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A NOTE ON ENNOLA RELATION

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Abstract. Ennola gives an example of a relation among the cyclotomic units which is not a combination of elementary relations. He also proves that twice any relation among the cyclotomic units is a consequence of elementary relations. In the sense of the distribution, the torsion part of the universal even punctured distribution $\left(A_n^0\right)^+$ is a 2-torsion group. In particular, when n has three distinct prime divisors, $\left(A_n^0\right)^+$ has a unique 2-torsion element. The aim of this paper is to find an algorithm to produce the unique 2-torsion element when n has three distinct odd prime divisors.

1. Introduction

For a positive integer n ($n \not\equiv 2 \mod 4$), let $\zeta_n = e^{2\pi i/n}$ be a primitive n^{th} root of 1 in $\mathbb{C}$. For an integer k with $n \nmid k$, put $a_k = \log|1-\zeta_n^k|$, which is (the logarithm of) a cyclotomic number. It is well known that there are two types of relations among the cyclotomic numbers:

$$a_k = a_{n-k} \quad \text{for } n \nmid k \tag{1.1}$$

$$a_{(n/m)k} = \sum_{i=0}^{n/m-1} a_{k+mi} \text{ for } m \mid n \text{ and } m, n \nmid k. \tag{1.2}$$

We call these relations the elementary relations. In [2], Ennola gives a relation for $n = 105$ which is not a combination of elementary relations:

$$a_1 + a_2 + a_{17} + a_{43} + a_{44} + a_{46} - a_3 + a_9 + a_{36} + a_{25} + a_{40} + a_{28} = 0.$$

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We call such a relation an Ennola relation.

Let $\left(A_n^0\right)^+$ be the universal even punctured distribution. Namely, $\left(A_n^0\right)^+$ is the abelian group generated by

$$\left\{g\Big(\frac{x}{n}\Big)\Big|\frac{x}{n}\in\frac{1}{n}\mathbb{Z}/\mathbb{Z},\frac{x}{n}\neq 0\right\}$$

with the relations:

$$g\Big(\frac{-x}{n}\Big)=g\Big(\frac{x}{n}\Big)\ \text{for}\ \frac{x}{n}\neq 0 \tag{1.3}$$

$$g\Big(\frac{x}{m}\Big)=\sum_{i=0}^{n/m-1}g\Big(\frac{x+mi}{n}\Big)\ \text{for}\ m\mid n\ \text{and}\ \frac{x}{n},\frac{x}{m}\neq 0. \tag{1.4}$$

The structure of $\left(A_n^0\right)^+$ is known to be ([4, Theorem 12.18])

$$\left(A_n^0\right)^+\simeq\mathbb{Z}^{\varphi(n)/2+r-1}\oplus(\mathbb{Z}/2\mathbb{Z})^{2^{r-1}-r},$$

where r is the number of distinct prime divisors of n. Moreover, the map $g(x/n)\mapsto a_x$ induces an isomorphism

$$\left(A_n^0\right)^+/(\mathbb{Z}/2\mathbb{Z})^{2^{r-1}-r}\simeq\langle\log|1-\zeta_n^a|\rangle.$$

Thus from the 2-torsion elements of $\left(A_n^0\right)^+$, we can obtain Ennola relations. In particular, $\left(A_n^0\right)^+$ has a unique 2-torsion element when $n=p_1^{e_1}p_2^{e_2}p_3^{e_3}$ has three distinct prime divisors.

The aim of this paper is to find an algorithm to produce an Ennola relation when n has three distinct odd prime divisors. Namely, we will find the 2-torsion element of the universal even punctured distribution. Although there is another algorithm to find Ennola relations ([1]), it seems that our result is more explicit and efficient once the generators of $(\mathbb{Z}/p_i^{e_i}\mathbb{Z})^\times$ are given.

2. Preliminaries and Notations

Let $n=p_1^{e_1}p_2^{e_2}p_3^{e_3}$ be the prime factorization of n which is odd. For each $i=1,2$ and 3, put $q_i=p_i^{e_i}$, $n_i=n/q_i$ and $m_i=\varphi(q_i)/2$, where φ is the Euler-phi function. We have

$$(\mathbb{Z}/n\mathbb{Z})^\times\simeq(\mathbb{Z}/q_1\mathbb{Z})^\times\times(\mathbb{Z}/q_2\mathbb{Z})^\times\times(\mathbb{Z}/q_3\mathbb{Z})^\times.$$

We fix a generator σ_i of the cyclic group $(\mathbb{Z}/q_i\mathbb{Z})^\times$. The unique integer $x \mod n$ satisfying $x\equiv\sigma_i \mod q_i$ and $x\equiv 1 \mod n_i$ is also denoted by σ_i. With these notations, the relations (2.1) and (2.2) below can be obtained from the relations (1.3) and (1.4), where p_i^{-1} is an integer satisfying $p_i^{-1}p_i\equiv 1 \mod n_i$:

$$g\Big(\frac{\sigma_1^{i_1+m_1}\sigma_2^{i_2+m_2}\sigma_3^{i_3+m_3}}{n}\Big)=g\Big(\frac{\sigma_1^{i_1}\sigma_2^{i_2}\sigma_3^{i_3}}{n}\Big) \tag{2.1}$$

$$\sum_{t=0}^{2m_i-1} g\Big(\frac{b\sigma_i^t}{n}\Big) = g\Big(\frac{b}{n_i}\Big) - g\Big(\frac{bp_i^{-1}}{n_i}\Big) \text{ for } \gcd(b,p_i)=1. \tag{2.2}$$

Throughout this paper we assume $\{1,2,3\} = \{\alpha,\beta,\gamma\}$. We define $I_\alpha(\beta)$ and $I'_\alpha(\beta)$ by

$$I_\alpha(\beta) = \text{ the index of } p_\beta^{-1} \text{ for the base } \sigma_\alpha, \text{ i.e., } {\sigma_\alpha}^{I_\alpha(\beta)} \equiv p_\beta^{-1} \mod q_\alpha.$$

$$I'_\alpha(\beta) = \begin{cases} I_\alpha(\beta) & \text{if } 0 \le I_\alpha(\beta) < m_\alpha, \\ I_\alpha(\beta) - m_\alpha & \text{if } m_\alpha \le I_\alpha(\beta) < \varphi(q_\alpha). \end{cases}$$

We also define δ_β^α by

$$\delta_\beta^\alpha = \begin{cases} 1 & \text{if } I_\beta(\alpha) = I'_\beta(\alpha), \\ -1 & \text{if } I_\beta(\alpha) \neq I'_\beta(\alpha). \end{cases}$$

Let

$$\mathcal{L}_\gamma^\alpha = \sum_{t=0}^{I_\gamma(\alpha)-1} \left(g\Big(\frac{\sigma_\gamma^t}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^t p_\beta^{-1}}{q_\gamma}\Big) \right) = \sum_{t=0}^{I_\gamma(\alpha)-1} \left(g\Big(\frac{\sigma_\gamma^t}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^{t+I_\gamma(\beta)}}{q_\gamma}\Big) \right)$$

and

$$\widetilde{\mathcal{L}}_\gamma^\alpha = \sum_{t=I'_\gamma(\alpha)}^{m_\gamma-1} \left(g\Big(\frac{\sigma_\gamma^t}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^t p_\beta^{-1}}{q_\gamma}\Big) \right) = \sum_{t=I'_\gamma(\alpha)}^{m_\gamma-1} \left(g\Big(\frac{\sigma_\gamma^t}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^{t+I_\gamma(\beta)}}{q_\gamma}\Big) \right).$$

In the summation above and for the rest of this paper, $\sum_{i=0}^{-1}(*)$ or $\sum_{i=1}^{0}(*)$ should be understood to be zero. Note that

$$\mathcal{L}_\gamma^\alpha = \sum_{i=0}^{2m_\beta-1} \sum_{j=0}^{I_\gamma(\alpha)-1} g\Big(\frac{\sigma_\beta^i \sigma_\gamma^j}{q_\beta q_\gamma}\Big)$$

and that

$$\mathcal{L}_\gamma^\alpha = \sum_{t=0}^{I'_\gamma(\alpha)-1} \left(g\Big(\frac{\sigma_\gamma^t}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^t p_\beta^{-1}}{q_\gamma}\Big) \right)$$

since

$$\sum_{t=I'_\gamma(\alpha)}^{I_\gamma(\alpha)-1} \left(g\Big(\frac{\sigma_\gamma^t}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^t p_\beta^{-1}}{q_\gamma}\Big) \right) = 0.$$

Lemma 2.1. *For integers α, β and γ, we have*

(i) $\widetilde{\mathcal{L}}_\gamma^\alpha = -\mathcal{L}_\gamma^\alpha$,

(ii) $\mathcal{L}_\gamma^\alpha = \mathcal{L}_\gamma^\beta$.

Proof. (i) It is not hard to check that

$$\sum_{t=0}^{m_\gamma-1}\left[g\Big(\frac{\sigma_\gamma^t}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^t\tau}{q_\gamma}\Big)\right] = 0 \text{ for all } \tau.$$

Thus $\mathcal{L}_\gamma^\alpha + \widetilde{\mathcal{L}}_\gamma^\alpha = 0$ with $\tau = p_\beta^{-1}$.

(ii) We have

$$\begin{aligned}
\mathcal{L}_\gamma^\alpha &= \sum_{t=0}^{I_\gamma(\alpha)-1}\left(g\Big(\frac{\sigma_\gamma^t}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^{t+I_\gamma(\beta)}}{q_\gamma}\Big)\right)\\
&= \sum_{t=0}^{I_\gamma(\alpha)-1}\sum_{s=0}^{I_\gamma(\beta)-1}\left(g\Big(\frac{\sigma_\gamma^{t+s}}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^{t+s+1}}{q_\gamma}\Big)\right)\\
&= \sum_{s=0}^{I_\gamma(\beta)-1}\sum_{t=0}^{I_\gamma(\alpha)-1}\left(g\Big(\frac{\sigma_\gamma^{s+t}}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^{s+t+1}}{q_\gamma}\Big)\right)\\
&= \sum_{s=0}^{I_\gamma(\beta)-1}\left(g\Big(\frac{\sigma_\gamma^s}{q_\gamma}\Big) - g\Big(\frac{\sigma_\gamma^{s+I_\gamma(\alpha)}}{q_\gamma}\Big)\right)\\
&= \mathcal{L}_\gamma^\beta.
\end{aligned}$$

■

3. A 2-Torsion Element in the Universal Even Punctured Distribution

This section is devoted to finding the 2-torsion element in $\left(A_n^0\right)^+$. Put

$$\mathcal{M}_1 = \sum_{i=0}^{2m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1} g\Big(\frac{\sigma_1^i\sigma_2^j\sigma_3^k}{n}\Big),$$

$$\mathcal{M}_2 = -\sum_{i=m_1}^{2m_1-1}\sum_{j=0}^{2m_2-1}\sum_{k=0}^{m_3-1} g\Big(\frac{\sigma_1^i\sigma_2^j\sigma_3^k}{n}\Big),$$

$$\mathcal{M}_3 = \sum_{i=m_1}^{2m_1-1}\sum_{j=m_2}^{2m_2-1}\sum_{k=0}^{2m_3-1} g\Big(\frac{\sigma_1^i\sigma_2^j\sigma_3^k}{n}\Big).$$

We also define $\mathcal{B}_\alpha$ by

$$\mathcal{B}_\alpha = \begin{cases} \mathcal{B}_{\beta\gamma}^{++} & \text{if } \delta_\beta^\alpha = 1, \delta_\gamma^\alpha = 1, \\ \mathcal{B}_{\beta\gamma}^{+-} & \text{if } \delta_\beta^\alpha = 1, \delta_\gamma^\alpha = -1, \\ \mathcal{B}_{\beta\gamma}^{-+} & \text{if } \delta_\beta^\alpha = -1, \delta_\gamma^\alpha = 1, \\ \mathcal{B}_{\beta\gamma}^{--} & \text{if } \delta_\beta^\alpha = -1, \delta_\gamma^\alpha = -1, \end{cases}$$

where

$$\mathcal{B}_{\beta\gamma}^{++} = \sum_{s=0}^{I'_\beta(\alpha)-1} \sum_{t=0}^{I'_\gamma(\alpha)+m_\gamma-1} g\Big(\frac{\sigma_\beta^s\sigma_\gamma^t}{n_\alpha}\Big) + \sum_{s=I'_\beta(\alpha)}^{m_\beta-1} \sum_{t=0}^{I'_\gamma(\alpha)-1} g\Big(\frac{\sigma_\beta^s\sigma_\gamma^t}{n_\alpha}\Big),$$

$$\mathcal{B}_{\beta\gamma}^{-+} = \sum_{s=0}^{I'_\beta(\alpha)-1} \sum_{t=0}^{I'_\gamma(\alpha)-1} g\Big(\frac{\sigma_\beta^s\sigma_\gamma^t}{n_\alpha}\Big) + \sum_{s=I'_\beta(\alpha)}^{m_\beta-1} \sum_{t=0}^{I'_\gamma(\alpha)+m_\gamma-1} g\Big(\frac{\sigma_\beta^s\sigma_\gamma^t}{n_\alpha}\Big),$$

$$\mathcal{B}_{\beta\gamma}^{+-} = \sum_{s=0}^{I'_\beta(\alpha)-1} \left[\sum_{t=0}^{m_\gamma-1} g\Big(\frac{\sigma_\beta^s\sigma_\gamma^t}{n_\alpha}\Big) + \sum_{t=m_\gamma+I'_\gamma(\alpha)}^{2m_\gamma-1} g\Big(\frac{\sigma_\beta^s\sigma_\gamma^t}{n_\alpha}\Big)\right] + \sum_{s=I'_\beta(\alpha)}^{m_\beta-1} \sum_{t=I'_\gamma(\alpha)}^{m_\gamma-1} g\Big(\frac{\sigma_\beta^s\sigma_\gamma^t}{n_\alpha}\Big),$$

$$\mathcal{B}_{\beta\gamma}^{--} = \sum_{s=0}^{I'_\beta(\alpha)-1} \sum_{t=I'_\gamma(\alpha)}^{m_\gamma-1} g\Big(\frac{\sigma_\beta^s\sigma_\gamma^t}{n_\alpha}\Big) + \sum_{s=I'_\beta(\alpha)}^{m_\beta-1} \left[\sum_{t=0}^{m_\gamma-1} g\Big(\frac{\sigma_\beta^s\sigma_\gamma^t}{n_\alpha}\Big) + \sum_{t=m_\gamma+I'_\gamma(\alpha)}^{2m_\gamma-1} g\Big(\frac{\sigma_\beta^s\sigma_\gamma^t}{n_\alpha}\Big)\right]$$

Lemma 3.1. *For integers α, β and γ, we have*

$$\mathcal{M}_\alpha + \delta_\beta^\alpha \mathcal{L}_\beta^\alpha + \delta_\gamma^\alpha \mathcal{L}_\gamma^\alpha = 2\mathcal{B}_\alpha.$$

Proof. First, we consider the case when $\alpha = 1$. Note that

$$\begin{aligned} \mathcal{M}_1 &= \sum_{i=0}^{2m_1-1} \sum_{j=0}^{m_2-1} \sum_{k=0}^{m_3-1} g\Big(\frac{\sigma_1^i\sigma_2^j\sigma_3^k}{n}\Big) \\ &= \sum_{j=0}^{m_2-1} \sum_{k=0}^{m_3-1} \left(g\Big(\frac{\sigma_2^j\sigma_3^k}{n_1}\Big) - g\Big(\frac{\sigma_2^j\sigma_3^k p_1^{-1}}{n_1}\Big)\right). \end{aligned}$$

Suppose that $\delta_2^1 = 1$ and $\delta_3^1 = 1$. Then we have

$$\begin{aligned} &\mathcal{M}_1 + \mathcal{L}_2^1 + \mathcal{L}_3^1 \\ &= \mathcal{M}_1 + \sum_{j=0}^{I'_2(1)-1} \left(g\Big(\frac{\sigma_2^j}{q_2}\Big) - g\Big(\frac{\sigma_2^j p_3^{-1}}{q_2}\Big)\right) + \sum_{k=0}^{I'_3(1)-1} \left(g\Big(\frac{\sigma_3^k}{q_3}\Big) - g\Big(\frac{\sigma_3^k p_2^{-1}}{q_3}\Big)\right) \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{M}_1 + \sum_{j=0}^{I_2'(1)-1} \sum_{k=0}^{2m_3-1} g\Big(\frac{\sigma_2^j \sigma_3^k}{n_1}\Big) + \sum_{k=0}^{I_3'(1)-1} \sum_{j=0}^{2m_2-1} g\Big(\frac{\sigma_2^j \sigma_3^k}{n_1}\Big) \\
&= \sum_{j=0}^{I_2'(1)-1} \sum_{k=0}^{m_3+I_3'(1)-1} 2g\Big(\frac{\sigma_2^j \sigma_3^k}{n_1}\Big) + \sum_{j=I_2'(1)}^{m_2-1} \sum_{k=0}^{I_3'(1)-1} 2g\Big(\frac{\sigma_2^j \sigma_3^k}{n_1}\Big) \\
&= \ 2\mathcal{B}_{23}^{++}.
\end{aligned}
$$

For $\delta_2^1 = \delta_3^1 = -1$, we have

$$
\mathcal{M}_1 - \mathcal{L}_2^1 - \mathcal{L}_3^1 = \mathcal{M}_1 + \mathcal{L}_2^1 + \mathcal{L}_3^1 - 2\mathcal{L}_2^1 - 2\mathcal{L}_3^1 = 2(\mathcal{B}_{23}^{++} + \widetilde{\mathcal{L}}_2^1 - \mathcal{L}_3^1) = 2\mathcal{B}_{23}^{--}
$$

since the meaning of $\mathcal{M}_1$ for $\delta_2^1 = \delta_3^1 = 1$ and that for $\delta_2^1 = \delta_3^1 = -1$ agree. When $\delta_2^1 = 1$ and $\delta_3^1 = -1$, we have

$$
\begin{aligned}
&\mathcal{M}_1 + \mathcal{L}_2^1 - \mathcal{L}_3^1 = \mathcal{M}_1 + \mathcal{L}_2^1 + \widetilde{\mathcal{L}}_3^1 \\
&= \mathcal{M}_1 + \sum_{j=0}^{I_2'(1)-1} \sum_{k=0}^{2m_3-1} g\Big(\frac{\sigma_2^j \sigma_3^k}{n_1}\Big) + \sum_{k=I_3'(1)}^{m_3-1} \sum_{j=0}^{2m_2-1} g\Big(\frac{\sigma_2^j \sigma_3^k}{n_1}\Big) \\
&= \sum_{j=0}^{I_2'(1)-1} \left[\sum_{k=0}^{m_3-1} 2g\Big(\frac{\sigma_2^j \sigma_3^k}{n_1}\Big) + \sum_{k=m_3+I_3'(2)}^{2m_3-1} 2g\Big(\frac{\sigma_2^j \sigma_3^k}{n_1}\Big) \right] + \sum_{j=I_2'(1)}^{m_2-1} \sum_{k=I_3'(2)}^{m_3-1} 2g\Big(\frac{\sigma_2^j \sigma_3^k}{n_1}\Big) \\
&= \ 2\mathcal{B}_{23}^{+-}.
\end{aligned}
$$

Finally, for $\delta_2^1 = -1$ and $\delta_3^1 = 1$, we have

$$
\mathcal{M}_1 - \mathcal{L}_2^1 + \mathcal{L}_3^1 = \mathcal{M}_1 + \widetilde{\mathcal{L}}_2^1 + \mathcal{L}_3^1 = 2\mathcal{B}_{23}^{-+}.
$$

The cases when $\alpha = 2$ or 3 can be similarly proved by using the identities

$$
\begin{aligned}
\mathcal{M}_2 &= -\sum_{i=m_1}^{2m_1-1} \sum_{k=0}^{m_3-1} \left(g\Big(\frac{\sigma_1^i \sigma_3^k}{n_2}\Big) - g\Big(\frac{\sigma_1^i \sigma_3^k p_2^{-1}}{n_2}\Big) \right) \\
&= \sum_{i=0}^{m_1-1} \sum_{k=0}^{m_3-1} \left(g\Big(\frac{\sigma_1^i \sigma_3^k}{n_2}\Big) - g\Big(\frac{\sigma_1^i \sigma_3^k p_2^{-1}}{n_2}\Big) \right)
\end{aligned}
$$

and

$$
\mathcal{M}_3 = \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} \left(g\Big(\frac{\sigma_1^i \sigma_2^j}{n_3}\Big) - g\Big(\frac{\sigma_1^i \sigma_2^j p_3^{-1}}{n_3}\Big) \right). \qquad \blacksquare
$$

Theorem 3.2. *Put*

$$\mathcal{M} = \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1} g\Big(\frac{\sigma_1^i\sigma_2^j\sigma_3^k}{n}\Big).$$

Then

$$\mathcal{R}_n = \mathcal{M} + \frac{(\delta_1^2+\delta_1^3)}{2}\mathcal{L}_1^2 + \frac{(\delta_2^1+\delta_2^3)}{2}\mathcal{L}_2^3 + \frac{(\delta_3^1+\delta_3^2)}{2}\mathcal{L}_3^1 - \mathcal{B}_1 - \mathcal{B}_2 - \mathcal{B}_3$$

is the 2-torsion element in $\big(A_n^0\big)^+$.

Proof. Observe that

$$\begin{aligned}\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3 &= \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1} g\Big(\frac{\sigma_1^i\sigma_2^j\sigma_3^k}{n}\Big) + \sum_{i=m_1}^{2m_1-1}\sum_{j=m_2}^{2m_2-1}\sum_{k=m_3}^{2m_3-1} g\Big(\frac{\sigma_1^i\sigma_2^j\sigma_3^k}{n}\Big)\\ &= 2\left(\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1} g\Big(\frac{\sigma_1^i\sigma_2^j\sigma_3^k}{n}\Big)\right)\\ &= 2\mathcal{M}.\end{aligned}$$

On the other hand, by Lemma 3.1, we have

$$\mathcal{M}_1+\delta_2^1\mathcal{L}_2^1+\delta_3^1\mathcal{L}_3^1+\mathcal{M}_2+\delta_1^2\mathcal{L}_1^2+\delta_3^2\mathcal{L}_3^2+\mathcal{M}_3+\delta_1^3\mathcal{L}_1^3+\delta_2^3\mathcal{L}_2^3 = 2\mathcal{B}_1+2\mathcal{B}_2+2\mathcal{B}_3.$$

Since $\mathcal{L}_1^2=\mathcal{L}_1^3$, $\mathcal{L}_2^1=\mathcal{L}_2^3$ and $\mathcal{L}_3^1=\mathcal{L}_3^2$, we have

$$2\mathcal{M} + (\delta_2^1+\delta_2^3)\mathcal{L}_2^1 + (\delta_3^1+\delta_3^2)\mathcal{L}_3^1 + (\delta_1^2+\delta_1^3)\mathcal{L}_1^2 - 2\mathcal{B}_1 - 2\mathcal{B}_2 - 2\mathcal{B}_3 = 0.$$

Hence

$$2\mathcal{R}_n = 0.$$

Finally, note that $\mathcal{R}_n \neq 0$ since the coefficient of $g(\frac{1}{n})$ in the expansion of $\mathcal{R}_n$ with respect to the basis of $\big(A_n^0\big)^+$ given in [3, Theorem 1] equals 1. ∎

4. Example

When $n = 105$, the theorem given in the previous section enables us to obtain the following Ennola relation.

Let $g(\frac{a}{n}) = g_n^a$ for simplicity. Put $p_1 = q_1 = 7$, $p_2 = q_2 = 5$ and $p_3 = q_3 = 3$. Then with $\sigma_1 = 3(31 \mod 105)$, $\sigma_2 = 3(43 \mod 105)$ and $\sigma_3 = 2(71 \mod 105)$, we have

$$\mathcal{M} = g_{105}^1 + g_{105}^{16} + g_{105}^{31} + g_{105}^{43} + g_{105}^{58} + g_{105}^{73}.$$

Since

$$\delta_1^2 = 1, \delta_1^3 = -1,$$
$$\delta_2^1 = 1, \delta_2^3 = -1,$$
$$\delta_3^1 = 1, \delta_3^2 = -1,$$

we have

$$\frac{1}{2}(\delta_1^2 + \delta_1^3)\mathcal{L}_1^2 = 0,$$
$$\frac{1}{2}(\delta_2^1 + \delta_2^3)\mathcal{L}_2^1 = 0,$$
$$\frac{1}{2}(\delta_3^1 + \delta_3^2)\mathcal{L}_3^1 = 0$$

and

$$\begin{aligned}\mathcal{B}_3 &= \mathcal{B}_{12}^{--} = g_{35}^8 + g_{35}^3 + g_{35}^{16} + g_{35}^{23} + g_{35}^2,\\ \mathcal{B}_2 &= \mathcal{B}_{13}^{+-} = g_{21}^1 + g_{21}^8 + g_{21}^{10} + g_{21}^{16},\\ \mathcal{B}_1 &= \mathcal{B}_{23}^{++} = g_{15}^1.\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{R}_{105} = &(g_{105}^1 + g_{105}^{16} + g_{105}^{31} + g_{105}^{43} + g_{105}^{58} + g_{105}^{73})\\ &-(g_{35}^8 + g_{35}^3 + g_{35}^{16} + g_{35}^{23} + g_{35}^2) - (g_{21}^1 + g_{21}^8 + g_{21}^{10} + g_{21}^{16}) - (g_{15}^1).\end{aligned}$$

To compare above relation with the one given by Ennola, we note that

$$\begin{aligned}\mathcal{R}_{105} = &g_{105}^1 + g_{105}^2 + g_{105}^{17} + g_{105}^{43} + g_{105}^{44} + g_{105}^{46}\\ &-g_{35}^1 + g_{35}^3 + g_{35}^{12} + g_{21}^5 + g_{21}^8 + g_{15}^4 + \mathbf{R}_1 + \mathbf{R}_2,\end{aligned}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are sums of elementary relations (1.3) and (1.4):

$$\begin{aligned}\mathbf{R}_1 = &-(g_{105}^2 + g_{105}^{44} + g_{105}^{23} + g_{105}^{86} + g_{21}^{13} - g_{21}^2)\\ &-(g_{105}^{17} + g_{105}^{59} + g_{105}^{38} + g_{105}^{101} + g_{21}^{16} - g_{21}^{17}) + (g_{105}^{23} + g_{105}^{58} + g_{35}^{31} - g_{35}^{23})\\ &+(g_{105}^{38} + g_{105}^{73} + g_{35}^1 - g_{35}^3) + (g_{105}^{31} + g_{105}^{101} + g_{35}^{22} - g_{35}^{31})\\ &+(g_{105}^{16} + g_{105}^{86} + g_{35}^{17} - g_{35}^{16}) - (g_{21}^1 + g_{21}^4 + g_{21}^{10} + g_{21}^{13} + g_{21}^{16} + g_{21}^{19})\\ &+(g_{21}^{13} + g_{21}^{20} + g_7^2 - g_7^6) - (g_{21}^1 + g_{21}^8 + g_7^5 - g_7^1) - (g_{15}^4 + g_5^3 + g_{15}^{14} - g_5^4)\\ &-(g_{35}^8 + g_{35}^3 + g_{35}^{23} + g_{35}^{13} + g_{35}^{18} + g_{35}^{33} + g_5^4 - g_5^3),\end{aligned}$$

and

$$\begin{aligned}\mathbf{R}_2 = &(g_{105}^{59} - g_{105}^{46}) + (g_{35}^{13} - g_{35}^{22}) + (g_{35}^{23} - g_{35}^{12}) + (g_{35}^{18} - g_{35}^{17}) + (g_{35}^{33} - g_{35}^2)\\ &+(g_{21}^{16} - g_{21}^5) + (g_{21}^{13} - g_{21}^8) + (g_{21}^{19} - g_{21}^2) + (g_{21}^4 - g_{21}^{17}) + (g_{21}^1 - g_{21}^{20})\\ &+(g_{15}^{14} - g_{15}^1) + (g_7^5 - g_7^2) + (g_7^6 - g_7^1).\end{aligned}$$

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