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INFINITELY MANY SOLUTIONS FOR FOURTH-ORDER ELLIPTIC EQUATIONS WITH SIGN-CHANGING POTENTIAL

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Abstract. In this paper, we study the following fourth-order elliptic equation

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where the potential $V \in C(\mathbb{R}^N, \mathbb{R})$ is allowed to be sign-changing. Under the weakest superquadratic conditions, we establish the existence of infinitely many solutions via variational methods for the above equation. Recent results from the literature are extended.

1. INTRODUCTION

This paper is concerned with the following fourth-order elliptic equation

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = f(x,u) \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

When Ω is a bounded domain of $\mathbb{R}^N$, the problem

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u) \text{ in } \Omega, \\ u = \Delta u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

which arises in the study of traveling waves in suspension bridges (see [1, 2, 3]) and the study of the static deflection of an elastic plate in a fluid, has been extensively

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investigated in recent years. For the results of multiple nontrivial and sign changing solutions of problem (1.2) we refer the readers to [4-19] and the references therein. In [5], An and Liu used the mountain pass theorem to get the existence results for problem (1.2), and by applying linking approach obtained at least three nontrivial solutions in Wang et al. [12]. In [17], Yang and Zhang considered the existence of positive, negative and sign-changing solutions. When $f(x,t)$ is odd in u and satisfies some additional conditions, Zhou and Wu [18] established the existence and multiplicity of sign-changing solutions by using the sign-changing critical theorems. In [14], Zhang and Wei obtained the existence of infinitely many solutions via variant fountain theorem established in Zou [40] when the nonlinearity $f(x,u)$ involves a combination of superlinear and asymptotically linear terms. In [19], Pu et al. proved the existence and multiplicity of solutions for the fourth-order Navier boundary value problem with indefinite nonlinearity, applying the least action principle, the Ekeland variational principle and the mountain pass theorem.

Recently, the problems in the whole space $\mathbb{R}^N$ were considered in some works. For example, see [20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. In [23], Chabrowski and Marcos do Ó studied the following fourth-order elliptic problem involving critical growth

$$\begin{cases} \Delta^2 u - \lambda g(x)u = f(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \in D^{2,2}(\mathbb{R}^N)\backslash\{0\}, \end{cases} \tag{1.3}$$

where $\lambda > 0$, $p = \frac{2N}{N-4}(N > 4)$ is the critical Sobolev exponent, the coefficient $f(x)$ is a continuous bounded function varying in sign and $g \in L^{\frac{N}{4}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\backslash\{0\}$ is a nonnegative locally Hölder continuous function. Existence and multiplicity of solutions for problem (1.3) were obtained by variational method. In [29], the authors dealt with the fourth-order problem

$$\begin{cases} \varepsilon^4\Delta^2 u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where $\varepsilon > 0$, $N \geq 5$, and $V : \mathbb{R}^N \to \mathbb{R}$ is such that there exists a bounded domain $\Omega \subset \mathbb{R}^N$ and $x_0 \in \Omega$ with $0 < V(x_0) = \inf_{\mathbb{R}^N} V < \inf_{\partial\Omega} V$. A family of solutions was proved to exist and to concentrate at a point in the limit. For other related results on sublinear case, we refer the readers to [26, 27, 28] and the references therein.

More recently, there are a lot of papers devoted to the study of existence and multiplicity of solutions for problem (1.1) when $f(x,u)$ is superquadratic at infinity in u, see for instance [24, 25, 26, 27]. We would like to point out that, on the one hand, the papers in [24, 25] assumed the following classic Ambrosetti-Rabinowitz condition, i.e.,

(S_1) there exists $\mu > 2$ such that

(1.4) $$\mu F(x,u) \le uf(x,u),\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where $F(x,u) = \int_0^u f(x,s)ds$.

It is well known that the role of (1.4) is to ensure the boundedness of the (PS) sequences of the energy functional. Later, the papers in [26, 27] considered more general assumption than (S_1) condition, i.e.,

(S_2) $\frac{F(x,u)}{|u|^2} \to \infty$ as $|u| \to \infty$ uniformly in x, and the (PS) condition is replaced by the Cerami $(C)_c$ condition. Hence, the results in [26, 27] unified and generalized the results in [24, 25], respectively. On the other hand, the above papers always assumed the potential V is positive. However, to the best of our knowledge, for the sign-changing potential case, there are no previous results for problem (1.1). For the potential sign-changing case in semilinear Schrődinger equations, we refer the readers to [32, 33, 34, 35, 36, 37] and the references therein.

Motivated by the above fact, we continue to consider the superquadratic case with sign-changing potential, and establish the existence of infinitely many solutions by symmetric Mountain Pass Theorem in [39]. More precisely, we make the following assumptions:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V(x) > -\infty$;

(V_2) there exists a constant $d_0 > 0$ such that

$$\lim_{|y|\to\infty} meas\left(\{x \in \mathbb{R}^N : |x-y| \le d_0, V(x) \le M\}\right) = 0,\ \forall M > 0,$$

where $meas(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$;

(F_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist constants $c_1, c_2 > 0$ and $p \in (2, 2_*)$ such that

$$|f(x,u)| \le c_1|u| + c_2|u|^{p-1},\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where $2_* = +\infty$ if $N \le 4$ and $2_* = \frac{2N}{N-4}$ if $N > 4$;

(F_2) $\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^2} = \infty$, $a.e.\ x \in \mathbb{R}^N$, and there exists $r_0 \ge 0$ such that

$$F(x,u) \ge 0,\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},\ |u| \ge r_0;$$

(F_3) $\mathcal{F}(x,u) = \frac{1}{2}f(x,u)u - F(x,u) \ge 0$, and there exist $c_3 > 0$ and $\kappa > \max\{1, N/4\}$ such that

$$|F(x,u)|^\kappa \le c_3|u|^{2\kappa}\mathcal{F}(x,u),\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},\ |u| \ge r_0;$$

(F_4) there exist $\mu > 2$ and $\varrho > 0$ such that

$$\mu F(x,u) \le uf(x,u) + \varrho u^2\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R};$$

(F_5) $f(x,-u)=-f(x,u),\ \forall(x,u)\in\mathbb{R}^N\times\mathbb{R}$.

The main result of this paper are the following theorems.

Theorem 1.1. *Suppose that* (V_1)-(V_2), (F_1)-(F_3) *and* (F_5) *are satisfied. Then problem* (1.1) *has infinitely many solutions* $\{u_k\}$ *such that*

$$\frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}}(|\Delta u_k|^2+|\nabla u_k|^2+V(x)u_k^2)dx-\int_{\mathbb{R}^{\mathbb{N}}}F(x,u_k)dx\to\infty,\ \textit{as } k\to\infty.$$

Theorem 1.2. *Suppose that* (V_1)-(V_2), (F_1)-(F_2) *and* (F_4)-(F_5) *are satisfied. Then problem* (1.1) *has infinitely many solutions* $\{u_k\}$ *such that*

$$\frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}}(|\Delta u_k|^2+|\nabla u_k|^2+V(x)u_k^2)dx-\int_{\mathbb{R}^{\mathbb{N}}}F(x,u_k)dx\to\infty,\ \textit{as } k\to\infty.$$

Remark 1.3. Conditions like (V_1) and (V_2) have been given in [30] and [31], but there $\inf_{\mathbb{R}^N}V(x)>0$ is required. In the present paper, the potential $V(x)$ can be allowed to be sign-changing, which is weaker than the condition in [24, 26].

Remark 1.4. We note that the usual condition $\frac{f(x,u)}{u}\to 0$ as $u\to 0$ is not needed in our Theorems. Besides, the condition (F_2) is weaker than the conditions (S_1) and (S_2). From this, we can see that the condition (F_2) is the weakest superquadratic condition. Moreover, it is not difficult to find out that the following two functions $f(x,u)$ satisfy assumptions (F_1)-(F_5), for example:

$$f(x,u)=a(x)u\ln(\frac{1}{2}+u),$$

and

$$f(x,u)=a(x)[4|u|^3u+2u^2\sin u-4u\cos u],$$

where $a\in C(\mathbb{R}^N,\mathbb{R})$ and $0<\inf_{\mathbb{R}^N}a(x)\le\sup_{\mathbb{R}^N}a(x)<\infty$.

From Remarks 1.3 and 1.4, we know that Theorems 1.1 and 1.2 improve and generalize the results in [24, 26] by weakening the corresponding conditions.

2. Variational Setting and Proof of the Main Result

Before establishing the variational setting for our problem (1.1), we have the following

Remark 2.1. From (V_1), we know that there exists a constant $V_0>0$ such that $\bar{V}(x):=V(x)+V_0$ for all $x\in\mathbb{R}^N$. Let $\bar{f}(x,u):=f(x,u)+V_0u$ and consider the following new equation

$$\begin{cases}\Delta^2u-\Delta u+\bar{V}(x)u=\bar{f}(x,u), & x\in\mathbb{R}^N,\\ u\in H^2(\mathbb{R}^N).\end{cases}\tag{2.1}$$

Then problem (2.1) is equivalent to the problem (1.1). It is easy to check that the hypotheses (V_1), (V_2) and $(F_1 - F_5)$ still hold for $\bar{V}$ and $\bar{f}$ provided that those hold for V and f.

In view of Remark 2.1, now we will study the equivalent problem (2.1). Throughout the following sections, we make the following assumption instead of (V_1):

(V_1') $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V(x) > 0$.

We work in the Hilbert space

$$E = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)dx < +\infty\},$$

equipped with the inner product

$$(u, v) = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \cdot \nabla v + V(x)uv)dx,\ u, v \in E,$$

and the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2)dx\right)^{\frac{1}{2}},\ u \in E.$$

Evidently, E is continuously embedded into $H^2(\mathbb{R}^N)$ and hence continuously embedded into $L^s(\mathbb{R}^N)$ for $2 \le s < 2_*$, $i.e.$ there exists $\gamma_s > 0$ such that

$$\|u\|_s \le \gamma_s \|u\|,\ \forall\ u \in E, \tag{2.2}$$

where $\|u\|_s$ denotes the usual norm in $L^s(\mathbb{R}^N)$ for all $2 \le s < 2_*$. Motivated by Lemma 3.4 in [41], we can prove the following Lemma 2.2 in the same way. Here we omit it.

Lemma 2.2. *Under assumptions (V_1') and (V_2), the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is compact for any $s \in [2, 2_*)$.*

For each $u \in E$, we define

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N} F(x, u)dx. \tag{2.3}$$

Then we have the following lemma

Lemma 2.3. *If assumptions (V_1'), (V_2) and (F_1) hold, then $\Phi \in C^1(E, \mathbb{R})$ and*

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} [\Delta u \Delta v + \nabla u \cdot \nabla v + V(x)uv]dx - \int_{\mathbb{R}^N} f(x, u)vdx, \tag{2.4}$$

for all $u, v \in E$. *Moreover,* $\Psi' : E \to E^*$ *is compact, where* $\Psi(u) = \int_{\mathbb{R}^N} F(x,u)dx$.

Proof. To prove $\Phi \in C^1(E,\mathbb{R})$ and (2.4), it is sufficient to show that $\Psi \in C^1(E,\mathbb{R})$ and

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)vdx,\ \forall\ u,v \in E$$

First, we prove the existence of the Gateaux derivative of Ψ. By (F_1), we know

$$|F(x,u)| \le \frac{c_1}{2}|u|^2 + \frac{c_2}{p}|u|^p,\ \forall\ (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.5}$$

For any $u, v \in E$ and $0 < |t| < 1$, by mean value theorem and (F_1), there exists $0 < \theta < 1$ such that

$$\begin{aligned}\frac{|F(x,u+tv) - F(x,u)|}{|t|} &= |f(x,u+\theta tv)v| \\ &\le c_1|u+\theta tv||v| + c_2|u+\theta tv|^{p-1}|v| \\ &\le c_1|u||v| + c_1|v|^2 + c_2|u+\theta tv|^{p-1}|v| \\ &\le c_1|u||v| + c_1|v|^2 + 2^{p-1}c_2(|u|^{p-1}|v| + |v|^p).\end{aligned}$$

The Hölder inequality implies that

$$g(x) := c_1|u(x)||v(x)| + c_1|v(x)|^2 + 2^{p-1}c_2(|u(x)|^{p-1}|v(x)| + |v(x)|^p) \in L^1(\mathbb{R}^N).$$

Consequently, by the Lebesgue's Dominated Theorem, we have

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)vdx,\ \forall\ u,v \in E.$$

Next, we show that $\Psi' : E \to E^*$ is weak continuous. Assume that $u_n \rightharpoonup u$ in E, by Lemma 2.2, we get

$$u_n \to u \text{ in } L^s(\mathbb{R}^N),$$

for any $s \in [2, 2_*)$. Note that

$$\begin{aligned}\|\Psi'(u_n) - \Psi'(u)\|_{E^*} &= \sup_{\|v\|\le 1} |\langle \Psi'(u_n) - \Psi'(u), v\rangle| \\ &\le \sup_{\|v\|\le 1} \int_{\mathbb{R}^N} |f(x,u_n) - f(x,u)||v|dx.\end{aligned}$$

By the Hölder inequality and Theorem $A.4$ in [42], we have

$$\sup_{\|v\|\le 1} \int_{\mathbb{R}^N} |f(x,u_n) - f(x,u)||v|dx \to 0 \text{ as } n \to \infty.$$

Hence,

$$\|\Psi'(u_n) - \Psi'(u)\|_{E^*} \to 0 \text{ as } n \to \infty,$$

and then Ψ' is weakly continuous. Consequently, Ψ' is continuous and $\Psi \in C^1(E, \mathbb{R})$. Due to the form of Φ in (2.3), Φ' is also continuous and hence $\Phi \in C^1(E, \mathbb{R})$. ∎

We say that $I \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition if any sequence $\{u_n\}$ such that

$$I(u_n) \to c, \ \|I'(u_n)\|(1 + \|u_n\|) \to 0 \tag{2.6}$$

has a convergent subsequence.

Lemma 2.4. ([38, 39]). *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $I \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$, and*

- (I_1) $I(0) = 0, I(-u) = I(u)$ *for all* $u \in X$;
- (I_2) *there exist constants* $\rho, \alpha > 0$ *such that* $\Phi|_{\partial B_\rho \cap Z} \geq \alpha$;
- (I_3) *for any finite dimensional subspace* $\tilde{X} \subset X$, *there exists* $R = R(\tilde{X}) > 0$ *such that* $I(u) \leq 0$ *on* $\tilde{X} \setminus B_R$.

then I possesses an unbounded sequence of critical values.

Lemma 2.5. *Under assumptions (V_1'), (V_2) and $(F_1) - (F_3)$, any sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c > 0, \ \langle \Phi'(u_n), u_n \rangle \to 0 \tag{2.7}$$

is bounded in E.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, assume that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and $\|v_n\|_s \leq \gamma_s \|v_n\| = \gamma_s$ for $2 \leq s < 2_*$. Observe that for n large

$$c + 1 \geq \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n \rangle = \int_{\mathbb{R}^N} \mathcal{F}(x, u_n) dx. \tag{2.8}$$

From (2.3) and the form of $\|u\|$, we have

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u) dx, \ \forall \ u \in E, \tag{2.9}$$

and

$$\langle \Phi'(u), v \rangle = (u, v) - \int_{\mathbb{R}^N} f(x, u) v dx, \ \forall \ u, v \in E. \tag{2.10}$$

Combining (2.7) with (2.9), we get

$$\frac{1}{2} \leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{|F(x,u_n)|}{\|u_n\|^2} dx. \tag{2.11}$$

For $0 \leq a < b$, let

$$\Omega_n(a,b) = \{x \in \mathbb{R}^N : a \leq |u_n(x)| < b\}. \tag{2.12}$$

Passing to a subsequence, we may assume that $v_n \rightharpoonup v_1$ in E, then by Lemma 2.2, $v_n \to v_1$ in $L^s(\mathbb{R}^N)$ for all $s \in [2, 2_*)$, and $v_n(x) \to v_1(x)$ $a.e.$ on $\mathbb{R}^N$.

If $v_1 = 0$, then $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in [2, 2_*)$, and $v_n \to 0$ $a.e.$ on $\mathbb{R}^N$. Hence, it follows from (2.5) that

$$\begin{aligned}\int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 dx &\leq (\frac{c_1}{2} + \frac{c_2 r_0^{p-2}}{p}) \int_{\Omega_n(0,r_0)} |v_n|^2 dx \\ &\leq (\frac{c_1}{2} + \frac{c_2 r_0^{p-2}}{p}) \int_{\mathbb{R}^N} |v_n|^2 dx \\ &\to 0.\end{aligned} \tag{2.13}$$

Set $\kappa' = \frac{\kappa}{\kappa - 1}$. Since $\kappa > \max\{1, N/4\}$, we get $2\kappa' \in (2, 2_*)$. Hence, from (F_3) and (2.8), we have

$$\begin{aligned}&\int_{\Omega_n(r_0,\infty)} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 dx \\ &\leq \left(\int_{\Omega_n(r_0,\infty)} (\frac{|F(x,u_n)|}{|u_n|^2})^\kappa dx\right)^{\frac{1}{\kappa}} \left(\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'} dx\right)^{\frac{1}{\kappa'}} \\ &\leq c_4^{\frac{1}{\kappa}} \left(\int_{\Omega_n(r_0,\infty)} \mathcal{F}(x,u_n) dx\right)^{\frac{1}{\kappa}} \left(\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'} dx\right)^{\frac{1}{\kappa'}} \\ &\leq [c_4(c+1)]^{\frac{1}{\kappa}} \left(\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'} dx\right)^{\frac{1}{\kappa'}} \\ &\to 0.\end{aligned} \tag{2.14}$$

Combining (2.13) with (2.14), we get

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|F(x,u_n)|}{\|u_n\|^2} dx &= \int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 dx + \int_{\Omega_n(r_0,\infty)} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 dx \\ &\to 0,\end{aligned}$$

which contradicts (2.11).

If $v_1 \neq 0$, set

$$A := \{x \in \mathbb{R}^N : v_1(x) \neq 0\},$$

then $meas(A) > 0$. For $a.e.x \in A$, we have

$$\lim_{n\to\infty} |u_n(x)| = \infty.$$

Hence,

$$A \subset \Omega_n(r_0, \infty) \text{ for large } n \in \mathbb{N},$$

it follows from (2.5), (2.9), (F_2) and Fatou's Lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|u_n\|^2} = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^2} \\
&= \lim_{n\to\infty} [\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{u_n^2} v_n^2 dx] \\
&= \lim_{n\to\infty} [\frac{1}{2} - \int_{\Omega_n(0,r_0)} \frac{F(x, u_n)}{u_n^2} v_n^2 dx - \int_{\Omega_n(r_0,\infty)} \frac{F(x, u_n)}{u_n^2} v_n^2 dx] \\
&\leq \limsup_{n\to\infty} [\frac{1}{2} + (\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}) \int_{\mathbb{R}^N} |v_n|^2 dx - \int_{\Omega_n(r_0,\infty)} \frac{F(x, u_n)}{u_n^2} v_n^2 dx] \\
&\leq \frac{1}{2} + (\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2})\gamma_2^2 - \liminf_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{F(x, u_n)}{u_n^2} v_n^2 dx \\
&= \frac{1}{2} + (\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2})\gamma_2^2 - \liminf_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{F(x, u_n)}{u_n^2} [\chi_{\Omega_n(r_0,\infty)}(x)] v_n^2 dx \\
&\leq \frac{1}{2} + (\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2})\gamma_2^2 - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x, u_n)}{u_n^2} [\chi_{\Omega_n(r_0,\infty)}(x)] v_n^2 dx \\
&= -\infty,
\end{aligned} \tag{2.15}$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E. ∎

Lemma 2.6. *Under assumptions* (V_1'), (V_2) *and* $(F_1) - (F_3)$, *any sequence* $\{u_n\} \subset E$ *satisfying* (2.7) *has a convergent subsequence in* E.

Proof. From Lemma 2.5, we know that $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in E. By Lemma 2.2, $u_n \to u$ in $L^s(\mathbb{R}^N)$ for all $2 \leq s < 2_*$, thus

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |f(x,u_n)-f(x,u)||u_n-u|dx \\
\le\ & \int_{\mathbb{R}^N} (|f(x,u_n)|+|f(x,u)|)|u_n-u|dx \\
\le\ & \int_{\mathbb{R}^N} \left[(c_1|u_n|+c_2|u_n|^{p-1})+(c_1|u|+c_2|u|^{p-1})\right]|u_n-u|dx \\
\le\ & c_1\left(\int_{\mathbb{R}^N}(|u_n|+|u|)^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|u_n-u|^2dx\right)^{\frac{1}{2}} \\
&+c_2\left(\int_{\mathbb{R}^N}|u_n|^pdx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}|u_n-u|^pdx\right)^{\frac{1}{p}} \\
&+c_2\left(\int_{\mathbb{R}^N}|u|^pdx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}|u_n-u|^pdx\right)^{\frac{1}{p}} \\
\to\ & 0, \text{ as } n\to\infty.
\end{aligned}
\tag{2.16}
$$

Observe that

$$
\|u_n-u\|^2=\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle+\int_{\mathbb{R}^N}[f(x,u_n)-f(x,u)](u_n-u)dx. \tag{2.17}
$$

It is clear that

$$
\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle\to 0 \text{ as } n\to\infty. \tag{2.18}
$$

From (2.16), (2.17) and (2.18), we get

$$
\|u_n-u\|\to 0 \text{ as } n\to\infty.
$$

■

Lemma 2.7. *Under assumptions* (V_1'), (V_2), (F_1), (F_2) *and* (F_4), *any sequence* $\{u_n\}\subset E$ *satisfying* (2.7) *has a convergent subsequence in* E.

Proof. First, we prove that $\{u_n\}$ is bounded in E. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\|\to\infty$. Let $v_n=\frac{u_n}{\|u_n\|}$. Then $\|v_n\|=1$ and $\|v_n\|_s\le\gamma_s\|v_n\|=\gamma_s$ for all $2\le s<2_*$. By (2.7), (2.9), (2.10) and (F_4), we have

$$
\begin{aligned}
c+1 &\ge \Phi(u_n)-\frac{1}{\mu}\langle\Phi'(u_n),u_n\rangle \\
&=\frac{\mu-2}{2\mu}\|u_n\|^2+\int_{\mathbb{R}^N}[\frac{1}{\mu}f(x,u_n)u_n-F(x,u_n)]dx \\
&\ge\frac{\mu-2}{2\mu}\|u_n\|^2-\frac{\varrho}{\mu}\|u_n\|_2^2 \text{ for large } n\in\mathbb{N},
\end{aligned}
\tag{2.19}
$$

which implies

$$1 \le \frac{2\varrho}{\mu - 2} \limsup_{n\to\infty} \|v_n\|_2^2. \tag{2.20}$$

Passing to a subsequence, we may assume that $v_n \rightharpoonup v_1$ in E, then by Lemma 2.2, $v_n \to v_1$ in $L^s(\mathbb{R}^N)$ for all $2 \le s < 2_*$, and $v_n(x) \to v_1(x)$ $a.e.$ on $\mathbb{R}^N$. Hence, it follows from (2.20) that $v_1 \neq 0$. By a similar fashion as (2.15), we can conclude a contradiction. Thus, $\{u_n\}$ is bounded in E. The rest proof is the same as that in Lemma 2.6. ∎

Lemma 2.8. *Under assumptions (V_1'), (V_2), (F_1) and (F_2), for any finite dimensional subspace $\tilde{E} \subset E$, there holds*

$$\Phi(u) \to -\infty,\ \|u\| \to \infty,\ u \in \tilde{E}. \tag{2.21}$$

Proof. Arguing indirectly, assume that for some sequence $\{u_n\} \subset \tilde{E}$ with $\|u_n\| \to \infty$, there exists $M > 0$ such that $\Phi(u_n) \ge -M$ for all $n \in \mathbb{N}$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v_1$ in E. Since $\tilde{E}$ is finite dimensional, then $v_n \to v_1 \in \tilde{E}$ in E, $v_n(x) \to v_1(x)$ $a.e.$ on $\mathbb{R}^N$, and so $\|v_1\| = 1$. Hence, we can conclude a contradiction by a similar fashion as (2.15). ∎

Corollary 2.9. *Under assumptions (V_1'), (V_2), (F_1) and (F_2), for any finite dimensional subspace $\tilde{E} \subset E$, there exists $R = R(\tilde{E}) > 0$, such that*

$$\Phi(u) \le 0,\ \forall u \in \tilde{E},\ \|u\| \ge R. \tag{2.22}$$

Let $\{e_j\}$ is a total orthonormal basis of E and define

$$X_j = \mathbb{R}e_j,\ Y_k = \bigoplus_{j=1}^{k} X_j,\ Z_k = \bigoplus_{j=k+1}^{\infty} X_j\ k \in \mathbb{Z}. \tag{2.23}$$

Lemma 2.10. *Under assumptions (V_1') and (V_2), for $2 \le s < 2_*$,*

$$\beta_k(s) := \sup_{u \in Z_k, \|u\|=1} \|u\|_s \to 0,\ k \to \infty. \tag{2.24}$$

Proof. It is clear that $0 < \beta_{k+1} \le \beta_k$, so that $\beta_k \to \beta \ge 0 (k \to \infty)$. For every $k \in \mathbb{N}$, there exists $u_k \in Z_k$ with $\|u_k\| = 1$ such that $|u_k|_2 > \frac{\beta_k}{2}$. For any $v \in E$, writing $v = \sum_{j=1}^{\infty} c_j e_j$, we have, by the Cauchy-Schwartz inequality,

$$|(u_k, v)| = |(u_k, \sum_{j=1}^{\infty} c_j e_j)| = |(u_k, \sum_{j=k}^{\infty} c_j e_j)| \le \|u_k\| \cdot \|\sum_{j=k}^{\infty} c_j e_j\| = (\sum_{j=k}^{\infty} c_j^2)^{\frac{1}{2}} \to 0$$

as $k \to \infty$, which implies that $u_k \rightharpoonup 0$ in E. By Lemma 2.2, the compact embedding of $E \hookrightarrow L^s(\mathbb{R}^N)$ $(2 \le s < 2_*)$ implies that $u_k \to 0$ in $L^s(\mathbb{R}^N)$. Hence, letting $k \to \infty$, we get $\beta = 0$, which completes the proof. ■

By Lemma 2.10, we can choose an integer $m \ge 1$ such that

$$\|u\|_2^2 \le \frac{1}{2c_1}\|u\|^2, \ \|u\|_p^p \le \frac{p}{4c_2}\|u\|^p, \ \forall u \in Z_m. \tag{2.25}$$

Lemma 2.11. *Under assumptions* (V_1'), (V_2) *and* (F_1), *there exist constants* $\rho, \alpha > 0$ *such that* $\Phi|_{\partial B_\rho \cap Z_m} \ge \alpha$.

Proof. Combining (2.5), (2.9) with (2.25), for $u \in Z_m$, choosing $\rho := \|u\| = \frac{1}{2}$ we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \\
&\ge \frac{1}{2}\|u_n\|^2 - \frac{c_1}{2}\|u_n\|_2^2 - \frac{c_2}{p}\|u_n\|_p^p \\
&\ge \frac{1}{4}(\|u_n\|^2 - \|u_n\|^p) \\
&= \frac{2^{p-2}-1}{2^{p+2}} := \alpha > 0.
\end{aligned}$$

Thus, the proof is complete. ■

Proof of Theorem 1.1. Let $X = E, Y = Y_m$ and $Z = Z_m$. Obviously, $\bar{f}$ satisfies $(F_1) - (F_3)$, and (F_5). By Lemmas 2.5, 2.6, 2.11 and Corollary 2.9, all conditions of Lemma 2.4 are satisfied. Thus, problem (2.1) possesses infinitely many nontrivial solutions. By Remark 2.1, problem (1.1) also possesses infinitely many nontrivial solutions. ■

Proof of Theorem 1.2. Let $X = E, Y = Y_m$ and $Z = Z_m$. Obviously, $\bar{f}$ satisfies (F_1), (F_2), (F_4) and (F_5). The rest proof is the same as that of Theorem 1.1 by using Lemma 2.7 instead of Lemmas 2.5 and 2.6. ■

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