TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 4, pp. 1119-1127, August 2014
DOI: 10.11650/tjm.18.2014.3743
This paper is available online at http://journal.taiwanmathsoc.org.tw

ON THE DENSE UNBOUNDED DIVERGENCE OF THE DISCRETE BEST APPROXIMATION

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Abstract. A classic theorem of Approximation Theory states the uniform convergence of the best approximation polynomials, concerning the Banach space C of all real-valued continuous functions defined on the interval $[-1, 1]$ of $\mathbb{R}$, in supremum norm. By contrast, the main result of this paper highlights the phenomenon of double condensation of singularities (meaning unbounded divergence on large subsets of C and $[-1, 1]$, in topological sense) for the discrete best approximation on Chebyshev nodes.

1. Introduction

It is a classic theme in Approximation Theory that to find and to characterize the best approximation polynomials of a continuous function in the Chebyshev sense [2, 3, 6, 12]. However, it is often very hard to provide these polynomials explicitly, therefore the requirement to develop approximating methods in this field is of computational interest. One answer in this meaning consists in approximation over finite sets [1, 2, 3, 5, 9, 10, 12]. The present paper joins to this approach, as follows. Let us consider a node matrix $\mathcal{M}$ in the interval $[-1, 1]$ of $\mathbb{R}$ so that for each integer $n \geq 1$ the n-th row J_n of $\mathcal{M}$ contains at least n points. Denote by C the Banach space of all continuous functions $f : [-1, 1] \to \mathbb{R}$, endowed with the supremum norm $\| \cdot \|$.

Given an integer $n \geq 1$, let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n and let consider the operator $U_n = U_n(\mathcal{M})$ which associates to each f in C the unique polynomial $U_n f$ in $\mathcal{P}_{n-1}$ for which the infimum of the set $\{\max\{|f(x) - P(x)| : x \in J_n\} : P \in \mathcal{P}_{n-1}\}$ is attained [1, 2, 5, 9, 12]. If J_n contains n points, then $U_n f$ coincides with the Lagrange polynomial that interpolates f at the nodes of J_n. On the other hand, the theorem of Ch. de la Vallée-Poussin [1, 2, 6, 12] shows that

Received September 5, 2013, accepted December 23, 2013.
Communicated by Chong Li.
2010 *Mathematics Subject Classification*: 41A50, 41A10.
Key words and phrases: Chebyshev discrete best approximation, Chebyshev polynomials, Condensation of singularities, Superdense set.

the basic case for studying the best approximation over finite sets is that when J_n contains $n+1$ points; therefore, in what follows we will consider only this case, namely $J_n = \{x_{n+1}^k : \ 1 \le k \le n+1\}$, with $-1 \le x_{n+1}^1 < x_{n+1}^2 < x_{n+1}^3 < \ldots < x_{n+1}^{n+1} \le 1$. In this framework, the main result of [5] states that each operator U_n is a projection of C onto $\mathcal{P}_{n-1}$ and there exists a function g in C for which the sequence $(U_n g)_{n\ge 1}$ fails to converge uniformly on $[-1,1]$. In fact, the set of all functions f in C such that the sequence $(U_n f)_{n\ge1}$ unboundedly diverges, i.e. $\limsup_{n\to\infty} \|U_n f\| = \infty$, is superdense in the Banach space $(C, \|\cdot\|)$, [9]; we recall that a subset Y of a topological space X is said to be *superdense* in X if it is residual (namely, its complement is of first Baire category), uncountable and dense in X, [4]. These results contrast with the well known theorem concerning the uniform convergence of the best approximation polynomials in supremum norm, which states that each operator $\widetilde{U}_n : C \to \mathcal{P}_{n-1}$, $n \ge 1$, defined by $\|f - \widetilde{U}_n f\| = \inf\{\|f-P\| : \ P \in \mathcal{P}_{n-1}\}$, $f \in C$, is continuous, preserves all polynomials of $\mathcal{P}_{n-1}$ and the sequence $(\widetilde{U}_n f)_{n\ge1}$ is uniformly convergent to f, for each f in C [1, 2, 6, 12].

The aim of this paper is to highlight the phenomenon of double condensation of singularities, meaning unbounded divergence on large subsets of C and $[-1,1]$ in topological sense, with respect to the family of *Chebyshev projection operators* $U_n = U_n(\mathcal{T})$, $n \ge 1$, where the n-th row of the node matrix $\mathcal{T}$ contains the roots of Chebyshev polynomials T_{n+1}, with $T_n(x) = \cos(n \arccos x)$, $n \ge 1$, $-1 \le x \le 1$. To this purpose, the following principle of functional analysis, deriving from [4, Theorem 5.2], will be used.

Theorem 1.1. *Suppose that X is a nonzero Banach space, Y is a normed space and $\mathbb{T}$ is a nonvoid separable complete metric space without isolated points. Let $\{A_n : n \ge 1\}$ be a family of mappings of $X \times \mathbb{T}$ into Y satisfying the following conditions:*

1° *For each $n \ge 1$ and $t \in \mathbb{T}$, $A_n^t : X \to Y$, $A_n^t(x) = A_n(x,t)$, $\forall\ x \in X$, is a linear and continuous operator.*

2° *For each $n \ge 1$ and $x \in X$, $A_n^x : \mathbb{T} \to Y$, $A_n^x(t) = A_n(x,t)$, $\forall\ t \in \mathbb{T}$, is a continuous operator.*

3° *There exists a dense set $\mathbb{T}_0$ in $\mathbb{T}$ so that for each $t \in \mathbb{T}_0$ the relation $\sup\{\|A_n^t\| : n \ge 1\} = \infty$ is fulfilled.*

Then, there exists a superdense set D in X such that for each $x \in D$ the set $\{t \in \mathbb{T} : \sup\{\|A_n(x,t)\| : n \ge 1\} = \infty\}$ is superdense in $\mathbb{T}$.

As concerns recent works in this field, we cite [10] and [7, 8, 13], referring to the discrete best approximation over equidistant nodes and the more general topic of simultaneous best approximation, respectively. Otherwise, a future task for us is to investigate the problem of the simultaneous best approximation, in the framework described in this work.

The paper is organized as follows. In the next section, we present some needed notions and results concerning the discrete best approximation over arbitrary nodes. The third section contains two lemmas regarding the Chebyshev node matrix $\mathcal{T}$, which are essential to prove the unboundedness of the set of norms $\{\|U_n(\mathcal{T})\| : n \geq 1\}$ in the last section. The main result, which emphasizes the phenomenon of double condensation of singularities for Chebyshev projection operators $U_n(\mathcal{T})$, $n \geq 1$, is established in the fourth section.

2. Preliminaries Notions and Results

We start with some notations to be used in this paper, [1, 2, 3, 6, 14]. The notation $[a]$ stands for the largest integer n with $n \leq a$, $a \in \mathbb{R}$. Also, we denote by M_k, $k \geq 1$, some generic positive constants which are independent of any positive integer n and we write $a_n \sim b_n$ if the sequences of real numbers (a_n) and (b_n) satisfy the conditions $b_n \neq 0$ and $0 < M_1 \leq |a_n/b_n| \leq M_2$, $\forall\ n \geq 1$.

Further, let $n \geq 1$ be given and put $\omega_{n+1}(x) = \prod_{k=1}^{n+1}(x - x_{n+1}^k)$. Denote by $L_{n+1}f$ the Lagrange polynomial which interpolates a function $f \in C$ at the nodes of J_n, namely

$$(L_{n+1}f)(x) = \sum_{k=1}^{n+1} f(x_{n+1}^k) l_{n+1}^k(x), \quad -1 \leq x \leq 1,$$

with $l_{n+1}^k(x) = \omega_{n+1}(x)\ ((x - x_{n+1}^k)\omega_{n+1}'(x_{n+1}^k))^{-1}$, $1 \leq k \leq n+1$.

We associate to the n-th row J_n of $\mathcal{M}$ the corresponding Lebesgue function $\lambda_{n+1}(x) = \sum_{k=1}^{n+1} |l_{n+1}^k(x)|$, $-1 \leq x \leq 1$.

The coefficient of the leading term of $L_{n+1}f$ will be denoted by $a_n(f)$. If $f, g \in C$ and $a_n(g) \neq 0$, we set

$$q_n(f; g) = a_n(f)(a_n(g))^{-1}. \tag{2.1}$$

Now, let σ_{n+1} be a function in C satisfying the conditions $\sigma_{n+1}(x_{n+1}^k) = (-1)^k$, $1 \leq k \leq n+1$ (for instance, σ_{n+1} can be chosen as the unique polynomial of $\mathcal{P}_n$ satisfying the equalities $\sigma_{n+1}(x_{n+1}^k) = (-1)^k$, $1 \leq k \leq n+1$). By means of Theorem of Ch. de la Vallée-Poussin, [1, 2, 6, 12], the operator U_n can be written as:

$$U_n f = L_{n+1}f - q_n(f; \sigma_{n+1}) \cdot L_{n+1}\sigma_{n+1}, \quad f \in C, \tag{2.2}$$

see also [5, 9].

In order to point out the phenomenon of double condensation of singularities for the discrete Chebyshev best approximation, the following family of continuous functions $\{f_{n+1}^t\}$ will play an essential role. For each given point $t \in [-1,1]$, define $f_{n+1}^t \in C$ by

$$f_{n+1}^t(x) = \begin{cases} \mathrm{sgn} l_{n+1}^k(t), & \text{if } x = x_{n+1}^k,\ 1 \le k \le n+1 \\ 1, & \text{if } x \in \{-1,1\} \setminus J_n \\ \text{linear}, & \text{otherwise.} \end{cases} \tag{2.3}$$

With the notations

$$A_n = [x_{n+1}^1, x_{n+1}^{n+1}] \setminus J_n; \quad \tau_{n+1}^k = (\omega'_{n+1}(x_{n+1}^k))^{-1}, \quad 1 \le k \le n+1, \tag{2.4}$$

let us introduce the function $\delta_{n+1} : A_n \to [0,1]$ as follows: if $s \in \{1,2,3,\ldots,n\}$ and $t \in (x_{n+1}^s, x_{n+1}^{s+1})$, we set

$$\delta_{n+1}(t) = \left(\sum_{k=1}^{s} |\tau_{n+1}^k|\right)\left(\sum_{k=1}^{n+1} |\tau_{n+1}^k|\right)^{-1}. \tag{2.5}$$

The following statement holds.

Lemma 2.1. *Given $n \ge 1$, the equality*

$$|q_n(f_{n+1}^t, \sigma_{n+1})| = |1 - 2\delta_{n+1}(t)| \tag{2.6}$$

is satisfied for each $t \in A_n$.

Proof. Let $s \in \{1,2,3,\ldots,n\}$ and $t \in (x_{n+1}^s, x_{n+1}^{s+1})$.

The relations $\mathrm{sgn} l_{n+1}^k(t) = (-1)^{s-k}$ for $k \in \{1,2,\ldots,s\}$, $\mathrm{sgn} l_{n+1}^k(t) = (-1)^{k-s-1}$ for $k \in \{s+1, s+2, \ldots, n+1\}$, [2, 3], and $\mathrm{sgn}\tau_{n+1}^k = (-1)^{n+1-k}$ for $k \in \{1,2,\ldots,n+1\}$, combined with

$$a_n(f_{n+1}^t) = \sum_{k=1}^{n+1} \tau_{n+1}^k \mathrm{sgn} l_{n+1}^k(t) \tag{2.7}$$

and

$$a_n(\sigma_{n+1}) = (-1)^{n+1} \sum_{k=1}^{n+1} |\tau_{n+1}^k| \tag{2.8}$$

yield, after usual computations:

$$a_n(f_{n+1}^t) + (-1)^s a_n(\sigma_{n+1}) = 2 \cdot (-1)^{n+s+1} \sum_{k=1}^{s} |\tau_{n+1}^k|. \tag{2.9}$$

Dividing (2.9) by $a_n(\sigma_{n+1})$ and taking into account the relations (2.1) and (2.5), the equality (2.6) follows, which completes the proof.

3. Estimates Involving Chebyshev Node Matrix

Firstly, we point out an estimate concerning the Lebesgue functions associated to the Chebyshev matrix $\mathcal{T}$.

Lemma 3.1. *Given an arbitrary* $x \in (-1,1)$, *let* $\theta \in (0,\pi)$ *so that* $\cos\theta = x$ *and* $\eta = \dfrac{1}{2}\min(\theta, \pi-\theta)$. *Then, the inequality*

$$\lambda_n(x) \geq M_3(\sin\eta)\log n$$

holds for a proper sequence of positive integers.

Proof. The estimate $\lambda_n(x) - 1 \sim (\cos n\theta)\log n$ follows from [11, Th. 2]. On the other hand, similarly to [14, p. 330], the inequality $|\cos n\theta| \geq \sin\eta$ is satisfied if

$$\left(r+\frac{1}{2}\right)\pi - \eta < 2\eta n \leq \left(r+\frac{1}{2}\right)\pi + \eta, \quad r \geq 1.$$

Finally, we get:

$$\lambda_{n_r}(x) \geq M_3(\sin\eta)\log n_r, \quad \text{if} \quad n_r = 1 + \left[\frac{2r+1}{4\eta}\pi - \frac{1}{2}\right], \ r \geq 1, \tag{3.1}$$

which completes the proof. ∎

Further, let us consider the odd rows J_{2n-1} of $\mathcal{T}$, i.e.

$$J_{2n-1} = \left\{\cos\frac{2k-1}{4n}\pi : 1 \leq k \leq 2n\right\}, \quad n \geq 1.$$

According to the convention $x_{2n}^k < x_{2n}^{k+1}$, $1 \leq k \leq 2n-1$, it is easily seen that

$$x_{2n}^{2n+1-k} = -x_{2n}^k = \cos\frac{2k-1}{4n}\pi, \quad 1 \leq k \leq n \tag{3.2}$$

and

$$J_{2n-1} = \{\pm t_{2n}^k : 1 \leq k \leq n\}; \quad t_{2n}^k = x_{2n}^{n+k} = \sin\frac{2k-1}{4n}\pi; \quad 1 \leq k \leq n. \tag{3.3}$$

Now, noticing that $\omega_{2n}(x) = 2^{1-2n}T_{2n}(x)$, we derive from (2.4) and (3.2):

$$\tau_{2n}^{n+k} = (\omega_{2n}'(t_{2n}^k))^{-1} = (-1)^{n+k}\cdot\frac{2^{2n}}{4n}\cos\frac{2k-1}{4n}\pi, \quad 1 \leq k \leq n \tag{3.4}$$

and

$$\tau_{2n}^k + \tau_{2n}^{2n+1-k} = 0, \quad 1 \leq k \leq 2n. \tag{3.5}$$

The trigonometric identity $\sum_{k=1}^{m} \cos(2k-1)\alpha = \dfrac{\sin(2m\alpha)}{2\sin\alpha}$, $m \geq 1$, $\alpha \in \mathbb{R} \setminus (\pi\mathbb{Z})$, combined with (3.4), provides for $n \geq 1$ and $j \in \{1, 2, 3, \ldots, n\}$:

$$\sum_{k=n+1}^{n+j} |\tau_{2n}^{k}| = \sum_{k=1}^{j} |\tau_{2n}^{n+k}| = \frac{2^{2n}}{8n \sin \dfrac{\pi}{4n}} \cdot \sin \frac{j\pi}{2n}. \tag{3.6}$$

On the other hand, the relations (2.4), (2.5), (3.2) and (3.5) yield:

$$\delta_{2n}(t) + \delta_{2n}(-t) = 1, \text{ for all } t \in A_{2n-1}. \tag{3.7}$$

We are in a position to prove the following statement.

Lemma 3.2. *For each* $t \in (-1, 1) \setminus \mathcal{T}$ *there exists an integer* $m = m(t) \geq 2$ *satisfying the following relations:*

(i) $t \in A_{2mn-1}, \ \forall\, n \geq 2$*;*

(ii) $|1 - 2\delta_{2mn}(t)| \leq \cos \dfrac{\pi}{8m}, \ \forall\, n \geq 2.$

Proof. First step. Let $t \in [0, 1) \setminus \mathcal{T}$. The relations $\lim_{n\to\infty} t_{2n}^{1} = 0$ and $\lim_{n\to\infty} t_{2n}^{n} = 1$ imply the existence of some positive integers $m \geq 2$ and p, depending on t, such that:

$$1 \leq p \leq m - 1 \text{ and } t_{2m}^{p} < t < t_{2m}^{p+1}. \tag{3.8}$$

Further, let $m \geq 2$ be an arbitrary integer and define the integers i and j by

$$i = \left[pn - \frac{n}{2} + \frac{1}{2}\right] \quad \text{and} \quad j = \left[pn + \frac{n}{2} + \frac{1}{2}\right] + 1. \tag{3.9}$$

The inequalities $\dfrac{2i-1}{4mn}\pi \leq \dfrac{2p-1}{4m}\pi$ and $\dfrac{2p+1}{4m}\pi \leq \dfrac{2j-1}{4mn}\pi$, which follow from (3.9), combined with (3.3), give $t_{2mn}^{i} \leq t_{2m}^{p} < t_{2m}^{p+1} \leq t_{2mn}^{j}$. Consequently, in accordance with (3.8) and (3.9), there exists a positive integer $\nu = \nu(m, n)$ so that:

$$t \in (t_{2mn}^{\nu}, t_{2mn}^{\nu+1}) = (x_{2mn}^{mn+\nu}, x_{2mn}^{mn+\nu+1}), \ 1 \leq i \leq \nu \leq j - 1 \leq mn - 1. \tag{3.10}$$

The relation (3.10) proves the assertion (i) of this lemma for $t \in [0, 1) \setminus \mathcal{T}$.

Next, we derive from (2.5), (3.5), (3.6) and (3.10):

$$\delta_{2mn}(t) = \frac{1}{2}\left(1 + \sin \frac{\nu\pi}{2mn}\right), \quad n \geq 2. \tag{3.11}$$

A combination of the relations (3.8), (3.9), (3.10) and (3.11) yields:

$$\begin{aligned}\frac{1}{2} \le \delta_{2mn}(t) \le \frac{1}{2}\left(1+\sin\frac{j-1}{2mn}\pi\right) \le \frac{1}{2}\left(1+\sin\frac{2np+n+1}{4mn}\pi\right)\\ \le \frac{1}{2}\left(1+\sin\frac{2mn-n+1}{4mn}\pi\right) = \cos^2\left(\frac{\pi}{8m}-\frac{\pi}{8mn}\right),\end{aligned}$$

so we conclude:

$$\forall\ t\in[0,1)\setminus\mathcal{T},\ \exists\ m=m(t)\ge 2:\ \frac{1}{2}\le\delta_{2mn}(t)\le\cos^2\frac{\pi}{16m},\ \forall\ n\ge 2. \tag{3.12}$$

The inequalities (3.12) lead to:

$$|1-2\delta_{2mn}(t)| = 2\delta_{2mn}(t)-1 \le \cos\frac{\pi}{8m},\ \forall\ n\ge 2,$$

which proves the second assertion of Lemma 3.2 for $t\in[0,1)\setminus\mathcal{T}$.

Second step. Let us assume $t\in(-1,0)\setminus\mathcal{T}$. Applying the first step of this lemma for $-t\in(0,1)\setminus\mathcal{T}$, it results that there exists an integer $m=m(t)\ge 2$ satisfying the relations

$$-t\in A_{2mn-1},\ \forall\ n\ge 2 \tag{3.13}$$

and

$$|1-2\delta_{2mn}(-t)|\le\cos\frac{\pi}{8m},\ \forall\ n\ge 2. \tag{3.14}$$

Taking into account the symmetry of A_{2mn-1} with respect to the origin, the relation (3.13) implies $t\in A_{2mn-1}$, $\forall\ n\ge 2$. On the other hand, the relations (3.14) and (3.7) gives:

$$1-2\delta_{2mn}(-t)=2\delta_{2mn}(t)-1 \quad\text{and}\quad |1-2\delta_{2mn}(t)|\le\cos\frac{\pi}{8m},\ \forall\ n\ge 2.$$

Therefore, the assertions (i) and (ii) of Lemma 3.2 fulfil also for $t\in(-1,0)\setminus\mathcal{T}$, which completes the proof.

4. Double Condensation of Singularities for Chebyshev Projection Operators

The main result of this paper is stated as follows.

Theorem 4.1. *Let $U_n=U_n(\mathcal{T})$, $n\ge 1$, be the Chebyshev projection operators. Then, there exists a superdense set D in C so that for each f in D, the set of unbounded divergence of the family $\{U_nf : n\ge 1\}$, i.e.*

$$\{t\in[-1,1] : \sup\{|(U_nf)(t)| : n\ge 1\}=\infty\}$$

is superdense in the interval $[-1,1]$ of $\mathbb{R}$.

Proof. We use Theorem 1.1, with $X = C$, $\mathbb{T} = [-1, 1]$, $Y = \mathbb{R}$ and $A_n : C \times [-1, 1] \to \mathbb{R}$, $A_n(f; x) = (U_n f)(x)$, $\forall\, f \in C$ and $x \in [-1, 1]$. By means of (2.2), a more explicit expression of $A_n(f; x)$ can be obtained, that is:

$$(4.1) \qquad A_n(f; x) = \sum_{k=1}^{n+1} d_{n+1}^k(f) l_{n+1}^k(x); \quad f \in C,\ x \in [-1, 1],\ n \geq 1,$$

where $d_{n+1}^k(f) = f(x_{n+1}^k) + (-1)^{k+1} q_n(f; \sigma_{n+1})$, $1 \leq k \leq n + 1$.

The hypotheses 1° and 2° of Theorem 1.1 are obviously fulfilled. As to the third hypothesis of Theorem 1.1, let $\mathbb{T}_0 = (-1, 1) \setminus \mathcal{T}$; it is clear that $\mathbb{T}_0$ is dense in the interval $[-1, 1]$, because $\mathcal{T}$ is a countable set. Further, let $t_0 \in \mathbb{T}_0$ be arbitrarily given; the function $f_{n+1}^{t_0}$, defined by (2.3), satisfies the inequality:

$$(4.2) \qquad \|A_n^{t_0}\| \geq |A_n^{t_0}(f_{n+1}^{t_0}|,\ \forall\, n \geq 1.$$

In order to estimate $|A_n^{t_0}(f_{n+1}^{t_0})|$, we derive from (4.1) and (2.3):

$$(4.3) \qquad A_n^{t_0}(f_{n+1}^{t_0}) = \sum_{k=1}^{n+1} \alpha_{n+1}^k(f_{n+1}^{t_0}) |l_{n+1}^k(t_0)|,$$

with $\alpha_{n+1}^k(f_{n+1}^{t_0}) = 1 + (-1)^{k+1} q_n(f_{n+1}^{t_0}; \sigma_{n+1}) \mathrm{sgn} l_{n+1}^k(t_0)$.

According to (2.1), (2.7) and (2.8) we obtain $|q_n(f_{n+1}^{t_0}; \sigma_{n+1})| \leq 1$, which together with (4.3) and (2.6) yields:

$$\alpha_{n+1}^k(f_{n+1}^{t_0}) \geq 1 - |q_n(f_{n+1}^{t_0}; \sigma_{n+1})| \geq 0$$

and

$$(4.4) \qquad |A_n^{t_0}(f_{n+1}^{t_0})| = A_n^{t_0}(f_{n+1}^{t_0}) \geq (1 - |1 - 2\delta_{n+1}(t_0)|)\lambda_{n+1}(t_0),\ \forall\, n \geq 1.$$

Now, it follows from Lemma 3.2 and (4.4) that there exists an integer $m_0 = m(t_0) \geq 2$ with $t_0 \in A_{2m_0 n - 1}$, $\forall\, n \geq 2$ and

$$(4.5) \qquad |A_{2m_0 n-1}^{t_0}(f_{2m_0 n}^{t_0})| \geq 2 \left(\sin^2 \frac{\pi}{16 m_0}\right) \lambda_{2m_0 n}(t_0),\ \forall\, n \geq 2.$$

Finally, denoting by $\theta_0 = \arccos t_0$ and $\eta_0 = \dfrac{1}{2} \min(\theta_0, \pi - \theta_0)$, a combination of the relations (4.2), (4.5) and (3.1) yields:

$$\begin{aligned} \sup\{\|A_n^{t_0}\| : n \geq 1\} &\geq \sup\{|A_n^{t_0}(f_{n+1}^{t_0})| : n \geq 1\} \\ &\geq \sup\{|A_{2m_0 n_r - 1}^{t_0}(f_{2m_0 n_r})| : r \geq 1\} \\ &\geq M_4(\sin \eta_0) \left(\sin^2 \frac{\pi}{16 m_0}\right) \sup\{\ln n_r : r \geq 1\} = \infty. \end{aligned}$$

Therefore, the hypothesis 3° of Theorem 1.1 is also satisfied, which completes the proof of this theorem.

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