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OPTIMALITY CONDITIONS FOR EFFICIENCY ON NONSMOOTH MULTIOBJECTIVE PROGRAMMING PROBLEMS

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Abstract. In this paper, a nonsmooth multiobjective programming problem is introduced and studied. By using the generalized Guignard constraint qualification, some stronger Kuhn-Tucker type necessary optimality conditions for efficiency in terms of convexificators are established, in which we are not assuming that the objective functions are directionally differentiable. Moreover, some conditions which ensure that a feasible solution is an efficient solution to nonsmooth multiobjective programming problems are also given. The results presented in this paper improve the corresponding results in the literature.

1. INTRODUCTION

In recent years, stronger Kuhn-Tucker type necessary optimality conditions have received much attention by many authors. Maeda [16] obtained stronger Kuhn-Tucker type necessary optimality conditions for multiobjective programming problems where the objective and constraint functions are continuously differentiable. Later, Preda and Chitescu [19] extended the results obtained by Maeda from the continuously differentiable case to the directionally differentiable case. In the framework of the locally Lipschitz case, Li [11] and Giorgi et al. [6] derived some results of stronger Kuhn-Tucker type necessary optimality conditions in terms of the Clarke subdifferentials. Huang et al. [7] and Luu [14] obtained Kuhn-Tucker necessary conditions of efficiency for multiobjective programming problems in terms of the Michel-Penot subdifferentials.

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On the other hand, the idea of convexificators has been used to extend, unify, and sharpen various results in nonsmooth analysis and optimization due to convexificators are in general closed sets unlike the well-known subdifferentials which are convex and compact (see, for example, [2, 3, 4, 5, 9, 10, 13] and the references therein). It has been shown in [10] that for a locally Lipschtiz function, most known subdifferentials such as the subdifferentials of Clarke [1], Michel-Penot [17], Ioffe-Morduchovich [8, 18], and Treiman [20] are convexificators and these known subdifferentials may contain the convex hull of a convexificator. Therefore, from the viewpoint of optimization and applications, the descriptions of the optimality conditions, calculus rules, and the characterizations of generalized convex functions in terms of convexificators provide sharper results. Recently, under the assumption of directional differentiability, Li and Zhang [12] derived stronger Kuhn-Tucker type necessary optimality conditions for multiobjective programming problems in terms of upper convexificators, where the objective functions are directionally differentiable and the directional derivatives of the objective function and inequality constraints are sublinear in the second variable.

In this paper, we consider a nonsmooth multiobjective programming problem without assuming that the objective functions are directionally differentiable. By using the generalized Guignard constraint qualification, we obtain stronger Kuhn-Tucker type necessary optimality conditions for this problem. Moreover, we give some conditions which ensure that a feasible solution is an efficient solution to nonsmooth multiobjective programming problems. The results presented in this paper improve the corresponding results obtained by Li and Zhang [12].

2. Preliminaries

Throughout this paper, we assume that X is a real Banach space. The dual space of X is denoted by X^* and it equipped with weak* topology. For any set $A \subset X$, we denote by cl$A$, co$A$, and clco$A$ as the closed hull, convex hull, and closed convex hull of the set A respectively. The contingent cone or Bouligand cone [21] to the subset A at $x \in \text{cl}A$ is the set defined by

$$T(A, x) = \{d \in X : \exists\ (t_n, d_n) \to (0^+, d) \text{ such that } x + t_n d_n \in A\}.$$

Note that $T(A, x)$ is a closed cone in X.

Let $f : X \to R$ be a real-valued function. The lower and upper Dini directional derivatives of f at $x \in X$ in the direction $d \in X$ are defined, respectively, by

$$f^-(x; d) = \liminf_{t \downarrow 0} \frac{f(x + td) - f(x)}{t},$$
$$f^+(x; d) = \limsup_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}.$$

It is worth noting that, in the case where f is locally Lipschitz at x, $f^-(x;d)$ and $f^+(x;d)$ are continuous in d. A function f is said to be directionally differentiable at $x \in X$ if, for every direction $d \in X$, the usual one-sided directional derivative

$$f'(x;d) = \lim_{t \downarrow 0} \frac{f(x+td) - f(x)}{t}$$

of f at x in the direction d exists and is finite. Obviously, if f is directionally differentiable at $x \in X$, then for every $d \in X$,

$$f'(x;d) = f^-(x;d) = f^+(x;d).$$

We recall now some definitions that will be used in the sequel.

Definition 2.1. [10]. A function $f : X \to R$ is said to admit a lower convexificator $\partial^* f(x) \subseteq X^*$ at $x \in X$ if $\partial_* f(x)$ is weak* closed and

$$f^+(x;d) \geq \inf_{x^* \in \partial_* f(x)} \langle x^*, d \rangle, \ \forall \ d \in X.$$

Definition 2.2 [10] A function $f : X \to R$ is said to admit an upper convexificator $\partial^* f(x) \subseteq X^*$ at $x \in X$ if $\partial^* f(x)$ is weak* closed and

$$f^-(x;d) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, d \rangle, \ \forall \ d \in X.$$

A weak* closed set $\partial^* f(x)$ is said to be a convexificator of f at x if it is both upper and lower convexificator of f at x.

Remark 2.1. It is important to note that convexificators are not necessary weak* compact or convex [5]. For instance, the function $f : R \to R$, defined by

$$f_1(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0; \\ -\sqrt{-x} & \text{if } x < 0, \end{cases}$$

admits noncompact convexificators at 0 of the form $[\alpha, \infty)$ with $\alpha \in R$. On the other hand, the function $f : R \to R$, defined by $f(x) = -|x|$ admits a nonconvex convexificator $\partial^* f(0) = \{1, -1\}$ at 0.

Definition 2.3. [3]. A function $f : X \to R$ is said to admit an upper semiregular convexificator $\partial^* f(x) \subseteq X^*$ at $x \in X$ if $\partial^* f(x)$ is weak* closed and

$$f^+(x;d) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, d \rangle, \ \forall \ d \in X.$$

Remark 2.2. Since $f^-(x;d) \leq f^+(x;d)$, for all $d \in X$, an upper semiregular convexificator is an upper convexificator of f at x. But the converse is not true (see

Example 2.1 in [4]). If f is directionally differentiable at x in every direction d, then every upper convexificator is also an upper semiregular convexificator of f at x.

Let f be a locally Lipschitz function at $x \in X$. The Clarke [1] generalized directional derivative of f at x in the direction $d \in X$ is defined by

$$f^\circ(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}$$

and the Clarke [1] generalized gradient of f at x is denoted by

$$\partial_C f(x) = \{\xi \in X^* \mid f^\circ(x; d) \geq \langle \xi, d \rangle, \forall\, d \in X\}.$$

It follows that

$$f^\circ(x; d) = \sup_{\xi \in \partial_C f(x)} \langle \xi, d \rangle, \ \forall\, d \in X.$$

Note that, for every fixed $x \in X$, $\partial_C f(x)$ is a nonempty weak* compact subset of X^*. Moreover, for every x and d in X, since

$$f^-(x; d) \leq f^+(x; d) \leq f^\circ(x; d),$$

the Clarke subdifferential $\partial_C f(x)$ is a weak* compact and convex upper semiregular convexificator of f at x. On the other hand, Example 2.1 of [10] shows that the convex hull of a upper convexificator of a locally Lipschitz function may be strictly contained in the Clarke subdifferential. Therefore, for optimization problems involving locally Lipschitz functions, the results of the necessary optimality conditions expressed in terms of upper or upper semiregular convexificators are sharper than those expressed in terms of Clarke subdifferentials.

3. Optimality Conditions

Let R^n be the n-dimensional Euclidean space. In the sequel, we will use the following conventions for vectors in Euclidean space R^n:

$$x > y \Leftrightarrow x_i > y_i, \quad i = 1, 2, \cdots, n;$$
$$x \geqq y \Leftrightarrow x_i \geqq y_i, \quad i = 1, 2, \cdots, n;$$
$$x \geq y \Leftrightarrow x_i \geqq y_i, \quad i = 1, 2, \cdots, n, \text{ but } x \neq y.$$

In this paper, we consider the following nonsmooth multiobjective programming problem:

(MP) Minimize $\quad f(x) = (f_1(x), f_2(x), \cdots, f_p(x)),$
s.t. $\quad x \in S = \{x \in X : g(x) = (g_1(x), g_2(x), \cdots, g_m(x)) \leqq 0\},$

where the real-valued functions $f_i : X \to R$, $i \in I := \{1, 2, \cdots, p\}$, and $g_j : X \to R$, $j \in J := \{1, 2, \cdots, m\}$ are locally Lipschitz functions on X. Denote by $J(x)$ the set of indices of all the constraints active at $x \in S$; i.e., $J(x) = \{j \in J : g_j(x) = 0\}$.

Definition 3.1. A vector $x_0 \in S$ is said to be an efficient solution for (MP) if there does not exist $x \in S$ such that $f(x) \leq f(x_0)$.

Definition 3.2. A vector-valued function$f : X \to R^p$ is said to be strong pseudoconvex at $x_0 \in X$ if, for all $x \in X$,

$$f(x) \leq f(x_0) \Rightarrow f^+(x_0; x - x_0) \leq 0.$$

Definition 3.3. A vector-valued function$f : X \to R^p$ is said to be quasiconvex at $x_0 \in X$ if, for all $x \in X$,

$$f(x) \leqq f(x_0) \Rightarrow f^+(x_0; x - x_0) \leqq 0.$$

As in [12], for each $i \in I$, define the sets

$$\begin{aligned} Q(x) &= \{y \in X : f(y) \leqq f(x) \text{ and } g(y) \leqq 0\}, \\ Q^i(x) &= \{y \in X : f_k(y) \leqq f_k(x),\ k \in I \backslash \{i\} \text{ and } g(y) \leqq 0\}, \\ Q^i(x) &= Q(x), \text{ if } p = 1, \\ C(Q(x), x) &= \{d \in X : f_i^-(x; d) \leqq 0,\ i \in I, \text{ and } g_j^-(x; d) \leqq 0,\ j \in J(x)\}, \\ C(Q^i(x), x) &= \{d \in X : f_k^-(x; d) \leqq 0,\ k \in I \backslash \{i\}, \text{ and } g_j^-(x; d) \leqq 0,\ j \in J(x)\}. \end{aligned}$$

The following result shows that the relationship between the tangent cones $T(Q^i(x), x)$ and the set $C(Q(x), x)$.

Proposition 3.1. *Let $x \in S$. If $f_i^-(x; \cdot)$ and $g_j^-(x; \cdot)$, with $i \in I$ and $j \in J(x)$, are convex functions on X, then,*

$$\bigcap_{i \in I} \text{clco} T(Q^i(x), x) \subseteq C(Q(x), x).$$

Proof. First, we shall show that $C(Q^i(x), x)$ is closed and convex for all $i \in I$.

Let $\alpha \geqq 0$ and $d \in C(Q^i(x), x)$. Then, $\alpha d \in C(Q^i(x), x)$ as $f_k^-(x; \alpha d) = \alpha f_k^-(x; d) \leqq 0$, $k \in I \backslash \{i\}$ and $g_j^-(x; \alpha d) \leqq 0$, $j \in J(x)$. Now, let $d_1, d_2 \in C(Q^i(x), x)$, and let $\lambda \in [0, 1]$. Since $f_i^-(x; \cdot)$ and $g_j^-(x; \cdot)$ are convex functions, we have, for $i \in I$,

$$f_i^-(x; \lambda d_1 + (1 - \lambda) d_2) \leqq \lambda f_i^-(x; d_1) + (1 - \lambda) f_i^-(x; d_2) \leqq 0$$

and similarly, for $j \in J(x)$,

$$g_j^-(x; \lambda d_1 + (1-\lambda)d_2) \leqq 0.$$

Thus, $C(Q^i(x), x)$ is convex for all $i \in I$.

Since f_i and g_i are locally Lipschitz and $f_i^-(x;\cdot)$ and $g_j^-(x;\cdot)$, with $i \in I$ and $j \in J(x)$, are convex, we know that $f_i^-(x;\cdot)$ and $g_j^-(x;\cdot)$ are continuous. It follows that we can easily prove $C(Q^i(x), x)$ is closed for all $i \in I$.

By the definitions of $C(Q(x), x)$ and $C(Q^i(x), x)$,

$$C(Q(x), x) = \bigcap_{i \in I} C(Q^i(x), x).$$

Therefore,

$$C(Q(x), x) = \bigcap_{i \in I} \text{clco}C(Q^i(x), x).$$

Second, we shall show that, for each $i \in I$,

$$T(Q^i(x), x) \subseteq C(Q^i(x), x).$$

The rest of the proof is similar to those of Proposition 3.1 in [12] and so we omit it. This completes the proof. ∎

Remark 3.1. Note that a sublinear function is a convex function, but the converse is not true. For example, the function $f : [-1, 1] \to R$ defined by $f(x) = \sqrt{1 - x^2}$ is a convex function but not sublinear on $[-1, 1]$. Therefore, Proposition 3.1 improve Proposition 3.1 of Li and Zhang [12].

Remark 3.2. If $f^-(x; d) = f^+(x; d)$ for all $d \in X$, then Proposition 3.1 improve Proposition 3.1 of Preda and Chitescu [19] because the condition that f is quasiconvex at x is removed.

In order to obtain the necessary conditions that a feasible solution to problem (MP) be an efficient solution, we need the following constraint qualification and Lemma.

Definition 3.1. [12]. We shall say the problem (MP) satisfies the generalized Guignard constraint qualification (GGCQ) at $x \in S$ if

$$C(Q(x), x) \subseteq \bigcap_{i \in I} \text{clco}T(Q^i(x), x).$$

Lemma 3.1. *[12] Let x be an efficient solution to problem (MP). If $f_{i_0}^+(x;\cdot)$ is concave for some $i_0 \in I$, then*

$$\{d \in X : f_{i_0}^+(x;\cdot) < 0\} \cap \bigcap_{i \in I} \text{clco}T(Q^i(x), x) = \emptyset.$$

Now, we establish the following stronger Kuhn-Tucker type necessary optimality conditions.

Theorem 3.1. *Let $x_0 \in S$ be an efficient solution to problem (MP). Suppose that*

(i) *constraint qualification (GGCQ) holds at x_0;*

(ii) *f_i and g_i admit respectively the upper semiregular convexificators $\partial^* f_i(x_0)$ and upper convexificators $\partial^* g_j(x_0)$, with $i \in I$ and $j \in J$;*

(iii) *$f_{i_0}^+(x_0; \cdot)$ is concave on X for some $i_0 \in I$;*

(iv) *$f_i^+(x_0; \cdot)$ is convex on X for all $i \in I$;*

(v) *$g_j^-(x_0; \cdot)$ is convex on X for all $j \in J(x_0)$;*

(vi) *there exists $d \in X$ such that $g_j^-(x_0; d) < 0$ for all $j \in J(x_0)$.*

Then, there exist real numbers $\alpha = (\alpha_1, \cdots, \alpha_p) \in R_+^p$ with $\alpha \neq 0$ and $\beta = (\beta_1, \cdots, \beta_m) \in R_+^m$ such that

$$0 \in \mathrm{cl}(\sum_{i \in I} \alpha_i \mathrm{co} \partial^* f_i(x_0) + \sum_{j \in J} \beta_j \mathrm{co} \partial^* g_j(x_0)),$$

$$\beta_j g_j(x_0) = 0, j = 1, 2, \cdots, m.$$

Proof. Since $x_0 \in S$ is an efficient solution to problem (MP), we have the following system

$$f_{i_0}^+(x_0; d) < 0,$$
$$f_k^+(x_0; d) \leqq 0, \ k \in I \backslash \{i_0\},$$
$$g_j^-(x_0; d) \leqq 0, \ j \in J(x_0),$$

has no solution $d \in X$. In fact, suppose by contradiction that there exists $v \in X$ which solves the system. This implies that the system

$$f_{i_0}^-(x_0; d) < 0,$$
$$f_k^-(x_0; d) \leqq 0, \ k \in I \backslash \{i_0\},$$
$$g_j^-(x_0; d) \leqq 0, \ j \in J(x_0),$$

has a solution $v \in X$. It follows that $v \in C(Q(x_0), x_0)$. This fact together with condition (i) yields

$$v \in \{d \in X : f_{i_0}^+(x_0; d) < 0\} \cap \bigcap_{i \in I} \mathrm{clco} T(Q^i(x_0), x_0),$$

which contradicts the result of Lemma 3.1.

By conditions (iv) and (v) and the Farkas theorem [15] in the convex case, there exist real numbers $\alpha = (\alpha_1, \cdots, \alpha_p) \in R^p_+$ and $\beta_j \geqq 0$ with $j \in J(x_0)$, not all zero, such that

$$\sum_{i\in I} \alpha_i f_i^+(x_0; d) + \sum_{j\in J(x_0)} \beta_j g_j^-(x_0; d) \geqq 0, \text{ for all } d \in X.$$

Now we prove that $\alpha \neq 0$. Indeed, if $\alpha = 0$, then there exists $j \in J(x_0)$ such that $\beta_j > 0$ and

$$\sum_{j\in J(x_0)} \beta_j g_j^-(x_0; d) \geqq 0, \text{ for all } d \in X. \tag{1}$$

By condition (vi), there exists $d_0 \in X$ such that

$$\sum_{j\in J(x_0)} \beta_j g_j^-(x_0; d_0) < 0,$$

which contradicts (1). Therefore, $\alpha \neq 0$.

Using condition (b), one has

$$\sum_{i\in I}(\alpha_i \sup_{x^*\in\partial^* f_i(x_0)} \langle x^*, d\rangle) + \sum_{j\in J(x_0)} (\beta_j \sup_{y^*\in\partial^* g_j(x_0)} \langle y^*, d\rangle) \geqq 0, \text{ for all } d \in X.$$

Denote by

$$C(x_0) = \sum_{i\in I} \alpha_i \partial^* f_i(x_0) + \sum_{j\in J(x_0)} \beta_j \partial^* g_j(x_0).$$

It follows that

$$\begin{aligned} &\sup_{z^*\in C(x_0)} \langle z^*, d\rangle \\ &= \sum_{i\in I}(\alpha_i \sup_{x^*\in\partial^* f_i(x_0)} \langle x^*, d\rangle) + \sum_{j\in J(x_0)} (\beta_j \sup_{y^*\in\partial^* g_j(x_0)} \langle y^*, d\rangle) \\ &\geqq 0, \text{ for all } d \in X. \end{aligned}$$

By the usual calculus of support functions,

$$0 \in \text{clco}(\sum_{i\in I} \alpha_i \partial^* f_i(x_0) + \sum_{j\in J(x_0)} \beta_j \partial^* g_j(x_0)),$$

which implies

$$0 \in \text{cl}(\sum_{i\in I} \alpha_i \text{co}\partial^* f_i(x_0) + \sum_{j\in J(x_0)} \beta_j \text{co}\partial^* g_j(x_0)).$$

By setting $\beta_j = 0$, $j \notin J(x_0)$, the result is derived. This completes the proof. ∎

Remark 3.3. In [12], Li and Zhang gave a sufficient condition guaranteeing the stronger Kuhn-Tucker type condition holds. The assumption that f_i is directional differentiable is required in [12]. However, Theorem 3.1 does not require this assumption.

Remark 3.4. We observe also that condition (v) in Theorem 3.1 is weaker than that condition (c) in Theorem 3.1 of Li and Zhang [12] as the reason mentioned in Remark 3.1. Therefore, Theorem 3.1 improve and generalize Theorem 3.1 of Li and Zhang [12].

The following example illustrates that the condition of Theorem 3.1 holds, whereas the condition of Theorem 3.1 in [12] does not hold.

Example 3.1. Let $\mathbb{Q}$ denote the set of rationals. We consider the following multiobjective programming problem:

$$\begin{aligned} \text{(MP) Min} \quad & (f_1(x), f_2(x)), \\ \text{s.t.} \quad & g(x) \leqq 0, \end{aligned}$$

where $f_i : R \to R$, $i = 1, 2$, and $g : R \to R$ are given by

$$f_1(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}; \end{cases}$$
$$f_2(x) = -x,$$
$$g(x) = x.$$

Obviously, $x_0 = 0$ is an efficient solution to problem (MP). By simple calculations, we have

$$f_1^+(0; d) = \max\{0, \frac{1}{2}d\},$$
$$f_1^-(0; d) = \min\{0, \frac{1}{2}d\},$$
$$f_2^+(0; d) = f_2^-(0; d) = -d,$$
$$g^+(0; d) = g^-(0; d) = d.$$

It is easy to check that the conditions (iii)-(vi) of Theorem 3.1 are satisfied. For $x_0 = 0$, one has

$$Q^1(0) = \{x \in R : x = 0\},$$
$$Q^2(0) = \{x \in R : x \leq 0\},$$
$$C(Q(0), 0) = 0,$$
$$T(Q^1(0), 0) = 0,$$
$$T(Q^2(0), 0) = \{x \in R : x \leq 0\}.$$

Thus, constraint qualification (GGCQ) holds at $x_0 = 0$.

Consider the set $\partial^* f_1(0) = \{0, 1\}$, $\partial^* f_2(0) = \{-1, 1\}$ and $\partial^* g(0) = \{1\}$. Observe that

$$\sup_{x^* \in \partial^* f_1(0)} \langle x^*, d\rangle = \begin{cases} d & \text{if } d \geq 0; \\ 0 & \text{if } d < 0. \end{cases}$$

It is clear that $\partial^* f_1(0) = \{0, 1\}$ is an upper semiregular convexificator of f_1 at $x_0 = 0$. Similarly, we can verify that $\partial^* f_2(0) = \{-1, 1\}$ and $\partial^* g(0) = \{1\}$ are an upper semiregular convexificator of f_2 at $x_0 = 0$ and an upper convexificator of g at $x_0 = 0$, respectively. This implies that condition (b) of Theorem 3.1 is satisfied.

Therefore, all the conditions of Theorem 3.1 are satisfied. Then, by setting

$$\alpha_1 = \alpha_2 = 1, \quad \beta = 0,$$

we have

$$0 \in \text{cl}(\sum_{i=1}^{2} \alpha_i \text{co}\partial^* f_i(x_0) + \beta \text{co}\partial^* g(x_0)) = [-1, 2].$$

It is easy to see that, for any $d \neq 0$, $f_1^+(0; d) \neq f_1^-(0; d)$, i.e., f_1 is not directional differentiable. Thus, Theorem 3.1 of Li and Zhang [12] can not be used.

From the proof of Theorem 3.1, we can easily obtain the following result.

Theorem 3.2. *Let $x_0 \in X$ be an efficient solution to problem (MP). Suppose that*

(*i*) *constraint qualification (GGCQ) holds at x_0;*

(*ii*) *$f_{i_0}^+(x_0; \cdot)$ is concave on X for some $i_0 \in I$;*

(*iii*) *$f_i^+(x_0; \cdot)$ is convex for all $i \in I$;*

(*iv*) *$g_j^-(x_0; \cdot)$ is convex on X for all $j \in J(x_0)$;*

(*v*) *there exists $d \in X$ such that $g_j^-(x_0; d) < 0$ for all $j \in J(x_0)$.*

Then, there exist real numbers $\alpha = (\alpha_1, \cdots, \alpha_p) \in R_+^p$ with $\alpha \neq 0$ and $\beta = (\beta_1, \cdots, \beta_m) \in R_+^m$ such that

$$\sum_{i \in I} \alpha_i f_i^+(x_0; d) + \sum_{j \in J} \beta_j g_j^-(x_0; d) \geqq 0, \text{ for all } d \in X,$$
$$\beta_j g_j(x_0) = 0, j = 1, 2, \cdots, m.$$

In the following theorem, we give the sufficient condition for a feasible solution to be an efficient solution to problem (MP).

Theorem 3.3. *Let $x_0 \in S$ be a feasible solution to problem (MP). Assume that functions f and g are strong pseudoconvex and quasiconvex at x_0, respectively. If there exist real numbers $\alpha_i > 0$ and $\beta_j \geqq 0$ with $i \in I$ and $j \in J$ such that*

$$\sum_{i\in I}\alpha_i f_i^+(x_0;d)+\sum_{j\in J}\beta_j g_j^-(x_0;d)\geqq 0,\ \textit{for all } d\in X, \tag{2}$$

$$\beta_j g_j(x_0)=0, j=1,2,\cdots,m. \tag{3}$$

Then x_0 is an efficient solution for problem (MP).

Proof. Suppose by contradiction that x_0 is not an efficient solution for problem (MP). Then there exists $y\in S$ such that

$$f(y)\leq f(x_0), \tag{4}$$

$$g_{J(x_0)}(y)\leqq 0. \tag{5}$$

Since f and g are strong pseudoconvex and quasiconvex respectively at x_0, (4) and (5) yield

$$f^+(x_0;y-x_0)\leq 0 \tag{6}$$

and

$$g^-_{J(x_0)}(x_0;y-x_0)\leqq 0. \tag{7}$$

Let $d=y-x_0$. Note that $\alpha_i>0$ and $\beta_j\geqq 0$ with $i\in I$ and $j\in J(x_0)$. This fact combining with (6) and (7) yields

$$\sum_{i\in I}\alpha_i f_i^+(x_0;d)+\sum_{j\in J(x_0)}\beta_j g_j^-(x_0;d)<0.$$

By (3), we obtain $\beta_j=0$ for $j\notin J(x_0)$. It follows that

$$\sum_{i\in I}\alpha_i f_i^+(x_0;d)+\sum_{j\in J}\beta_j g_j^-(x_0;d)<0,$$

which contradicts to (2). This completes the proof. ∎

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