda z}, \tag{1.2}$$

where $a(z)$ is a polynomial. If $a(z)$ is not a constant, then the equation (1.2) does not have any transcendental entire solution of finite order.

Remark 1.2. In Theorems A, C and D, it is required that $n \geq 3$, and in Theorem B, though n can be equal to 2, it is required that α_1 and α_2 are positive numbers, with α_1/α_2 being irrational.

In this note, we shall tackle differential or differential-difference equations in the form (1.1) with $n = 2,\ \alpha_1/\alpha_2 = -1$, and obtain the following results.

Theorem 1.1. *Let $p(z),\ h(z),\ g(z)$ be polynomials, such that either p and h are linearly independent, or there is one and only one of p and h being identically equal to zero, and let $c,\ d_1,\ d_2,\ \lambda \in \mathbb{C}$ be constants such that $d_1d_2\lambda \neq 0$ and $e^{\lambda c} \neq 1$.*

Then the differential-difference equation

$$f(z)^2 + p(z)f(z+c) + h(z)f'(z) + g(z) = d_1e^{\lambda z} + d_2e^{-\lambda z} \tag{1.3}$$

has no entire solution of finite order.

Example. The equation

$$f(z)^2 + \frac{1}{2}if(z+\pi i) + f'(z) - 2 = e^z + e^{-z}$$

has a solution $f = e^{\frac{z}{2}} + e^{-\frac{z}{2}}$, where $p(z) = \frac{1}{2}i$ and $h(z) = 1$ are linearly dependent. This shows that the condition "p and h are linearly independent, or that there is one and only one of p and h being identically equal to zero" in Theorem 1.1 can not be omitted.

Theorem 1.2. *Let $h(z)$ $(\not\equiv 0)$, $g(z)$ be polynomials, and let $d_1,\ d_2,\ \lambda \in \mathbb{C}$ be constants such that $d_1 d_2 \lambda \neq 0$. Then the differential equation*

$$f(z)^2 + h(z)f'(z) + g(z) = d_1 e^{\lambda z} + d_2 e^{-\lambda z} \tag{1.4}$$

has no entire solution.

Corollary 1.3. *Let $p(z),\ h(z),\ g(z)$ be polynomials, such that $\deg p \neq \deg h$, and let $c,\ d_1,\ d_2,\ \lambda \in \mathbb{C}$ be constants such that $d_1 d_2 \lambda \neq 0$.*

Then the differential-difference equation (1.3) has no entire solution of finite order.

Corollary 1.4 *Let $p(z)$ $(\not\equiv 0)$, $g(z)$ be polynomials, and let $d_1,\ d_2,\ \lambda \in \mathbb{C}$ be constants such that $d_1 d_2 \lambda \neq 0$. Then the difference equation*

$$f(z)^2 + p(z)f(z+c) + g(z) = d_1 e^{\lambda z} + d_2 e^{-\lambda z} \tag{1.5}$$

has no entire solution of finite order.

Remark 1.3. If one follows the proofs of the theorems carefully, then it is not difficult to see that Theorems 1.1 and 1.2 remain to be valid if the term f' in the equations of the two theorems is replaced by any linear differential polynomial or differential-difference polynomial ($\not\equiv 0$), respectively.

2. Proofs of Theorems

Lemma 2.1. (see e.g. [1, p. 69-70]). *Suppose that $n \geq 2$ and let $f_j(z),\ j = 1, \cdots, n$, be meromorphic functions and $g_j(z),\ j = 1, \cdots, n$, be entire functions such that*

(*i*) $\sum_{j=1}^{n} f_j(z) \exp\{g_j(z)\} \equiv 0$*;*

(*ii*) *when* $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ *is not constant;*

(*iii*) *when* $1 \leq j \leq n,\ 1 \leq h < k \leq n,$

$$T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\}\ (r \to \infty,\ r \notin E), \tag{2.1}$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0,\ j = 1, \cdots, n$.

Lemma 2.2. (see [3]). *Let f be a nonconstant finite-order meromorphic solution of*

$$f^n P(f) = Q(f),$$

where $P(f),\ Q(f)$ are difference polynomials in f with small meromorphic coefficients, and let $\delta < 1$. If the total degree of $Q(f)$ as a polynomial in f and its shifts is at most n, then

$$m(r, P(f)) = o\left(\frac{T(r+|c|, f)}{r^\delta} + o(T(r,f)\right) \tag{2.2}$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Remark 2.1. In Lemma 2.2, if f is transcendental with $\sigma(f) < \infty$, and $P(f),\ Q(f)$ are differential-difference polynomials in f, then by using a similar method as in the proof of Lemma 2.4.2 of [5], we see that a similar conclusion of Lemma 2.2 holds. Moreover, we see that if the coefficients of $P(f)$ and $Q(f)$ ($\sigma(f) < \infty$) are polynomials or rational functions $a_j(z),\ j = 1,\ \cdots,\ k$, then (2.2) can be replaced by

$$m(r, P(f)) = S(r,\ f) + O\left(\sum_{j=1}^{k} m(r,\ a_j)\right)$$

where r is sufficiently large.

Lemma 2.3. *Let λ denote a nonzero constant, and $H(z)$ a nonvanishing polynomial. Then the differential equation*

$$4y''(z) - \lambda^2 y(z) = H(z) \tag{2.3}$$

has a special solution $y_0(z)$ which is a nonvanishing polynomial.

Proof. If $H(z)$ is a nonzero constant, then clearly $y_0(z) = -\frac{H(z)}{\lambda^2}$ is a special solution of (2.3).

Now suppose that

$$H(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $n \geq 1$ is an integer, $a_n \neq 0,\ a_{n-1},\ \cdots,\ a_0$ are constants.

We use the method of undetermined coefficients, to derive the polynomial solution $y_0(z)$ satisfying (2.3) by $\lambda,\ a_n,\ a_{n-1},\ \cdots,\ a_0$. Clearly, by (2.3), we see that $\deg y_0 = \deg H$. For $n = 1,$ or 2, clearly, equation (2.3) has a polynomial solution

$$y_0(z) = -\frac{1}{\lambda^2}(a_1 z + a_0),$$

or

$$y_0(z) = -\frac{1}{\lambda^2}(a_2 z^2 + a_1 z + a_0 + 8\frac{a_2}{\lambda^2}).$$

In a general case, for $n \geq 3$, (2.3) has a polynomial solution

$$y_0(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_j z^j + \cdots + b_1 z + b_0$$

where

$$b_n = -\frac{1}{\lambda^2} a_n, \quad b_{n-1} = -\frac{1}{\lambda^2} a_{n-1},$$

$$b_j = -\frac{1}{\lambda^2}(a_j - 4(j+2)(j+1)b_{j+2}) \qquad j = n-2, \ \cdots, \ 0.$$

Hence, (2.3) has a nonvanishing polynomial solution $y_0(z)$.

Lemma 2.4. (see [2]). *Suppose that $f(z)$ is a meromorphic function of finite order. Then*

$$T(r+1, \ f) = T(r, \ f) + S(r, \ f).$$

Remark 2.2. it follows that $\sigma(f(z+c)) = \sigma(f(z))$, for any constant $c \in \mathbb{C}$.

2.1. Proof of Theorem 1.1

Clearly, $\sigma\left(d_1 e^{\lambda z} + d_2 e^{-\lambda z}\right) = 1$. By Lemma 2.4 and Remark 2.2, it follows right away from the equation (1.3) that $\sigma(f) \geq 1$.

Differentiating both sides of (1.3), we obtain

$$2f(z)f'(z) + Q_{11}(f) = d_1 \lambda e^{\lambda z} + d_2(-\lambda)e^{-\lambda z}, \tag{2.4}$$

where $Q_{11}(f)$ and the following $Q_{12}(f), \ Q_{13}(f), \ \cdots$ denote differential-difference polynomials in $f(z)$, with a total degree ≤ 1, and with the polynomials as the coefficients.

By eliminating $e^{-\lambda z}$ from the equations (1.3) and (2.4), we have

$$\lambda f(z)^2 + 2f(z)f'(z) + Q_{12}(f) = 2\lambda d_1 e^{\lambda z}. \tag{2.5}$$

Similarly, by eliminating $e^{\lambda z}$ from the equations (1.3) and (2.4), we have

$$\lambda f(z)^2 - 2f(z)f'(z) + Q_{13}(f) = 2\lambda d_2 e^{-\lambda z}. \tag{2.6}$$

Again by eliminating $e^{\lambda z}$ and $e^{-\lambda z}$ from the equations (2.5) and (2.6), we obtain

$$\lambda^2 f(z)^4 - 4f(z)^2 f'(z)^2 + Q_{31}(f) = 4\lambda^2 d_1 d_2$$

or

$$\lambda^2 f(z)^4 - 4f(z)^2 f'(z)^2 + Q_{32}(f) = 0, \tag{2.7}$$

where $Q_{31}(f), \ Q_{32}(f)$ and the following $Q_{33}(f), \ \cdots$ denote differential-difference polynomials in $f(z)$, with total degree ≤ 3.

Differentiating both sides of (2.4), we have

$$2f'(z)^2 + 2f(z)f''(z) + Q_{14}(f) = d_1\lambda^2 e^{\lambda z} + d_2\lambda^2 e^{-\lambda z}. \tag{2.8}$$

Combining (1.3) with (2.8), we obtain

$$2f'(z)^2 + 2f(z)f''(z) - \lambda^2 f(z)^2 + Q_{15}(f) = 0,$$

that is

$$f'(z)^2 = \frac{1}{2}\lambda^2 f(z)^2 - f(z)f''(z) - Q_{15}(f). \tag{2.9}$$

Substituting (2.9) into (2.7), we obtain

$$\lambda^2 f(z)^4 - 4f(z)^2\left(\frac{1}{2}\lambda^2 f(z)^2 - f(z)f''(z) - Q_{15}(f)\right) + Q_{32}(f) = 0$$

or

$$f(z)^3\left(4f''(z) - \lambda^2 f(z)\right) = Q_{33}(f). \tag{2.10}$$

Now we consider the equation (2.10) in two cases, Case 1: $Q_{33}(f) \not\equiv 0$ and Case 2: $Q_{33}(f) \equiv 0$.

Case 1. In this case, since $f(z)$ is a transcendental entire function of finite order, we see that (2.10) satisfies conditions of Lemma 2.2 and Remark 2.1. Thus, we have

$$\begin{aligned} &m\left(r,\ 4f''(z) - \lambda^2 f(z)\right) \\ &= S(r,\ f) + O(m(r,\ p) + m(r,\ h) + m(r,\ g)) = O(\log\ r), \end{aligned} \tag{2.11}$$

which implies that $4f''(z)-\lambda^2 f(z)$ is a polynomial. Thus, from (2.10) and $Q_{33}(f) \not\equiv 0$, we have

$$4f''(z) - \lambda^2 f(z) = H(z), \tag{2.12}$$

that $H(z)$ is a nonvanishing polynomial, but $H(z)$ may be a nonzero constant. By Lemma 2.3, we see that equation (2.12) must have a nonvanishing polynomial solution, say, $f_0(z)$.

Since the differential equation

$$4f''(z) - \lambda^2 f(z) = 0 \tag{2.13}$$

has two fundamental solutions

$$f_1(z) = e^{\frac{\lambda}{2}z}, \quad f_2(z) = e^{-\frac{\lambda}{2}z}.$$

It follows that the general entire solution $f(z)$ of (2.12) can be expressed as

$$f(z) = c_1 e^{\frac{\lambda}{2}z} + c_2 e^{-\frac{\lambda}{2}z} + f_0(z), \tag{2.14}$$

where c_1 and c_2 are constants, $f_0(z)$ is a nonvanishing polynomial.

Substituting this into (1.3), we obtain

$$\begin{aligned} &(c_1^2 - d_1)e^{\lambda z} + (c_2^2 - d_2)e^{-\lambda z} + c_1 \left[2f_0(z) + e^{\frac{\lambda}{2}c}p(z) + \frac{\lambda}{2}h(z)\right] e^{\frac{\lambda}{2}z} \\ &+c_2 \left[2f_0(z) + e^{\frac{-\lambda}{2}c}p(z) - \frac{\lambda}{2}h(z)\right] e^{-\frac{\lambda}{2}z} \\ &+f_0(z)^2 + 2c_1c_2 + p(z)f_0(z) + h(z)f_0'(z) + g(z) = 0. \end{aligned} \tag{2.15}$$

It follows from Lemma 2.1 that

$$c_1^2 = d_1 \neq 0, \quad c_2^2 = d_2 \neq 0, \quad f_0(z)^2 + 2c_1c_2 + p(z)f_0(z) + h(z)f_0'(z) + g(z) \equiv 0,$$

and

$$2f_0(z) + e^{\frac{\lambda}{2}c}p(z) + \frac{\lambda}{2}h(z) \equiv 0, \quad 2f_0(z) + e^{\frac{-\lambda}{2}c}p(z) - \frac{\lambda}{2}h(z) \equiv 0. \tag{2.16}$$

Clearly, if $h(z) \equiv 0$, then $p(z) \not\equiv 0$ by assumption of the theorem. Thus, by (2.16), we obtain $2f_0(z) = -e^{\frac{\lambda c}{2}}p(z) = -e^{-\frac{\lambda c}{2}}p(z)$, which leads to $e^{\lambda c} = 1$, which is a contradiction with the assumption that $e^{\lambda c} \neq 1$. If $p(z) \equiv 0$, then $h(z) \not\equiv 0$. Again by (2.16), we have $2f_0(z) \equiv -\frac{\lambda}{2}h(z) \equiv \frac{\lambda}{2}h(z)$, which is also a contradiction.

Now suppose that $p(z)$ and $h(z)$ are linearly independent, which implies neither $p(z)$ nor $h(z)$ can be identically zero. Then, by (2.16), we deduce that

$$\left(e^{\frac{\lambda c}{2}} - e^{-\frac{\lambda c}{2}}\right)p(z) + \lambda h(z) \equiv 0. \tag{2.17}$$

This also contradicts the assumptions of the theorem.

Case 2. In this case, by (2.10) and $Q_{33}(z) \equiv 0$, we have that

$$4f''(z) - \lambda^2 f(z) \equiv 0. \tag{2.18}$$

By the fact that every entire solution $f(z)(\not\equiv 0)$ of (2.18) can be expressed as

$$f(z) = c_1 e^{\frac{\lambda}{2}z} + c_2 e^{-\frac{\lambda}{2}z}, \tag{2.19}$$

where c_1 and c_2 are constants, with at least one of them being not equal to zero. Substituting this into (1.3), we obtain

$$(c_1^2 - d_1)e^{\lambda z} + (c_2^2 - d_2)e^{-\lambda z} + c_1 \left[e^{\frac{\lambda}{2}c}p(z) + \frac{\lambda}{2}h(z)\right] e^{\frac{\lambda}{2}z}$$

$$+c_2\left[e^{\frac{-\lambda}{2}c}p(z)-\frac{\lambda}{2}h(z)\right]e^{-\frac{\lambda}{2}z}+g(z)=0. \tag{2.20}$$

Again by Lemma 2.1, we conclude

$$c_1^2=d_1\neq 0,\quad c_2^2=d_2\neq 0,\quad 2c_1c_2+g(z)\equiv 0,$$

and

$$e^{\frac{\lambda}{2}c}p(z)+\frac{\lambda}{2}h(z)\equiv 0,\quad e^{\frac{-\lambda}{2}c}p(z)-\frac{\lambda}{2}h(z)\equiv 0. \tag{2.21}$$

Thus, we have $p(z)\equiv 0$ and $h(z)\equiv 0$, which contradicts with the assumptions of the theorem. Similarly, if $p(z)$ and $h(z)$ are linearly independent, then by the same arguments used in Case 1, we can also derive a contradiction. Theorem 1.1 is thus proved.

2.2. Proof of Theorem 1.2

Now we are going to show that any entire solution f of the equation (1.4) must be of finite order. By (1.4), we have that

$$T(r,\ f^2+hf'+g)=T(r,\ d_1e^{\lambda z}+d_2e^{-\lambda z})\leq 2T(r,\ e^{\lambda z})+O(1). \tag{2.22}$$

On the other hand, by the fact that

$$T(r,\ f')=m(r,\ f)\leq m\left(r,\ \frac{f'}{f}\right)+T(r,\ f)\leq T(r,\ f)+S(r,\ f)$$

we obtain

$$\begin{aligned}T(r,\ f^2+hf'+g)&\geq T(r,\ f^2)-T(r,hf'+g)\\ &\geq 2T(r,\ f)-(T(r,\ f')+T(r,\ h)+T(r,\ g)\\ &\geq 2T(r,\ f)-T(r,\ f)+S(r,\ f)=T(r,\ f)+S(r,\ f).\end{aligned} \tag{2.23}$$

This and (2.22) lead to

$$T(r,\ f)+S(r,\ f)\leq 2T(r,\ e^{\lambda z})+O(1).$$

It follows that $\sigma(f)\leq\sigma(e^{\lambda z})=1$. Thus, $\sigma(f)$ is finite. This contradicts Theorem 1.1 that the equation (1.4) has no entire solution of finite order. The theorem is thus proved.

Finally, we would like to conclude the paper with the following:

Conjecture. Let q_1 and q_2 denote any two nonconstant polynomials, with $q_1/q_2\neq$ rational number, and $P_1(f)$ denote any differential or differential-difference polynomial, with $P_1(0)\neq 0$, then the equation

$$f(z)^2+P_1(f)=q_3e^{q_1(z)}+q_4e^{q_2(z)}$$

has no entire solutions, for any two polynomials q_3 and q_4 with $q_3q_4\not\equiv 0$.

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