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CERTAIN COMBINATORIAL CONVOLUTION SUMS INVOLVING DIVISOR FUNCTIONS PRODUCT FORMULA

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Abstract. It is known that certain combinatorial convolution sums involving two divisor functions product formulae of arbitrary level can be explicitly expressed as a linear combination of divisor functions. In this article we deal with cases for certain combinatorial convolution sums involving three, four, six and twelve divisor functions product formula and obtain explicit expressions.

1. INTRODUCTION

Let N, d, k, l be positive integers. Throughout this paper, we define

$$\sigma_l(N) := \sum_{d|N} d^l, \quad \sigma_l^*(N) := \sigma_l^*(N;2) = \sum_{\substack{d|N \\ N/d \text{ odd}}} d^l,$$

$$\sum_{N=1}^{\infty} c(N)q^N := q\prod_{N=1}^{\infty}(1-q^N)^8(1-q^{2N})^8,$$

and

$$(a)_N := (a)(a-1)\cdots(a-N+1).$$

The exact evaluation of the basic convolution sum

$$\sum_{m=1}^{N-1}\sigma_1(m)\sigma_1(N-m)$$

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first appeared in a letter from Besge to Liouville in 1862([10, p.125]). The evaluation of such sums also appears in the works of Glaisher, Lahiri, Lehmer, Ramanujan, Skoruppa and Williams. For instance, Ramanujan([9]) obtained

$$\begin{aligned} &\sum_{m=1}^{N-1}\sigma_1(m)\sigma_1(N-m)=\frac{1}{12}\{5\sigma_3(N)+(1-6N)\sigma_1(N)\} \quad \text{and} \\ &\sum_{m=1}^{N-1}\sigma_1(m)\sigma_3(N-m)=\frac{1}{240}\{21\sigma_5(N)+(10-30N)\sigma_3(N)-\sigma_1(N)\} \end{aligned} \tag{1.1}$$

using only elementary arguments.

Glaisher [4, 5, 6] extended Besge's formula by replacing $\sigma_1(N)$ with other arithmetical functions in the convolution sum (1.1), for example, he obtained,

$$\sum_{m=1}^{N-1}\sigma_1^*(m)\sigma_1^*(N-m)=\frac{1}{4}\{\sigma_3^*(N)-N\sigma_1^*(N)\}$$

and

$$\sum_{m=1}^{N-1}\sigma_1^*(m)\sigma_3^*(N-m)=\frac{1}{16}\{\sigma_5^*(N)-N\sigma_3^*(N)\}.$$

Recently, D. Kim *et al.* [8, Table 5 and Theorem 3.5] proved that

$$\sum_{m=1}^{N-1}\sigma_1^*(m)\sigma_5^*(N-m)=\frac{1}{408}\{12\sigma_7^*(N)-17N\sigma_5^*(N)+5c(N)\} \tag{1.2}$$

and

$$\sum_{m_1+m_2+m_3=N}\sigma_1^*(m_1)\sigma_1^*(m_2)\sigma_1^*(m_3)=\frac{1}{64}\left\{\sigma_5^*(N)-3N\sigma_3^*(N)+2N^2\sigma_1^*(N)\right\}. \tag{1.3}$$

The study of convolution sums and their applications is a classical topic and they play an important role in number theory. In this paper we focus on the combinatorial convolution sums. For positive integers l and N, the combinatorial convolution sum

$$\sum_{\substack{a_1+a_2+\cdots a_n=2l+1\\ a_1,a_2,\cdots,a_n \text{ odd}}}\binom{2l+1}{a_1,a_2,\cdots,a_n}\sum_{\substack{m_1+m_2+\cdots+m_s=N\\ m_i,\cdots,m_j \text{ odd}\\ m_{i'},\cdots,m_{j'} \text{ even}}}\sigma_{a_1}(m_1)\cdots\sigma_{a_s}(m_s) \tag{1.4}$$

can be evaluated explicitly in terms of divisor functions. We are motivated by Ramanujan's recursion formula for sums of the product of two Eisenstein series [1, Entry 14, p. 332] and its proof, and also the following propositions.

Proposition 1. ([10]). *Let k, N be positive integers. Then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(N-m)$$
$$=\frac{2k+3}{4k+2}\sigma_{2k+1}(N)+\left(\frac{k}{6}-N\right)\sigma_{2k-1}(N)+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(N),$$

where B_j is the j-th Bernoulli number.

Proposition 2. ([2]). *For any integers $k \geq 1$ and $N \geq 3$, we have*

$$\sum_{r=0}^{2k}\binom{2k}{r}\sum_{i=1}^{[\frac{N-1}{2}]}\sum_{m=1}^{n-1}\sigma_{2k-r}(m;i,N)\sigma_r(n-m;i,N)$$
$$=\sigma^*_{2k+1}(n;N)-\frac{2}{N}n\sigma^*_{2k-1}(n;N)-\frac{1}{N}\sum_{i=1}^{[\frac{N-1}{2}]}(N-2i)\sigma_{2k}(n;i,N)$$
$$-\frac{1+(-1)^N}{2}\left(\sigma^*_{2k+1}\left(\frac{2n}{N};2\right)-\frac{2}{N}n\sigma^*_{2k-1}\left(\frac{2n}{N};2\right)\right),$$

where

$$\sigma_r(n;i,N)=\sum_{\substack{d|n\\ \frac{n}{d}\equiv i(N)}}d^r-(-1)^r\sum_{\substack{d|n\\ \frac{n}{d}\equiv -i(N)}}d^r \quad \text{and} \quad \sigma^*_r(n;N)$$
$$=\sum_{\substack{d|n\\ \frac{n}{d}\not\equiv 0(N)}}d^r=\sigma_r(n)-\sigma_r(n/N).$$

The aim of this article is to study six certain combinatorial convolution sums of the analogous type (1.4). More precisely, we prove the following theorems.

Theorem 3. *Let $N \geq 4$ be an even integer with $l \in \mathbb{N}$. Then we have*

$$\sum_{\substack{a+b+c=2l+1\\ a,b,c \text{ odd}}}a\binom{2l+1}{a,b,c}\sum_{\substack{m_1+m_2+m_3=N\\ m_3 \text{ even}}}(-1)^{m_1+1}\sigma^*_a(m_1)\sigma^*_b(m_2)\sigma^*_c(m_3)$$
$$=\frac{(2l+1)N}{32}\left\{\sigma^*_{2l+1}(N)-2N\sigma^*_{2l-1}(N)\right\}.$$

Theorem 4. *Let $N \geq 4$ be an even integer with $l \in \mathbb{N}$. Then we have*

$$\sum_{\substack{a+b+c=2l+1\\ a,b,c \text{ odd}}}\binom{2l+1}{a,b,c}\sum_{\substack{m_1+m_2+m_3=N\\ m_1,m_2 \text{ odd}}}\sigma^*_a(m_1)\sigma^*_b(m_2)\sigma^*_c(m_3)$$
$$=\frac{1}{32}\left\{\sigma^*_{2l+3}(N)-2N\sigma^*_{2l+1}(N)-16\sum_{M<\frac{N}{2}}\sigma^*_1(M)\sigma^*_{2l+1}(N-2M)\right\}.$$

Theorem 5. *Let $N \geq 4$ be an even integer with $l \in \mathbb{N}$. Then we have*

$$\sum_{\substack{a+b+c+d=2l\\ a,b,c,d \text{ odd}}} (c+d+1)^{-1}\binom{2l}{a,b,c,d} \sum_{\substack{m_1+m_2+m_3+m_4=N\\ m_1,m_2,m_3,m_4 \text{ odd}}} \sigma_a(m_1)\sigma_b(m_2)\sigma_c(m_3)\sigma_d(m_4)$$

$$= \frac{1}{128(2l+1)}\left\{\sigma^*_{2l+3}(N) - 2N\sigma^*_{2l+1}(N) - 32(l+1)\sum_{M<\frac{N}{2}} \sigma^*_1(M)\sigma^*_{2l+1}(N-2M)\right\}.$$

Theorem 6. (*1*) *Let $N \geq 4$ be an even integer with $l \in \mathbb{N}$. Then we have*

$$\sum_{\substack{a+b+c+d=2l\\ a,b,c,d \text{ odd}}} \frac{a+b}{c+d+1}\cdot\binom{2l}{a,b,c,d} \sum_{\substack{m_1+m_2+m_3+m_4=N\\ m_3,m_4 \text{ odd}}} (-1)^{m_1+1}\sigma^*_a(m_1)\sigma^*_b(m_2)\sigma^*_c(m_3)\sigma^*_d(m_4)$$

$$= \frac{1}{64}\left\{N\sigma^*_{2l+1}(N) - 2N^2\sigma^*_{2l-1}(N) - 64l\sum_{M<\frac{N}{2}} \left(\frac{N}{2} - M\right)\sigma^*_1(M)\sigma^*_{2l-1}(N-2M)\right\}.$$

(*2*) *Let $N \geq 4$ be a positive integer with $\alpha, \beta, \gamma, \delta \in \mathbb{N} \cup \{0\}$. Then we obtain*

$$\sum_{m_1+m_2+m_3+m_4=N} \sigma^*_1(2^\alpha m_1)\sigma^*_1(2^\beta m_2)\sigma^*_1(2^\gamma m_3)\sigma^*_1(2^\delta m_4)$$

$$= \frac{2^{\alpha+\beta+\gamma+\delta}}{32640}\{15\sigma^*_7(N) - 85N\sigma^*_5(N) + 153N^2\sigma^*_3(N) - 85N^3\sigma^*_1(N) + 2c(N)\}.$$

Remark 7. From Theorem 3 to the first part of Theorem 6, we give formula of multinomial convolution sums as the sum of divisor functions or their binomial convolution sums. In each case that is not a combinatorial convolution sum, we need coefficients of new forms. The second part of Theorem 6 is an evidence.

For a positive even integer k,

$$G_k(z) := -\frac{B_k}{2k} + \sum_{N=1}^{\infty} \sigma_{k-1}(N)q^N \quad \text{and} \quad G^0_k(z) := \sum_{N=1}^{\infty} \sigma^*_{k-1}(N)q^N,$$

where $q = e^{2\pi iz}$. Note that D and U_r are operators defined by $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$ so that

$$Df(z) = \sum Na(N)q^N \text{ and } U_r f(z) = \sum a(rN)q^N$$

for $f(z) = \sum a(N)q^N$. G_k ($k \geq 4$) is the weight k Eisenstein series and G_2 is weight 2 quasi-modular form for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. The G^0_k is the Eisenstein series at the cusp 0, since there are two nonequivalent cusps, 0 and ∞ on the subgroup $\Gamma_0(2)$, where $\Gamma_0(m) = \left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : m|c\right\}$.

Remark 8. In the language of modular form our first four theorems give some identities between modular forms as follows, respectively. Let

$$L_1(z) := \sum_{\substack{a+b+c=2l+1\\ 2\nmid a,b,c}} a\binom{2l+1}{a,b,c}\big(G^0_{a+1}(z) - 2U_2G^0_{a+1}(2z)\big)\left(G^0_{b+1}(z)\right)\big(G^0_{c+1}(z) - U_2G^0_{c+1}(2z)\big),$$
$$L_2(z) := \sum_{\substack{a+b+c=2l+1\\ 2\nmid a,b,c}} \binom{2l+1}{a,b,c}\big(G^0_{a+1}(z) - U_2G^0_{a+1}(2z)\big)\big(G^0_{b+1}(z) - U_2G^0_{b+1}(2z)\big)\left(G^0_{c+1}(z)\right),$$
$$L_3(z) := \sum_{\sum_{j=1}^4 a_j = 2l, 2\nmid a_j} \frac{1}{a_3+a_4+1}\binom{2l}{a_1,a_2,a_3,a_4}\prod_{j=1}^{4}\left(G_{a_j+1}(z) - U_2G_{a_j+1}(2z)\right),$$
$$L_4(z) := \sum_{\substack{a+b+c+d=2l\\ 2\nmid a,b,c,d}} \frac{a+b}{c+d+1}\binom{2l}{a,b,c,d}\big(G^0_{a+1}(z) - 2U_2G^0_{a+1}(2z)\big)\left(G^0_{b+1}(z)\right)$$
$$\times\big(G^0_{c+1}(z) - U_2G^0_{c+1}(2z)\big)\big(G^0_{d+1}(z) - U_2G^0_{d+1}(2z)\big)$$

and

$$R_1(z) := \frac{2l+1}{32}\left\{DG^0_{2l+2}(z) + 2D^2G^0_{2l}(z)\right\},$$
$$R_2(z) := \frac{1}{32}G^0_{2l+4}(z) - \frac{1}{16}DG^0_{2l+2}(z) - \frac{1}{2}G^0_2(2z)G^0_{2l+2}(z),$$
$$R_3(z) := \frac{1}{128(2l+1)}\left\{G^0_{2l+4}(z) - 2DG^0_{2l+2}(z) - 32(l+1)G^0_2(2z)G^0_{2l+2}(z)\right\},$$
$$R_4(z) := \frac{1}{64}DG^0_{2l+2}(z) - \frac{1}{32}D^2G^0_{2l}(z) - \frac{l}{2}DG^0_{2l}(z)G^0_2(2z).$$

We may interpret Theorems 3–6 (1) as results for U_2-operator of modular form. Then the identities of Theorem 3–6 (1) are equivalent to

$$U_2L_i(z) = U_2R_i(z)(i = 1, 2, 3, 4).$$

Theorem 9. *Let* $l, N \in \mathbb{N}$ *with* $4|N$ $(N \geq 8)$*, the ordered subset of* 6*-tuples*

$$S_N := \{\boldsymbol{m} = (m_1, m_2, m_3, m_4, m_5, m_6) \in \mathbb{N}^6$$
$$: 2|m_1+m_2, 2|m_3+m_4, 4|m_5+m_6 \text{ and } \sum_{i=1}^{6} m_i = N\}$$

and for $\boldsymbol{m} \in S_N$*,*

$$\alpha_{\boldsymbol{m}} := \frac{(-1)^{\frac{m_1+m_2}{2}+m_1+m_3+m_5}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)}.$$

Then we have

$$\sum_{\sum_{i=1}^{6} a_i=2l, 2\nmid a_i} (a_1+a_2-1)(a_1+a_2)(a_3+a_4)(a_5+a_6)\binom{2l}{a_1,a_2,a_3,a_4,a_5,a_6}$$
$$\times \sum_{\boldsymbol{m}\in S_N} \alpha_{\boldsymbol{m}} \cdot \prod_{j=1}^{6} \sigma^*_{a_j}(m_j)$$
$$= 2^{-9}(2l)_4 N \left\{\sigma^*_{2l-3}(N) - 4N\sigma^*_{2l-5}(N)\right\}.$$

Theorem 10. *Let $l, N \in \mathbb{N}$ with $8|N$ $(N \geq 16)$, the ordered subset of 12-tuples*

$$C_N := \Big\{\boldsymbol{m}=(m_1,m_2,\cdots m_{11},m_{12}) \in \mathbb{N}^{12} : 2|(m_1+m_2),\ 2|(m_3+m_4),\ 2|(m_5+m_6),$$
$$2|(m_7+m_8),\ 2|(m_9+m_{10}),\ 2|(m_{11}+m_{12}),\ 4|(m_1+m_2+m_3+m_4),$$
$$4|(m_5+m_6+m_7+m_8),\ 8|(m_9+m_{10}+m_{11}+m_{12}) \text{ and } \sum_{i=1}^{12} m_i = N\Big\}$$

and for $\boldsymbol{m} \in C_N$,

$$\beta_{\boldsymbol{m}} := \frac{(-1)^{m_1+m_3+m_5+m_7+m_9+m_{11}}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)(m_7+m_8)(m_9+m_{10})(m_{11}+m_{12})}$$
$$\times \frac{(-1)^{\frac{m_1+m_2+m_3+m_4}{4}+\frac{m_1+m_2}{2}+\frac{m_5+m_6}{2}+\frac{m_9+m_{10}}{2}}}{(m_1+m_2+m_3+m_4)(m_5+m_6+m_7+m_8)(m_9+m_{10}+m_{11}+m_{12})}.$$

Then we have

$$\sum_{\substack{\sum_{i=1}^{12} a_i=2l \\ a_i \text{ odd}}} (a_1+a_2)(a_3+a_4)(a_5+a_6)(a_7+a_8)(a_9+a_{10})(a_{11}+a_{12})$$
$$\times (a_1+a_2+a_3+a_4-3)(a_1+a_2+a_3+a_4-2)(a_5+a_6+a_7+a_8-2)$$
$$\times (a_9+a_{10}+a_{11}+a_{12}-2)\binom{2l}{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}}$$
$$\times \sum_{\boldsymbol{m}\in C_N} \beta_{\boldsymbol{m}} \prod_{i=1}^{12} \sigma^*_{a_i}(m_i)$$
$$= 2^{-16}(2l)_{10} N \left\{\sigma^*_{2l-9}(N) - 8N\sigma^*_{2l-11}(N)\right\}.$$

2. Calculation of Convolution Sums

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ or $\Gamma_0(2)$. For an even integer $k > 2$, the space $M_k(\Gamma)$ of modular forms on Γ of weight k is a finite dimensional vector space over $\mathbb{C}$. More precisely, $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is spanned by G_k and Hecke eigenforms and $M_k(\Gamma_0(2))$ by G_k, G_k^0 and Hecke eigenforms.

The following proposition and lemma are obtained by using this for a given modular form on $\Gamma_0(m)$ for $m = 1, 2, 4$.

Proposition 11. *Let $N \geq 2$ be a positive integer. Then*

(1)
$$\sum_{m=1}^{N-1} m\sigma_1^*(m)\sigma_1^*(N-m) = \frac{N}{8}\sigma_3^*(N) - \frac{N^2}{8}\sigma_1^*(N).$$

(2)
$$\sum_{m=1}^{N-1} m\sigma_3^*(m)\sigma_1^*(N-m) = \frac{N}{24}\sigma_5^*(N) - \frac{N^2}{20}\sigma_3^*(N) + \frac{1}{120}c(N).$$

(3)
$$\sum_{m=1}^{N-1} m^2\sigma_1^*(m)\sigma_1^*(N-m) = \frac{3N^2}{40}\sigma_3^*(N) - \frac{N^3}{12}\sigma_1^*(N) + \frac{1}{120}c(N).$$

Proof. It is well-known that the space of quasi-modular forms is a finite demensional vector space.

(1) $G_2^0 \times DG_2^0$ is a quasi-modular form in the space generated by $D^2G_2, D^2G_2^0, DG_4, DG_4^0, G_6, G_6^0$. Thus $G_2^0 \times DG_2^0$ is $-\frac{1}{8}D^2G_2^0 + \frac{1}{8}DG_4^0$.

Consider the space $\widetilde{M_8}(\Gamma_0(2))$ of quasi-modular form of weight 8 on $\Gamma_0(2)$ generated by

$$D^3G_2, D^3G_2^0, D^2G_4, D^2G_4^0, DG_6, DG_6^0, G_8, G_8^0, \Delta_8,$$

where $\Delta_8(z) = q\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8$.

(2) Since $DG_4^0(z) \times G_2^0(z)$ is $-\frac{1}{20}D^2G_4^0(z) + \frac{1}{24}DG_6^0(z) + \frac{1}{120}\Delta_8(z)$, we get the result.

(3) From the fact that $D^2G_2^0(z) \times G_2^0(z) = -\frac{1}{12}D^3G_2^0(z) + \frac{3}{40}D^2G_4^0(z) + \frac{1}{120}\Delta_8(z)$ we desire our result. ■

Lemma 12. *Let $N \geq 3$ be a positive integer. Then*

(1)
$$\sum_{m<\frac{N}{2}} \sigma_1^*(m)\sigma_1^*(N-2m) = \frac{1}{16}\sigma_3^*(N) - \frac{N}{16}\sigma_1^*(N) - \frac{N}{8}\sigma_1^*(\frac{N}{2}).$$

(2)
$$\sum_{m<\frac{N}{2}} \sigma_1^*(m)\sigma_3^*(N-2m) = \frac{1}{64}\sigma_5^*(N) - \frac{N}{32}\sigma_3^*(N) + \frac{1}{64}c_6(N),$$

where $q\prod_{N=1}^{\infty}(1-q^{2N})^{12} = \sum_{N=1}^{\infty} c_6(N)q^N$.

(3)

$$\sum_{m<\frac{N}{2}} \sigma_1^*(m)\sigma_5^*(N-2m) = \frac{1}{136}\sigma_7^*(N) - \frac{N}{48}\sigma_5^*(N) + \frac{11}{816}c(N) + \frac{1}{2}c(\frac{N}{2}).$$

Proof. We omit the proof of this lemma because the calculation is routine. Actually, we used the modular form on $\Gamma_0(4)$. ■

Remark 13. When the coefficients $c(N)$ is an N-th coefficient of normalized new form $\Delta_8(z)$ of weight 8 on $\Gamma_0(2)$ then

$$\begin{aligned} &c(p^{r+1}) = c(p^r)c(p) - p^7 c(p^{r-1}), \quad p \neq 2 \\ &c(2^r) = c(2)^r \\ &c(MN) = c(M)c(N), \qquad\qquad \text{if } \gcd(M,N) = 1, \end{aligned}$$

and its expansion is $q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 + 1016q^7 - 512q^8 - 2043q^9 + 1680q^{10} + O(q^{11})$.

Remark 14. Let α be a fixed integer with $\alpha \geq 3$, and let

$$\text{Pyr}_\alpha(x) = \frac{1}{6}x(x+1)\left((\alpha-2)x + 5 - \alpha\right)$$

denote the αth order pyramidal number [3]. These combinatorial numbers play an important role in number theory and discrete mathematics. In fact, if $N = 2q+1$ is a prime number, then from Lemma 12 (1), we obtain

$$\sum_{m<\frac{N}{2}} \sigma_1^*(m)\sigma_1^*(N-2m) = \frac{1}{2}q^2(q+1) = \text{Pyr}_5(q).$$

3. Proofs of Theorems

Before starting our proofs, we need the following lemma.

Lemma 15. (*1*) *Let N be a positive integer. Then*

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2N-1} (-1)^{m+1} \sigma_{2k-2s-1}^*(m)\sigma_{2s+1}^*(2N-m) = N\sigma_{2k-1}^*(2N).$$

(*2*) *Let $N \geq 2$ be a positive integer. Then*

$$\sum_{k=0}^{l-1} \binom{2l}{2k+1} \sum_{m=1}^{N-1} \sigma_{2l-2k-1}^*(m)\sigma_{2k+1}^*(N-m) = \frac{1}{2}\left\{\sigma_{2l+1}^*(N) - N\sigma_{2l-1}^*(N)\right\}.$$

(3) *Let* N *be a positive integer. Then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N}\sigma_{2k-2s-1}(2m-1)\sigma_{2s+1}(2N-2m+1)=\frac{1}{4}\sigma^*_{2k+1}(2N).$$

(4) *Let* $N\geq 4$ *be an even positive integer. Then*

$$\sum_{k=1}^{l}\binom{2l}{2k-1}\sum_{m<\frac{N}{2}}m\sigma^*_{2k-1}(m)\sigma^*_{2l+1-2k}(\frac{N}{2}-m)=\frac{N}{2^{2l+4}}\{\sigma^*_{2l+1}(N)-2N\sigma^*_{2l-1}(N)\}.$$

Proof. (1) is in [7, (16)], (2) in [7, (10)] and (3) in [7, (15)]. From that

$$\sum_{m<\frac{N}{2}}m\sigma^*_{2k-1}(m)\sigma^*_{2l+1-2k}(\frac{N}{2}-m)=\sum_{m<\frac{N}{2}}(\frac{N}{2}-m)\sigma^*_{2k-1}(\frac{N}{2}-m)\sigma^*_{2l+1-2k}(m),$$

the left hand side of (4) is

$$\frac{N}{4}\sum_{k=1}^{l}\binom{2l}{2k-1}\sum_{m<\frac{N}{2}}\sigma^*_{2k-1}(m)\sigma^*_{2l+1-2k}(\frac{N}{2}-m)=\frac{N}{8}\{\sigma^*_{2l+1}(\frac{N}{2})-\frac{N}{2}\sigma^*_{2l-1}(\frac{N}{2})\}$$
$$=\frac{N}{2^{2l+4}}\{\sigma^*_{2l+1}(N)-2N\sigma^*_{2l-1}(N)\}.$$

The first equality is same to (2) if we consider $\frac{N}{2}$ instead of N. ■

Proof of the Theorem 3. Consider

$$\begin{aligned}
&\sum_{\substack{a+b+c=2l+1\\ a,b,c\text{ odd}}}(a+b)\binom{2l+1}{a,b,c}\sum_{\substack{m_1+m_2+m_3=N\\ m_3\text{ even}}}(-1)^{m_1+1}\sigma^*_a(m_1)\sigma^*_b(m_2)\sigma^*_c(m_3)\\
&=\sum_{\substack{a+b+c=2l+1\\ a,b,c\text{ odd}}}\frac{(2l+1)!}{(a+b-1)!c!}\frac{(a+b)!}{a!b!}\\
&\quad\times\sum_{M<\frac{N}{2}}\left(\sum_{m_1+m_2=2M}(-1)^{m_1+1}\sigma^*_a(m_1)\sigma^*_b(m_2)\right)\sigma^*_c(N-2M)\\
&=\sum_{\substack{2k+c=2l+1\\ c\text{ odd}}}\frac{(2l+1)!}{(2k-1)!c!}\\
&\quad\times\sum_{M<\frac{N}{2}}\left(\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m_1=1}^{2M-1}(-1)^{m_1+1}\sigma^*_{2k-2s-1}(m_1)\sigma^*_{2s+1}(2M-m_1)\right)\sigma^*_c(N-2M),
\end{aligned}\tag{3.1}$$

where we put $a+b=2k$ and $b=2s+1$. By Lemma 15 (1), (3.1) becomes

$$\begin{aligned}
&\sum_{\substack{2k+c=2l+1\\ c\text{ odd}}} \frac{(2l+1)!}{(2k-1)!c!} \sum_{M<\frac{N}{2}} M\sigma^*_{2k-1}(2M)\sigma^*_c(N-2M)\\
&= 2^{2l}(2l+1)\sum_{k=1}^{l}\binom{2l}{2k-1}\sum_{M<\frac{N}{2}} M\sigma^*_{2k-1}(M)\sigma^*_{2l+1-2k}(\frac{N}{2}-M)\\
&= \frac{(2l+1)N}{16}\left\{\sigma^*_{2l+1}(N)-2N\sigma^*_{2l-1}(N)\right\}.
\end{aligned}$$

Lemma 15 (4) is used in last equality.

On the other hand,

$$\begin{aligned}
&\sum_{\substack{a+b+c=2l+1\\ a,b,c\text{ odd}}}(a+b)\binom{2l+1}{a,b,c}\sum_{\substack{m_1+m_2+m_3=N\\ m_3\text{ even}}}(-1)^{m_1+1}\sigma^*_a(m_1)\sigma^*_b(m_2)\sigma^*_c(m_3)\\
&=2\sum_{\substack{a+b+c=2l+1\\ a,b,c\text{ odd}}} a\binom{2l+1}{a,b,c}\sum_{\substack{m_1+m_2+m_3=N\\ m_3\text{ even}}}(-1)^{m_1+1}\sigma^*_a(m_1)\sigma^*_b(m_2)\sigma^*_c(m_3),
\end{aligned}$$

Our assertion turns out. ∎

Proof of the Theorem 4. The equation to solve can be written as

$$\begin{aligned}
&\sum_{\substack{a+b+c=2l+1\\ a,b,c\text{ odd}}}\binom{2l+1}{a,b,c}\sum_{\substack{m_1+m_2+m_3=N\\ m_1,m_2\text{ odd}}}\sigma^*_a(m_1)\sigma^*_b(m_2)\sigma^*_c(m_3)\\
&=\sum_{\substack{a+b+c=2l+1\\ a,b,c\text{ odd}}}\frac{(2l+1)!}{(a+b)!c!}\cdot\frac{(a+b)!}{a!b!}\\
&\qquad\times\sum_{M<\frac{N}{2}}\left(\sum_{\substack{m_1+m_2=2M\\ m_1,m_2\text{ odd}}}\sigma^*_a(m_1)\sigma^*_b(m_2)\right)\sigma^*_c(N-2M) \qquad (3.2)\\
&=\sum_{\substack{2k+c=2l+1\\ c\text{ odd}}}\frac{(2l+1)!}{(a+b)!c!}\sum_{M<\frac{N}{2}}\left(\sum_{s=0}^{k-1}\binom{2k}{2s+1}\right.\\
&\qquad\left.\sum_{m_1=1}^{M}\sigma^*_{2k-2s-1}(2m_1-1)\sigma^*_{2s+1}(2M-2m_1+1)\right)\sigma^*_c(N-2M),
\end{aligned}$$

where we put $a+b=2k$ and $b=2s+1$. Using Lemma 15 (4), (3.2) becomes

$$\sum_{k=1}^{l}\binom{2l+1}{2k}\sum_{M<\frac{N}{2}}\frac{1}{4}\sigma^*_{2k+1}(2M)\sigma^*_{2l+1-2k}(N-2M)$$

$$=\sum_{k=1}^{l}\binom{2l+1}{2k}\sum_{M<\frac{N}{2}}\frac{1}{4}\cdot 2^{2k+1}\sigma^*_{2k+1}(M)\cdot 2^{2l+1-2k}\sigma^*_{2l+1-2k}(\frac{N}{2}-M)$$

$$=\frac{2^{2l}}{2l+2}\sum_{k=1}^{l}\binom{2l+2}{2k+1}(2k+1)\sum_{M<\frac{N}{2}}\sigma^*_{2k+1}(M)\sigma^*_{2l+1-2k}(\frac{N}{2}-M)$$

$$=\frac{2^{2l}}{2l+2}\sum_{k=0}^{l}\binom{2l+2}{2k+1}(2k+1)\sum_{M<\frac{N}{2}}\sigma^*_{2k+1}(M)\sigma^*_{2l+1-2k}(\frac{N}{2}-M)$$

$$-\frac{2^{2l}}{2l+2}\binom{2l+2}{1}\sum_{M<\frac{N}{2}}\sigma^*_{1}(M)\sigma^*_{2l+1}(\frac{N}{2}-M)$$

$$=\frac{2^{2l}}{2l+2}\sum_{k=0}^{l}\binom{2l+2}{2k+1}(2k+1)\sum_{M<\frac{N}{2}}\sigma^*_{2k+1}(M)\sigma^*_{2l+1-2k}(\frac{N}{2}-M)$$

$$-2^{2l}\sum_{M<\frac{N}{2}}\sigma^*_{1}(M)\sigma^*_{2l+1}(\frac{N}{2}-M).$$

Here the first sum can be written as

$$2^{2l-1}\sum_{k=0}^{l}\binom{2l+2}{2k+1}\sum_{M<\frac{N}{2}}\sigma^*_{2k+1}(M)\sigma^*_{2l+1-2k}(\frac{N}{2}-M)=\frac{1}{32}\sigma^*_{2l+3}(N)-\frac{N}{16}\sigma^*_{2l+1}(N)$$

by Lemma 15 (3). Therefore

$$\sum_{\substack{a+b+c=2l+1\\ a,b,c\text{ odd}}}\binom{2l+1}{a,b,c}\sum_{\substack{m_1+m_2+m_3=N\\ m_1,m_2\text{ odd}}}\sigma^*_a(m_1)\sigma^*_b(m_2)\sigma^*_c(m_3)$$
$$=\frac{1}{32}\sigma^*_{2l+3}(N)-\frac{N}{2}\sigma^*_{2l+1}(N)-\frac{1}{2}\sum_{M<\frac{N}{2}}\sigma^*_1(M)\sigma^*_{2l+1}(N-2M).$$

$$\sum_{\substack{a+b+c=2l+1\\ a,b,c\text{ odd}}}\binom{2l+1}{a,b,c}\sum_{\substack{m_1+m_2+m_3=N\\ m_1,m_2\text{ odd}}}\sigma^*_a(m_1)\sigma^*_b(m_2)\sigma^*_c(m_3)$$
$$=\frac{1}{32}\left\{\sigma^*_{2l+3}(N)-2N\sigma^*_{2l+1}(N)-16\sum_{M<\frac{N}{2}}\sigma^*_1(M)\sigma^*_{2l+1}(N-2M)\right\}.\qquad\blacksquare$$

Proof of the Theorem 5. The left hand side is

$$\sum_{\substack{a+b+c+d=2l\\ a,b,c,d \text{ odd}}} (c+d+1)^{-1}\binom{2l}{a,b,c,d} \sum_{\substack{m_1+m_2+m_3+m_4=N\\ m_1,m_2,m_3,m_4 \text{ odd}}} \sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)\sigma_d^*(m_4)$$

$$= \sum_{\substack{a+b+c+d=2l\\ a,b,c,d \text{ odd}}} \frac{2l!}{(a+b)!(c+d+1)!}\cdot\frac{(a+b)!}{a!b!}\cdot\frac{(c+d)!}{c!d!}$$

$$\times \sum_{M+L=\frac{N}{2}} \left(\sum_{\substack{m_1+m_2=2M\\ m_1,m_2 \text{ odd}}} \sigma_a^*(m_1)\sigma_b^*(m_2)\right)\left(\sum_{\substack{m_3+m_4=2L\\ m_3,m_4 \text{ odd}}} \sigma_c^*(m_3)\sigma_d^*(m_4)\right)$$

$$= \sum_{k_1+k_2=l} \frac{2l!}{2k_1!(2k_2+1)!} \times \sum_{M+L=\frac{N}{2}} \sum_{s_1=0}^{k_1-1} \binom{2k_1}{2s_1+1}$$

$$\sum_{t_1=1}^{M} \sigma_{2k_1-2s_1-1}^*(2t_1-1)\sigma_{2s_1+1}^*(2M-2t_1+1)$$

$$\times \sum_{s_2=0}^{k_2-1}\binom{2k_2}{2s_2+1}\sum_{t_2=1}^{L}\sigma_{2k_2-2s_2-1}^*(2t_2-1)\sigma_{2s_2+1}^*(2L-2t_2+1) \tag{3.3}$$

$$= \sum_{k_1+k_2=l} \frac{2l!}{(2k_1)!(2k_2+1)!} \sum_{M+L=\frac{N}{2}} \frac{1}{16}\sigma_{2k_1+1}^*(2M)\sigma_{2k_2+1}^*(2L)$$

$$= 2^{2l-2}\sum_{k_1=1}^{l-1}\binom{2l+2}{2k_1+1}\frac{2k_1+1}{(2l+1)(2l+2)}\sum_{M<\frac{N}{2}}\sigma_{2k_1+1}^*(M)\sigma_{2l-2k_1+1}^*(\frac{N}{2}-M)$$

$$= \frac{2^{2l-2}}{(2l+1)(2l+2)}\sum_{k_1=0}^{l}\binom{2l+2}{2k_1+1}(2k_1+1)\sum_{M<\frac{N}{2}}\sigma_{2k_1+1}^*(M)\sigma_{2l-2k_1+1}^*(\frac{N}{2}-M)$$

$$-\frac{2^{2l}}{4(2l+1)(2l+2)}\left[(2l+2)\sum_{M<\frac{N}{2}}\sigma_1^*(M)\sigma_{2l+1}^*(\frac{N}{2}-M)+(2l+2)(2l+1)\right.$$

$$\left.\sum_{M<\frac{N}{2}}\sigma_{2l+1}^*(M)\sigma_1^*(\frac{N}{2}-M)\right].$$

The third equality is due to Lemma 15 (3). Since the first sum is same in (3.3), we get

$$\frac{1}{128(2l+1)}\left\{\sigma_{2l+3}^*(N)-2N\sigma_{2l+1}^*(N)\right\}-\frac{l+1}{4(2l+1)}\sum_{M<\frac{N}{2}}\sigma_1^*(M)\sigma_{2l+1}^*(N-2M).\blacksquare$$

Proof of the Theorem 6.

(1) We prove this in similar way to one of previous theorem. Note that

$$\sum_{\substack{a+b+c+d=2l\\ a,b,c,d \text{ odd}}} \frac{a+b}{c+d+1}\cdot\binom{2l}{a,b,c,d} \sum_{\substack{m_1+m_2+m_3+m_4=N\\ m_3,m_4 \text{ odd}}} (-1)^{m_1+1}\sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)\sigma_d^*(m_4)$$

$$= \sum_{\substack{a+b+c+d=2l\\ a,b,c,d \text{ odd}}} \frac{(2l)!}{(a+b-1)!(c+d+1)!}\cdot\frac{(a+b)!}{a!b!}\cdot\frac{(c+d)!}{c!d!}$$

$$\times \sum_{M+L=\frac{N}{2}}\left(\sum_{m_1+m_2=2M}(-1)^{m_1+1}\sigma_a^*(m_1)\sigma_b^*(m_2)\right)\left(\sum_{\substack{m_3+m_4=2L\\ m_3,m_4 \text{ odd}}}\sigma_c^*(m_3)\sigma_d^*(m_4)\right) \tag{3.4}$$

$$= \sum_{k_1+k_2=l}\frac{2l!}{(2k_1-1)!(2k_2+1)!}$$

$$\times \sum_{M+L=\frac{N}{2}}\left(\sum_{s_1=0}^{k_1-1}\binom{2k_1}{2s_1+1}\sum_{m_1=1}^{2M-1}(-1)^{m_1+1}\sigma_{2k_1-2s_1-1}^*(m_1)\sigma_{2s_1+1}^*(2M-m_1)\right)$$

$$\times\left(\sum_{s_2=0}^{k_2-1}\binom{2k_2}{2s_2+1}\sum_{m_3=1}^{L}\sigma_{2k_2-2s_2-1}^*(2m_3-1)\sigma_{2s_2+1}^*(2L-2m_3+1)\right)$$

when we let $a+b=2k_1$, $b=2s_1+1$, $c+d=2k_2$, and $d=2s_2+1$. Thus, by Lemma 15 (1) and (3), we can write (3.4) as

$$\sum_{k_1+k_2=l}\binom{2l}{2k_1-1}\sum_{M+L=\frac{N}{2}}\frac{M}{4}\sigma_{2k_1-1}^*(2M)\sigma_{2k_2+1}^*(2L)$$

$$= 2^{2l-2}\sum_{k_1=1}^{l-1}\binom{2l}{2k_1-1}\sum_{M<\frac{N}{2}}M\sigma_{2k_1-1}^*(M)\sigma_{2l-2k_1+1}^*(\frac{N}{2}-M)$$

$$= 2^{2l-2}\sum_{k_1=1}^{l}\binom{2l}{2k_1-1}\sum_{M<\frac{N}{2}}M\sigma_{2k_1-1}^*(M)\sigma_{2l-2k_1+1}^*(\frac{N}{2}-M)$$

$$-2^{2l-1}\cdot l\sum_{M<\frac{N}{2}}M\sigma_{2l-1}^*(M)\sigma_1^*(\frac{N}{2}-M)$$

$$= 2^{2l-2}\sum_{k_1=1}^{l}\binom{2l}{2k_1-1}\sum_{M<\frac{N}{2}}M\sigma_{2k_1-1}^*(M)\sigma_{2l-2k_1+1}^*(\frac{N}{2}-M)$$

$$-2^{2l-1}\cdot l\sum_{M<\frac{N}{2}}\left(\frac{N}{2}-M\right)\sigma_1^*(M)\sigma_{2l-1}^*(\frac{N}{2}-M).$$

So, by Lemma 15 (4), we obtain the proof.

(2) Note that $\sigma_s^*(2N) = 2^s\sigma_s^*(N)$ (see [8, (11)]). Thus we obtain

$$\begin{aligned}
&\sum_{m_1+m_2+m_3+m_4=N} \sigma_1^*(2^\alpha m_1)\sigma_1^*(2^\beta m_2)\sigma_1^*(2^\gamma m_3)\sigma_1^*(2^\delta m_4)\\
&= 2^{\alpha+\beta+\gamma+\delta} \sum_{m_1+m_2+m_3+m_4=N} \sigma_1^*(m_1)\sigma_1^*(m_2)\sigma_1^*(m_3)\sigma_1^*(m_4)\\
&= 2^{\alpha+\beta+\gamma+\delta} \sum_{u=1}^{N-3} \sigma_1^*(u) \sum_{m_1+m_2+m_3=N-u} \sigma_1^*(m_1)\sigma_1^*(m_2)\sigma_1^*(m_3)\\
&= 2^{\alpha+\beta+\gamma+\delta-6}\{\sum_{u=1}^{N-3} \sigma_1^*(u)\sigma_5^*(N-u) - 3\sum_{m=1}^{N-3} m\sigma_1^*(N-m)\sigma_3^*(m)\\
&\quad +2\sum_{m=1}^{N-3} m^2\sigma_1^*(N-m)\sigma_1^*(m)\}
\end{aligned}$$

by (1.3). From (1.2), (2) and (3) in Proposition 11, our assertion is true. ∎

Proof of the Theorem 9. For the convenience we use (a, b, c, d, e, f) instead of $(a_1, a_2, a_3, a_4, a_5, a_6)$. First since $4|N$, we have $\frac{N}{2}$ is even. Let $a + b := 2k_1$, $c + d := 2k_2$, and $e + f := 2k_3$. And from the conditions $2|(m_1 + m_2)$, $2|(m_3 + m_4)$, and $4|(m_5 + m_6)$, we can write $m_1 + m_2 = 2N_1$, $m_3 + m_4 = 2N_2$, and $m_5 + m_6 = 2N_3$ for integers N_1, N_2 and even integer N_3. Then using Lemma 15 (1), the left hand side of our assertion is

$$\begin{aligned}
&\sum_{\substack{a+b+c+d+e+f=2l\\ a,b,c,d,e,f \text{ odd}}} \frac{(a+b-1)(2l)!}{(a+b-1)!(c+d-1)!(e+f-1)!}\cdot\frac{(a+b)!}{a!b!}\cdot\frac{(c+d)!}{c!d!}\cdot\frac{(e+f)!}{e!f!}\\
&\times\sum_{\mathbf{m}\in C_N} \frac{(-1)^{\frac{m_1+m_2}{2}+1}\cdot[(-1)^{m_1+1}\sigma_a^*(m_1)\sigma_b^*(m_2)]\cdot[(-1)^{m_3+1}\sigma_c^*(m_3)\sigma_d^*(m_4)]\cdot[(-1)^{m_5+1}\sigma_e^*(m_5)\sigma_f^*(m_6)]}{(m_1+m_2)(m_3+m_4)(m_5+m_6)}\\
&= \sum_{\substack{k_1+k_2+k_3=l\\ k_1,k_2,k_3\geq 1}} \frac{(2k_1-1)(2l)!}{(2k_1-1)!(2k_2-1)!(2k_3-1)!}\\
&\quad \sum_{\substack{N_1+N_2+N_3=\frac{N}{2}\\ N_3 \text{ even}\\ \frac{N}{2} \text{ even}}} \frac{(-1)^{N_1+1}}{8N_1N_2N_3}\cdot N_1\sigma_{2k_1-1}^*(2N_1)\cdot N_2\sigma_{2k_2-1}^*(2N_2)\cdot N_3\sigma_{2k_3-1}^*(2N_3)\\
&= 2^{2l-6}(2l)_3 \sum_{\substack{k_1+k_2+k_3=l\\ k_1,k_2,k_3\geq 1}} (2k_1-1)\binom{2l-3}{2k_1-1, 2k_2-1, 2k_3-1}\\
&\quad \sum_{\substack{N_1+N_2+N_3=\frac{N}{2}\\ 2|\frac{N}{2},N_3}} (-1)^{N_1+1}\sigma_{2k_1-1}^*(N_1)\sigma_{2k_2-1}^*(N_2)\sigma_{2k_3-1}^*(N_3).
\end{aligned}$$

So by Theorem 3, the above is

$$\begin{aligned}
&2^{-9}(2l)(2l-1)(2l-2)(2l-3)N\left\{\sigma_{2l-3}^*(N) - 4N\sigma_{2l-5}^*(N)\right\}\\
&= 2^{-9}(2l)_4N\left\{\sigma_{2l-3}^*(N) - 4N\sigma_{2l-5}^*(N)\right\}.
\end{aligned}$$

∎

Proof of the Theorem 10. After rewriting of left hand side of our theorem as the sum of $2^{2k_j-1}N_j\sigma^*_{2k_j-1}(N_j)$, put $2k_j = a_{2j-1} + a_{2j}$ and $2N_j = m_{2j-1} + m_{2j}$ for $j = 1, \cdots, 6$. By considering $2k_j - 1$ and N_j as a_j and m_j in Theorem 9 we get the right hand side. ■

Example 16. (1) When $l = 1$ in Theorem 3, we obtain

$$\sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even} \\ N \text{ even}}} (-1)^{m_1+1}\sigma^*_1(m_1)\sigma^*_1(m_2)\sigma^*_1(m_3) = \frac{N}{64}\left\{\sigma^*_3(N) - 2N\sigma^*_1(N)\right\}.$$

(2) If $l = 1$ in Theorem 4, from the fact that Lemma 12 (2) and $c_6(N) = 0$ for even integer N, we have

$$\sum_{\substack{m_1+m_2+m_3=N \\ m_1,m_2 \text{ odd} \\ N \text{ even}}} \sigma^*_1(m_1)\sigma^*_1(m_2)\sigma^*_1(m_3) = \frac{1}{256}\left\{\sigma^*_5(N) - 2N\sigma^*_3(N)\right\}.$$

(3) In the case of $l = 2$ in Theorem 5, Lemma 12 (3) shows that

$$\sum_{\substack{m_1+m_2+m_3+m_4=N \\ m_1,m_2,m_3,m_4 \text{ odd} \\ N \text{ even}}} \sigma_1(m_1)\sigma_1(m_2)\sigma_1(m_3)\sigma_1(m_4) = \frac{1}{87040}\left\{5\sigma^*_7(N) - 22c(N) - 816c(\frac{N}{2})\right\}.$$

(4) Similarly, if $l = 2$ in Theorem 6,

$$\sum_{\substack{m_1+m_2+m_3+m_4=N \\ 2\nmid m_3,m_4}} (-1)^{m_1+1}\sigma^*_1(m_1)\sigma^*_1(m_2)\sigma^*_1(m_3)\sigma^*_1(m_4) = \frac{N}{15360}\left\{5\sigma^*_5(N) - 6N\sigma^*_3(N) - 128c(\frac{N}{2})\right\}.$$

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