TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 3, pp. 895-907, June 2014
DOI: 10.11650/tjm.18.2014.3724
This paper is available online at http://journal.taiwanmathsoc.org.tw

SINGULAR VALUE INEQUALITIES OF LEWENT TYPE

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Abstract. Let A_i be strictly contractive matrices and let λ_i be nonnegative real numbers with $\sum_{i=1}^{m} \lambda_i = 1,\ \ i = 1, \ldots, m$. We prove that

$$s\left(\frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i}\right) \prec_{\mathrm{wlog}} \prod_{i=1}^{m} s\left(\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}\right),$$

which generalizes a Lewent type determinantal inequality due to Lin [M. Lin, A Lewent type determinantal inequality, Taiwanese J. Math. 17(2013), 1303-1309]. On the other hand, we also prove

$$s\left(\frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i}\right) \prec_{\mathrm{wlog}} \sum_{i=1}^{m} \lambda_i s\left(\frac{I+|A_i|}{I-|A_i|}\right).$$

Here "$\prec_{\mathrm{wlog}}$" stands for weakly log-majorization. In addition, some other related inequalities are also obtained.

1. INTRODUCTION

Let M_n denote the vector space of all complex $n \times n$ matrices and let H_n be the set of all Hermitian matrices of order n. We always denote the eigenvalues of $A \in H_n$ in decreasing order by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and denote $\lambda(A) =$

Received August 31, 2013, accepted November 22, 2013.
Communicated by Wen-Wei Lin.
2010 *Mathematics Subject Classification*: 47A30, 47B15, 15A45, 15A60.
Key words and phrases: Lewent inequality, Weak log-majorization, Singular values, Strict contractions.
The work was supported by the NSFC grant 11371145 and the Science Foundation for the Excellent Youth Scholars of Anhui Education Ministry (No. 2011SQRL070).

$(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$. The *singular values* of $A \in M_n$ are defined to be the nonnegative square roots of the eigenvalues of A^*A. The absolute value of $A \in M_n$ is defined and denoted by $|A| = (A^*A)^{\frac{1}{2}}$. Thus the singular values of A are the eigenvalues of $|A|$. We always denote the singular values of $A \in M_n$ by $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ and denote $s(A) = (s_1(A), s_2(A), \ldots, s_n(A))$. Denote by $\|\cdot\|_\infty$ the spectral norm. For $A \in M_n$, $\|A\|_\infty = s_1(A)$. For $A, B \in H_n$, we use the notation $A \leq B$ or $B \geq A$ to mean that $B - A$ is positive semidefinite. Clearly, " $\leq$ " and " $\geq$ " define two partial orders on H_n, each of which is called *Löwner partial order*. In particular, $B \geq 0$ means that B is positive semidefinite. Recall that a complex matrix C is called a *contraction* if $\|C\|_\infty \leq 1$, or equivalently $C^*C \leq I$. Moreover, C is called a *strict contraction* if $\|C\|_\infty < 1$. Given a real vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we rearrange its components as $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$.

Definition 1. For $x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]},\ \ k = 1, 2, \ldots, n,$$

then we say that x is *weakly majorized* by y and denote $x \prec_{\mathrm{w}} y$. If $x \prec_{\mathrm{w}} y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then we say that x is *majorized* by y and denote $x \prec y$.

Definition 2. Let the components of $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be nonnegative. If

$$\prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]},\ \ k = 1, 2, \ldots, n,$$

then we say that x is *weakly log-majorized* by y and denote $x \prec_{\mathrm{wlog}} y$. If $x \prec_{\mathrm{wlog}} y$ and $\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i$, then we say that x is *log-majorized* by y and denote $x \prec_{\log} y$.

In 1908, by using the power-series method Lewent [7] proved the following numerical inequality:

$$\frac{1 + \sum_{i=1}^{m} \lambda_i x_i}{1 - \sum_{i=1}^{m} \lambda_i x_i} \leq \prod_{i=1}^{m} \left(\frac{1 + x_i}{1 - x_i} \right)^{\lambda_i} \tag{1}$$

where $x_i \in [0, 1)$ and the nonnegative real numbers λ_i, $i = 1, \ldots, m$, are (scalar) weights with $\sum_{i=1}^{m} \lambda_i = 1$.

Recently, Lin [5] proved an interesting analogue of (1) for the determinant of strict contractions: Let $A_i,\ i = 1, \ldots, m$, be strictly contractive matrices. Then

$$\left| \det \left(\frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i} \right) \right| \leq \prod_{i=1}^{m} \det \left(\frac{I + |A_i|}{I - |A_i|} \right)^{\lambda_i}, \tag{2}$$

where each $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$.

Here $\frac{I+A}{I-A}$ is understood as $(I+A)(I-A)^{-1}$, which is also equal to $(I-A)^{-1}(I+A)$.

For simplicity, we state our results for matrices, but these results still hold for trace class operators on a complex separable Hilbert space via limiting arguments.

For $B_i \in M_n,\ i = 1, \ldots, m$, we always denote $\prod_{i=1}^{m} s(B_i) := \left(\prod_{i=1}^{m} s_1(B_i), \cdots, \prod_{i=1}^{m} s_n(B_i) \right)$. In this paper, we shall prove the following inequalities: Let $A_i \in M_n,\ i = 1, \ldots, m$, be strictly contractive matrices and let λ_i be nonnegative real numbers with $\sum_{i=1}^{m} \lambda_i = 1,\ i = 1, \ldots, m$.. Then

$$s \left(\frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i} \right) \prec_{\text{wlog}} \prod_{i=1}^{m} s \left(\left(\frac{I + |A_i|}{I - |A_i|} \right)^{\lambda_i} \right),$$

which generalizes (2). Meanwhile, we also prove

$$s \left(\frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i} \right) \prec_{\text{wlog}} \sum_{i=1}^{m} \lambda_i s \left(\frac{I + |A_i|}{I - |A_i|} \right).$$

Some other related results are also obtained.

2. Results and Proof

We start with several lemmas which will be used in our proof.

The following well known result is due to Ky Fan [3, 10].

Lemma 1. *Let* $A, B \in H_n$. *Then* $\lambda(A+B) \prec \lambda(A) + \lambda(B)$.

Denote by $H_n(\Omega)$ the set of $n \times n$ Hermitian matrices with the spectra in an interval Ω. We have the following

Lemma 2. [1] *Let* f *be a convex function on* Ω. *Then*

$$\lambda(f(\alpha A + (1-\alpha)B)) \prec_{\mathrm{w}} \lambda(\alpha f(A) + (1-\alpha) f(B))$$

for all $A, B \in H_n(\Omega)$ *and* $0 \leq \alpha \leq 1$.

Remark. Using an idea similar to that in [1], we can generalize Lemma 2 to m matrices:

$$\lambda(f(\alpha_1 A_1 + \cdots + \alpha_m A_m)) \prec_{\mathrm{w}} \lambda(\alpha_1 f(A_1) + \cdots + \alpha_m f(A_m)) \tag{3}$$

for $A_1, \ldots, A_m \in H_n(\Omega)$ and $\alpha_1, \ldots, \alpha_m \in [0,1]$ with $\sum_{i=1}^{m} \alpha_i = 1$.

Lemma 3. [3, 10]. *Let* $g(t)$ *be an increasing convex function. If* $x \prec_{\mathrm{w}} y$ *with* $x, y \in \mathbb{R}^n$, *then*

$$(g(x_1), \ldots, g(x_n)) \prec_{\mathrm{w}} (g(y_1), \ldots, g(y_n)).$$

Let f be a real valued function defined on an interval Ω. If f is positive and

$$f(\alpha s + (1-\alpha)t) \leq f(s)^{\alpha} f(t)^{1-\alpha},$$

for all $0 \leq \alpha \leq 1$, then f is called log-convex. The reader is referred to [8] for general properties of convex and log-convex functions.

Lemma 4. *Let* A_i, $i = 1, \ldots, m$, *be strictly contractive matrices. If* A_i, $i = 1, \ldots, m$, *are positive semidefinite, then*

$$s\left(\frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i}\right) \prec_{\mathrm{wlog}} \prod_{i=1}^{m} s\left(\left(\frac{I + A_i}{I - A_i}\right)^{\lambda_i}\right), \tag{4}$$

where each $\lambda_i \geq 0$ *and* $\sum_{i=1}^{m} \lambda_i = 1$.

Proof. First, we will show that $f(t) = \frac{1+t}{1-t}$ is log-convex on $[0,1)$. It is clear that $f(t)$ is positive on $[0,1)$. Let

$$g(t) := \log f(t).$$

It is equivalent to showing that $g(t)$ is convex on $[0, 1)$. Since $g(t)$ is continuous, we only need show $g^{''}(t) \geq 0$, for all $t \in [0, 1)$. A routine calculation shows that

$$g^{''}(t) = \frac{4t}{(1+t)^2(1-t)^2} \geq 0,$$

for all $t \in [0, 1)$. This shows that

$$f(t) = \frac{1\ +\ t}{1\ -\ t}$$

is log-convex on $[0, 1)$. Since the spectra of $\sum_{i=1}^{m}\lambda_i A_i$ and A_i are contained in $[0, 1)$, $i = 1, \ldots, m$, it follows that each A_i and $\sum_{i=1}^{m}\lambda_i A_i$ belong to $H_n([0, 1))$. By the spectral mapping theorem, the spectra of $f\left(\sum_{i=1}^{m}\lambda_i A_i\right)$ and $f(A_i)$ are contained in $[1, +\infty)$. For $\lambda_i \geq 0$ with $\sum_{i=1}^{m}\lambda_i = 1$, we have

$$\begin{aligned}
&\log\,\lambda\left(f\left(\sum_{i=1}^{m}\lambda_i A_i\right)\right) && \\
=\ &\lambda\left(\log\,f\left(\sum_{i=1}^{m}\lambda_i A_i\right)\right) && \text{by the Spectral Mapping Theorem}\\
\prec_{\mathrm{w}}\ &\lambda\left(\sum_{i=1}^{m}\lambda_i\,\log\,f\,(A_i)\right) && \text{by Lemma 2}\\
\prec\ &\sum_{i=1}^{m}\lambda\,(\lambda_i\,\log\,f\,(A_i)) && \text{by Lemma 1}\\
=\ &\sum_{i=1}^{m}\log\,\lambda(f(A_i)^{\lambda_i}). && \text{by the Spectral Mapping Theorem}
\end{aligned}$$

Then

$$\log\,\lambda\left(f\left(\sum_{i=1}^{m}\lambda_i A_i\right)\right) \prec_{\mathrm{w}} \sum_{i=1}^{m}\log\,\lambda(f(A_i)^{\lambda_i}).$$

Applying Lemma 3 to the above weak-majorization with the increasing convex function e^t, we obtain

$$\lambda\left(f\left(\sum_{i=1}^{m}\lambda_i A_i\right)\right) \prec_{\mathrm{wlog}} \prod_{i=1}^{m}\lambda(f(A_i)^{\lambda_i}).$$

Clearly, each $f(A_i)$ and $f\left(\sum_{i=1}^{m}\lambda_i A_i\right)$ are positive definite. Note that for positive definite matrices, singular values and eigenvalues are the same. Thus the inequality (4) holds. This completes the proof. ■

Remark. In [5], Lin proved the following result: Let $A_i,\ i=1,\dots,m$, be strictly contractive matrices. If $A_i,\ i=1,\dots,m$, are positive semidefinite, then

$$\det\left(\frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i}\right)\le\prod_{i=1}^{m}\det\left(\frac{I+A_i}{I-A_i}\right)^{\lambda_i},\tag{5}$$

where each $\lambda_i\ge 0$ and $\sum_{i=1}^{m}\lambda_i=1$. The author pointed out this result was also an application of Theorem 3.3 in [2]. Note that (5) is the special case $k=n$ of (4) in Lemma 4.

Let $\Phi: M_n\to M_n$ be a map. We say that Φ is 2-positive if whenever the 2×2 operator matrix $\begin{pmatrix} A & B\\ B^* & C\end{pmatrix}\ge 0$ then $\begin{pmatrix}\Phi(A) & \Phi(B)\\ \Phi(B^*) & \Phi(C)\end{pmatrix}\ge 0$. It is clear that any Liebian function is 2-positive [9].

Lemma 5. [5] $\Phi(t)=\frac{1+t}{1-t}$ *is* 2-positive *over the strictly contractive matrices.*

Lemma 6. [4, p.208] *The partitioned block matrix* $\begin{pmatrix} A & B\\ B^* & C\end{pmatrix}$ *is positive semidefinite if and only if both* A *and* C *are positive semidefinite and there exists a contraction* W *such that* $B=A^{\frac12}WC^{\frac12}$. *Moreover, we have*

$$s(B)\prec_{wlog} s(A^{\frac12})s(B^{\frac12}).$$

Theorem 7. *Let* $A_i\in M_n,\ i=1,\dots,m$, *be strictly contractive matrices. Then*

$$s\left(\frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i}\right)\prec_{\text{wlog}}\prod_{i=1}^{m}s\left(\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}\right),\tag{6}$$

where each $\lambda_i\ge 0$ *and* $\sum_{i=1}^{m}\lambda_i=1$.

Proof. Note that $A_i=|A_i^*|^{\frac12}U_i|A_i|,\ i=1,\dots,m$, with unitary U_i. By Lemma 6, we have $\begin{pmatrix}|A_i^*| & A_i\\ A_i^* & |A_i|\end{pmatrix}\ge 0$, for any i. For each $\lambda_i\ge 0$ with $\sum_{i=1}^{m}\lambda_i=1$, then we have

$$\begin{pmatrix} \sum_{i=1}^{m}\lambda_i|A_i^*| & \sum_{i=1}^{m}\lambda_i A_i \\ \sum_{i=1}^{m}\lambda_i A_i^* & \sum_{i=1}^{m}\lambda_i|A_i| \end{pmatrix} = \sum_{i=1}^{m}\lambda_i \begin{pmatrix} |A_i^*| & A_i \\ A_i^* & |A_i| \end{pmatrix} \geq 0.$$

Applying Lemma 5 to the above partitioned block matrix, we obtain the following 2×2 block matrix

$$\begin{pmatrix} \frac{I+\sum_{i=1}^{m}\lambda_i|A_i^*|}{I-\sum_{i=1}^{m}\lambda_i|A_i^*|} & \frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i} \\ \frac{I+\sum_{i=1}^{m}\lambda_i A_i^*}{I-\sum_{i=1}^{m}\lambda_i A_i^*} & \frac{I+\sum_{i=1}^{m}\lambda_i|A_i|}{I-\sum_{i=1}^{m}\lambda_i|A_i|} \end{pmatrix} \geq 0.$$

By Lemma 6, we have

$$s\left(\frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i}\right) \prec_{\text{wlog}} s\left(\left(\frac{I+\sum_{i=1}^{m}\lambda_i|A_i^*|}{I-\sum_{i=1}^{m}\lambda_i|A_i^*|}\right)^{\frac{1}{2}}\right) s\left(\left(\frac{I+\sum_{i=1}^{m}\lambda_i|A_i|}{I-\sum_{i=1}^{m}\lambda_i|A_i|}\right)^{\frac{1}{2}}\right).$$

Let $x\in\mathbb{R}^n$ be an vector with nonnegative components and denote $x^{\frac{1}{2}} := (x_1^{\frac{1}{2}},\ldots,x_n^{\frac{1}{2}})$. Then we have

$$\begin{aligned} & s\left(\frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i}\right) \\ \prec_{\text{wlog}} \; & s\left(\left(\frac{I+\sum_{i=1}^{m}\lambda_i|A_i^*|}{I-\sum_{i=1}^{m}\lambda_i|A_i^*|}\right)^{\frac{1}{2}}\right) s\left(\left(\frac{I+\sum_{i=1}^{m}\lambda_i|A_i|}{I-\sum_{i=1}^{m}\lambda_i|A_i|}\right)^{\frac{1}{2}}\right) \end{aligned}$$

$$
\begin{aligned}
&= s\left(\left(\frac{I+\sum_{i=1}^{m}\lambda_i|A_i^*|}{I-\sum_{i=1}^{m}\lambda_i|A_i^*|}\right)\right)^{\frac{1}{2}} s\left(\left(\frac{I+\sum_{i=1}^{m}\lambda_i|A_i|}{I-\sum_{i=1}^{m}\lambda_i|A_i|}\right)\right)^{\frac{1}{2}} \\
&\prec_{\mathrm{wlog}} \left[\prod_{i=1}^{m} s\left(\left(\frac{I+|A_i^*|}{I-|A_i^*|}\right)^{\lambda_i}\right)\right]^{\frac{1}{2}} \left[\prod_{i=1}^{m} s\left(\left(\frac{I+|A_i^*|}{I-|A_i^*|}\right)^{\lambda_i}\right)\right]^{\frac{1}{2}} \qquad \text{by Lemma 4} \\
&= \left[\prod_{i=1}^{m} s\left(\left(\frac{I+|A_i^*|}{I-|A_i^*|}\right)^{\lambda_i}\right) s\left(\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}\right)\right]^{\frac{1}{2}} \\
&= \prod_{i=1}^{m} s\left(\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}\right),
\end{aligned}
$$

where the last equality can be seen as follows. Using the spectral mapping theorem and $\lambda(|A_i|) = \lambda(|A_i^*|) = s(|A_i^*|) = s(|A_i|)$ for any i, we have

$$
s\left(\left(\frac{I+|A_i^*|}{I-|A_i^*|}\right)^{\lambda_i}\right) = \lambda\left(\left(\frac{I+|A_i^*|}{I-|A_i^*|}\right)^{\lambda_i}\right) = \lambda\left(\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}\right) = s\left(\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}\right),
$$

for any i. This completes the proof. ■

The following corollary is the main result [5], which follows by Theorem 7.

Corollary 8. [5]. *Let* $A_i,\ i = 1, \ldots, m,$ *be strictly contractive matrices. Then*

$$
\left|\det\left(\frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i}\right)\right| \le \prod_{i=1}^{m} \det\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i},
$$

where each $\lambda_i \ge 0$ *and* $\sum_{i=1}^{m}\lambda_i = 1$.

Proof. Denote $M = \frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i}$ and denote $M_i = \left(\frac{I+A_i}{I-A_i}\right)^{\lambda_i}$ for $i = 1, \ldots, m$. Suppose the eigenvalues of M is $\lambda_1(M), \ldots, \lambda_n(M)$ with $|\lambda_1(M)| \ge \cdots \ge |\lambda_n(M)|$

and denote $|\lambda(M)| = (|\lambda_1(M)|, \ldots, |\lambda_n(M)|)$. Using Weyl's Theorem [10, p.81] and Theorem 7, we have

$$|\lambda(M)| \prec_{\log} s(M) \prec_{\text{wlog}} \prod_{i=1}^{m} s\left(M_i\right).$$

Note that $M_j,\ j = 1, \ldots, m$, are positive definite. Letting $k = n$, we have

$$|\det\ M| = \prod_{i=1}^{n} |\lambda_i(M)| \leq \prod_{i=1}^{n}\prod_{j=1}^{m} s_i(M_j) = \prod_{j=1}^{m}\prod_{i=1}^{n} s_i(M_j) = \prod_{j=1}^{m} |\det\ M_j| = \prod_{j=1}^{m} \det M_j.$$

This completes the proof. ∎

Setting $k = 1$ in (6) of Theorem 7, we deduce an analogue of (1) for the spectral norm of strictly contractions:

Corollary 9. *Let $A_i,\ i = 1, \ldots, m$, be strictly contractive matrices. Then*

$$\left\| \frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i} \right\|_{\infty} \leq \prod_{i=1}^{m} \left\| \left(\frac{I + |A_i|}{I - |A_i|} \right)^{\lambda_i} \right\|_{\infty} \tag{7}$$

where each $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$.

Next, we derive another weak log-majorization involving contractive matrices and singular values.

Theorem 10. *Let $A_i,\ i = 1, \ldots, m$, be strictly contractive matrices. Then*

$$s\left(\frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i} \right) \prec_{\text{wlog}} \sum_{i=1}^{m} \lambda_i s\left(\frac{I + |A_i|}{I - |A_i|} \right) \tag{8}$$

where each $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$.

Proof. Denote $M = \frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i}$ and denote $M_i = \left(\frac{I + |A_i|}{I - |A_i|} \right)^{\lambda_i}$ for $i = 1, \ldots, m$.

Note that $\prod_{i=1}^{m} s\left(M_i\right) = \left(\prod_{i=1}^{m} s_1\left(M_i\right), \ldots, \prod_{i=1}^{m} s_n\left(M_i\right)\right)$. Let $x_1, \ldots, x_n$ and $\omega_1, \ldots, \omega_n$ be the nonnegative real numbers with $\sum_{i=1}^{n} \omega_i = 1$. Then the weighted arithmetic-geometric mean inequality says that

$$\prod_{i=1}^{n} x_i^{\omega_i} \leq \sum_{i=1}^{n} \omega_i x_i.$$

For each given j, we have

$$\prod_{i=1}^{m} s_j\left(M_i\right) = \prod_{i=1}^{m} s_j\left(\left(\frac{I+|A_i|}{I-|A_i|}\right)\right)^{\lambda_i} \leq \sum_{i=1}^{m} \lambda_i s_j\left(\left(\frac{I+|A_i|}{I-|A_i|}\right)\right), \tag{9}$$

where the first equality holds by the spectral mapping theorem and the last inequality holds by the weighted arithmetic-geometric mean inequality. Combining (9) and Theorem 7, we have

$$s(M) \prec_{\text{wlog}} \left(\sum_{i=1}^{m} \lambda_i s_1\left(\frac{I+|A_i|}{I-|A_i|}\right), \ldots, \sum_{i=1}^{m} \lambda_i s_n\left(\frac{I+|A_i|}{I-|A_i|}\right)\right),$$

i.e.,

$$s\left(\frac{I+\sum_{i=1}^{m} \lambda_i A_i}{I-\sum_{i=1}^{m} \lambda_i A_i}\right) \prec_{\text{wlog}} \sum_{i=1}^{m} \lambda_i s\left(\frac{I+|A_i|}{I-|A_i|}\right).$$

This completes the proof. ∎

Denote by $\mathbb{R}_+^n \downarrow$ the set of vectors in $\mathbb{R}^n$ whose components are nonnegative and are decreasingly ordered.

Lemma 11. [10, p. 74]. *Let $x, y, z \in \mathbb{R}^n$ with their components in decreasing order. If $x \prec_w y$ and $z \in \mathbb{R}_+^n \downarrow$, then*

$$\langle x, z\rangle \leq \langle y, z\rangle, \tag{10}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product.

Corollary 12. *Let $A_i,\ i = 1, \ldots, m$, be strictly contractive matrices. Then*

$$\left\|\frac{I+\sum_{i=1}^{m} \lambda_i A_i}{I-\sum_{i=1}^{m} \lambda_i A_i}\right\| \leq \sum_{i=1}^{m} \lambda_i \left\|\frac{I+|A_i|}{I-|A_i|}\right\| \tag{11}$$

for every unitarily invariant norm, where each $\lambda_i \geq 0$ *and* $\sum_{i=1}^{m} \lambda_i = 1$.

Proof. Let $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n) \in \mathbb{R}_+^n \downarrow$. Define $\|X\|_\alpha := \sum_{j=1}^{n} \alpha_j s_j(X)$ for $X \in M_n$. In other words, $\|X\|_\alpha = \langle s(X), \alpha \rangle$. It is known [10, p.56] that this $\|\cdot\|_\alpha$ is a unitarily invariant norm.

Note that for nonnegative vectors, weak log-majorization implies weak majorization [10, p.67]. By Theorem 10, we have

$$s\left(\frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i}\right) \prec_{\mathrm{w}} \sum_{i=1}^{m} \lambda_i s\left(\frac{I + |A_i|}{I - |A_i|}\right).$$

By Lemma 11, we have

$$\left\langle s\left(\frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i}\right), \alpha \right\rangle \leq \left\langle \sum_{i=1}^{m} \lambda_i s\left(\frac{I + |A_i|}{I - |A_i|}\right), \alpha \right\rangle,$$

i.e.,

$$\left\| \frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i} \right\|_\alpha \leq \sum_{i=1}^{m} \lambda_i \left\| \frac{I + |A_i|}{I - |A_i|} \right\|_\alpha.$$

As α was arbitrarily chosen, the inequality (11) follows from Corollary 3.5.9 in [4, p.206]. This completes the proof. ∎

Remark. Note that the spectral norm is a unitarily invariant norm. Then we have

$$\left\| \frac{I + \sum_{i=1}^{m} \lambda_i A_i}{I - \sum_{i=1}^{m} \lambda_i A_i} \right\|_\infty \leq \sum_{i=1}^{m} \lambda_i \left\| \frac{I + |A_i|}{I - |A_i|} \right\|_\infty \tag{12}$$

where each $\lambda_i \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$.

For $A \in M_n$, we denote by $\operatorname{tr} A$ the trace of A. We have

Corollary 13. *Let* $A_i,\ i = 1, \ldots, m$, *be strictly contractive matrices. Then*

$$\left|\operatorname{tr}\left(\frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i}\right)\right| \le \sum_{i=1}^{m}\lambda_i \operatorname{tr}\left(\frac{I+|A_i|}{I-|A_i|}\right), \tag{13}$$

where each $\lambda_i \ge 0$ *such that* $\sum_{i=1}^{m}\lambda_i = 1$.

Proof. Applying Corollary 12 to the trace norm and using Weyl's theorem, we have the inequality (13). This completes the proof. ■

Remark. By Corollary 13, we have

$$\left|\operatorname{tr}\left(\frac{I+\sum_{i=1}^{m}\lambda_i A_i}{I-\sum_{i=1}^{m}\lambda_i A_i}\right)\right| \le \max_i \left\{\operatorname{tr}\left(\frac{I+|A_i|}{I-|A_i|}\right)\right\}. \tag{14}$$

Acknowledgment

The author is indebted to the anonymous referee for his/her suggestions that improved the paper. The author is also grateful to Dr. Minghua Lin for his helpful discussions on Lemma 4.

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