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A PROBABILISTIC VERSION OF MEHLER'S FORMULA

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Abstract. In this paper, we use Lebesgue's dominated convergence theorem to give a probabilistic version of Mehler's formula for the well-known Rogers-Szegö polynomials. Applications of the probabilistic version are also given.

1. Introduction

Probabilistic method is a useful tool in the study of basic hypergeometric functions. There are some works available in the literature [4, 7, 11, 22, 23, 24]. In the present paper, we give a probabilistic version of Mehler's formula for the Rogers-Szegö polynomials.

We first recall some definitions, notation and known results in [1, 8, 16] which will be used in this paper. Throughout the whole paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.1}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, ..., a_m; q)_n = (a_1;q)_n(a_2;q)_n...(a_m;q)_n, \tag{1.2}$$

where n is an integer or ∞. The q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{1.3}$$

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The q-binomial theorem

$$\sum_{n=0}^{\infty}\frac{(a;q)_n x^n}{(q;q)_n}=\frac{(ax;q)_\infty}{(x;q)_\infty},\quad |x|<1, \tag{1.4}$$

and

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n}=(x;q)_\infty. \tag{1.5}$$

In [22, 23], the author established the following discrete probability distribution $\mathrm{W}(x;q)$:

$$\mathrm{p}(\xi=x^n q^k)=\frac{(-x)^n(x^{n-1}q^{k+1},x^n q^{k+1};q)_\infty q^k}{(q,q/x,x;q)_\infty}, \tag{1.6}$$

where $x<0$; $0<q<1$; $n=0,1$; $k=0,1,2,\cdots$, and gave some applications of this distribution in q-series. q-type distributions play an important role in applications. Various q-type distributions have appeared in physics literatures in the recent years [5, 6, 12, 13, 17]. In this paper, the probability distribution $\mathrm{W}(x;q)$ play an important role. The following formulas are given in [22]: let $-1<x<0$, $|a|<1, |b|<1$ and ξ denote a random variable having distribution $\mathrm{W}(x;q)$, then

$$\mathrm{E}\left\{\frac{\xi^m}{(a\xi;q)_\infty}\right\}=\frac{1}{(a,ax;q)_\infty}\,\varphi_m^{(a)}(x|q),\quad m=0,1,2,\cdots, \tag{1.7}$$

and

$$\mathrm{E}\left\{\frac{1}{(a\xi,b\xi;q)_\infty}\right\}=\frac{(abx;q)_\infty}{(a,b,ax,bx;q)_\infty}. \tag{1.8}$$

The Rogers-Szegö polynomials are defined as:

$$h_n(x|q)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}x^k. \tag{1.9}$$

The Rogers-Szegö polynomials play an important role in the theory of orthogonal polynomials, particularly in the study of the Askey-Wilson integral [2, 9, 10]. One of the important formulas for the Rogers-Szegö polynomials is Mehler's formula,

$$\sum_{n=0}^{\infty}h_n(x|q)h_n(y|q)\frac{s^n}{(q;q)_n}=\frac{(xys^2;q)_\infty}{(s,xs,ys,xys;q)_\infty}. \tag{1.10}$$

The Rogers-Szegö polynomials are the $a=0$ case of the Al-Salam-Carlitz polynomials $\varphi_n^{(a)}(x|q)$, which are defined as [19, 21]

$$\varphi_n^{(a)}(x|q)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}x^k(a;q)_k. \tag{1.11}$$

We also need to use the well-known theorems in this paper:

Analytic continuation theorem: If f and g are analytic at z_0 and agree at infinitely many points which include z_0 as an accumulation point, then $f = g$.

Lebesgue's dominated convergence theorem: Suppose that $\{X_n, n \geq 1\}$ is a sequence of random variables, that $X_n \to X$ pointwise almost everywhere as $n \to \infty$, and that $|X_n| \leq Y$ for all n, where the random variable Y is integrable. Then X is integrable, and

$$\lim_{n\to\infty} \mathrm{E}X_n = \mathrm{E}X. \tag{1.12}$$

Tannery's theorem [20] is a special case of Lebesgue's dominated convergence theorem on the sequence space L^1.

Tannery's theorem: If $s(n) = \sum\limits_{k\geq 0} f_k(n)$ is a finite sum (or a convergent series) for each n, $\lim\limits_{n\to\infty} f_k(n) = f_k$, $|f_k(n)| \leq M_k$, and $\sum\limits_{k=0}^{\infty} M_k < \infty$, then

$$\lim_{n\to\infty} s(n) = \sum_{k=0}^{\infty} f_k. \tag{1.13}$$

2. A Probabilistic Version of Mehler's Formula

The main result of this paper is the following expectation formula (2.1), which gives the relationship between the probability distribution $\mathrm{W}(x;q)$ and the Al-Salam-Carlitz polynomials $\varphi_n^{(a)}(x|q)$. In the next section, we will point out Mehler's formula is only a special case of the expectation formula given in this section. So, the expectation formula (2.1) can be thought of the probabilistic version of Mehler's formula. In fact, (2.1) tell us more than Mehler's formula.

Theorem 2.1. *Suppose ξ and η denote two independent random variables having distributions $W(x;q)$ and $W(y;q)$ respectively, then*

$$\begin{aligned} &E\left\{\frac{\xi^m\eta^n}{(a\xi, b\eta, s\xi\eta; q)_\infty}\right\} \\ &= \frac{1}{(a, ax, b, by; q)_\infty} \sum_{k=0}^{\infty} \varphi_{k+m}^{(a)}(x|q)\varphi_{k+n}^{(b)}(y|q)\frac{s^k}{(q;q)_k}, \end{aligned} \tag{2.1}$$

where $-1 < x, y < 0$ and $-1 < a, b, s < 1$.

Proof. Letting $a = 0, x = s\xi\eta$ in (1.4) gets

$$\sum_{k=0}^{\infty} \frac{(s\xi\eta)^k}{(q;q)_k} = \frac{1}{(s\xi\eta;q)_\infty}, \quad |s| < 1. \tag{2.2}$$

Multiplying both sides of (2.2) by $\frac{\xi^m\eta^n}{(a\xi,b\eta;q)_\infty}$, we obtain

$$\sum_{k=0}^{\infty}\frac{\xi^{k+m}}{(a\xi;q)_\infty}\frac{\eta^{k+n}}{(b\eta;q)_\infty}\frac{s^k}{(q;q)_k}=\frac{\xi^m\eta^n}{(a\xi,b\eta,s\xi\eta;q)_\infty}. \tag{2.3}$$

Applying the expectation operator E on both sides of (2.3), it follows that,

$$\mathrm{E}\left\{\sum_{k=0}^{\infty}\frac{\xi^{k+m}}{(a\xi;q)_\infty}\frac{\eta^{k+n}}{(b\eta;q)_\infty}\frac{s^k}{(q;q)_k}\right\}=\mathrm{E}\left\{\frac{\xi^m\eta^n}{(a\xi,b\eta,s\xi\eta;q)_\infty}\right\}. \tag{2.4}$$

Under the conditions of the theorem, it is obvious that $|\xi|\le 1$ and $|\eta|\le 1$.

Since,

$$\left|\frac{\xi^{k+m}}{(a\xi;q)_\infty}\frac{\eta^{k+n}}{(b\eta;q)_\infty}\frac{s^k}{(q;q)_k}\right|\le\frac{1}{(|a|,|b|;q)_\infty}\cdot\frac{|s|^k}{(q;q)_k} \tag{2.5}$$

and $\sum_{k=0}^{\infty}\frac{|s|^k}{(q;q)_k}$ is a convergent series, using Lebesgue dominated convergence theorem and (1.7), we obtain the left hand side of (2.4),

$$\begin{aligned}
&\mathrm{E}\left\{\sum_{k=0}^{\infty}\frac{\xi^{k+m}}{(a\xi;q)_\infty}\frac{\eta^{k+n}}{(b\eta;q)_\infty}\frac{s^k}{(q;q)_k}\right\}\\
&=\sum_{k=0}^{\infty}\mathrm{E}\left\{\frac{\xi^{k+m}}{(a\xi;q)_\infty}\frac{\eta^{k+n}}{(b\eta;q)_\infty}\right\}\frac{s^k}{(q;q)_k}\\
&=\sum_{k=0}^{\infty}\mathrm{E}\left\{\frac{\xi^{k+m}}{(a\xi;q)_\infty}\right\}\mathrm{E}\left\{\frac{\eta^{k+n}}{(b\eta;q)_\infty}\right\}\frac{s^k}{(q;q)_k}\\
&=\frac{1}{(a,ax,b,by;q)_\infty}\sum_{k=0}^{\infty}\varphi_{k+m}^{(a)}(x|q)\varphi_{k+n}^{(b)}(y|q)\frac{s^k}{(q;q)_k}.
\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.4) gives (2.1). ∎

3. Some Applications

The expectation formula (2.1) can be used to derive identities and transformation formulas about the Rogers-Szegö polynomials. In this section, we give some applications of it. We can easily get Mehler's formula from (2.1).

Theorem 3.1. *Let $|x|<1,|y|<1$ and $|s|<1$, then*

$$\sum_{k=0}^{\infty}h_k(x|q)h_k(y|q)\frac{s^k}{(q;q)_k}=\frac{(xys^2;q)_\infty}{(s,xs,ys,xys;q)_\infty}. \tag{3.1}$$

Proof. Let $a = b = 0$ and $m = n = 0$ in (2.1), we get

$$\sum_{k=0}^{\infty} h_k(x|q)h_k(y|q)\frac{s^k}{(q;q)_k} = \mathrm{E}\left\{\frac{1}{(s\xi\eta;q)_\infty}\right\}. \tag{3.2}$$

Using (1.7) and (1.8), we calculate the right hand side of (3.2),

$$\begin{aligned} &\mathrm{E}\left\{\frac{1}{(s\xi\eta;q)_\infty}\right\} = \mathrm{E}\left\{\mathrm{E}\left[\frac{1}{(s\xi\eta;q)_\infty} \mid \xi\right]\right\} \\ &= \mathrm{E}\left\{\frac{1}{(s\xi, sy\xi;q)_\infty}\right\} = \frac{(xys^2;q)_\infty}{(s, xs, ys, xys;q)_\infty}. \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2) gives

$$\sum_{k=0}^{\infty} h_k(x|q)h_k(y|q)\frac{s^k}{(q;q)_k} = \frac{(xys^2;q)_\infty}{(s, xs, ys, xys;q)_\infty}, \tag{3.4}$$

where $-1 < x < 0$, $-1 < y < 0$ and $|s| < 1$. By analytic continuation, we may replace the assumption $-1 < x < 0$ and $-1 < y < 0$ by $|x| < 1$ and $|y| < 1$, so we get (3.1). ∎

Then, we use (2.1) to derive the following transformations for the Al-Salam-Carlitz polynomials.

Theorem 3.2. *Let* $|a| < 1$, $|b| < 1$, $|s| < 1$, $|x| < 1$, $|y| < 1$, *then*

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{k+m}^{(a)}(x|q)\varphi_{k+m}^{(b)}(y|q)\frac{s^{m+k}q^k}{(q;q)_k} \\ &= (1-a)(1-ax)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{k+m}^{(aq)}(x|q)\varphi_{k}^{(b)}(y|q)\frac{a^m s^k}{(q;q)_k} \\ &= (1-b)(1-by)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{k}^{(a)}(x|q)\varphi_{k+m}^{(bq)}(y|q)\frac{b^m s^k}{(q;q)_k}. \end{aligned} \tag{3.5}$$

Proof. Let ξ and η denote two independent random variables having distributions $W(x;q)$ and $W(y;q)$ respectively. Using Lebesgue dominated convergence theorem and (2.1), we obtain

$$\begin{aligned} &\mathrm{E}\left\{\frac{1}{(a\xi, b\eta, s\xi\eta;q)_\infty}\right\} = \mathrm{E}\left\{\frac{1}{(aq\xi, b\eta, s\xi\eta;q)_\infty(1-a\xi)}\right\} \\ &= \mathrm{E}\left\{\frac{1}{(aq\xi, b\eta, s\xi\eta;q)_\infty}\sum_{m=0}^{\infty} a^m\xi^m\right\} = \sum_{m=0}^{\infty}\mathrm{E}\left\{\frac{a^m\xi^m}{(aq\xi, b\eta, s\xi\eta;q)_\infty}\right\} \\ &= \frac{1}{(aq, aqx, b, by;q)_\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{k+m}^{(aq)}(x|q)\varphi_{k}^{(b)}(y|q)\frac{a^m s^k}{(q;q)_k}. \end{aligned} \tag{3.6}$$

On exchanging a and b and exchanging x and y in (3.6), gives

$$
\begin{aligned}
&\mathrm{E}\left\{\frac{1}{(a\xi, b\eta, s\xi\eta; q)_\infty}\right\} \\
(3.7)\qquad &= \frac{1}{(a, ax, bq, bqy; q)_\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\varphi_k^{(a)}(x|q)\varphi_{k+m}^{(bq)}(y|q)\frac{b^m s^k}{(q;q)_k},
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\mathrm{E}\left\{\frac{1}{(a\xi, b\eta, s\xi\eta; q)_\infty}\right\} = \mathrm{E}\left\{\frac{1}{(a\xi, b\eta, sq\xi\eta; q)_\infty(1 - s\xi\eta)}\right\} \\
(3.8)\qquad &= \mathrm{E}\left\{\frac{1}{(a\xi, b\eta, sq\xi\eta; q)_\infty}\sum_{m=0}^{\infty} s^m\xi^m\eta^m\right\} = \sum_{m=0}^{\infty}\mathrm{E}\left\{\frac{s^m\xi^m\eta^m}{(a\xi, b\eta, sq\xi\eta; q)_\infty}\right\} \\
&= \frac{1}{(a, ax, b, by; q)_\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{k+m}^{(a)}(x|q)\varphi_{k+m}^{(b)}(y|q)\frac{s^{m+k}q^k}{(q;q)_k}.
\end{aligned}
$$

The left hand sides of (3.6), (3.7) and (3.8) are equal, so are right. Thus, we obtain

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{k+m}^{(a)}(x|q)\varphi_{k+m}^{(b)}(y|q)\frac{s^{m+k}q^k}{(q;q)_k} \\
(3.9)\qquad &= (1-a)(1-ax)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{k+m}^{(aq)}(x|q)\varphi_k^{(b)}(y|q)\frac{a^m s^k}{(q;q)_k} \\
&= (1-b)(1-by)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\varphi_k^{(a)}(x|q)\varphi_{k+m}^{(bq)}(y|q)\frac{b^m s^k}{(q;q)_k},
\end{aligned}
$$

where $-1 < x < 0$, $-1 < y < 0$. By analytic continuation, we may replace the assumption $-1 < x < 0$ and $-1 < y < 0$ by $|x| < 1$ and $|y| < 1$, so we get (3.5). ■

Theorem 3.3. *Let* $|a| < 1$, $|b| < 1$, $|s| < 1$, $|x| < 1$, $|y| < 1$, *then*

$$
(3.10)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} s^{n+k}}{(q;q)_n(q;q)_k}\varphi_{k+n}^{(a)}(x|q)\varphi_{k+n}^{(b)}(y|q) = 1.
$$

Proof. Let ξ and η denote two independent random variables having distributions $W(x;q)$ and $W(y;q)$ respectively. Using the formula (1.8), the q-binomial theorem (1.5), Lebesgue dominated convergence theorem and (2.1), we obtain

$$
\begin{aligned}
&\frac{1}{(a, b, ax, by; q)_\infty} = \mathrm{E}\left\{\frac{1}{(a\xi; q)_\infty}\right\}\mathrm{E}\left\{\frac{1}{(b\eta; q)_\infty}\right\} \\
(3.11)\qquad &= \mathrm{E}\left\{\frac{(s\xi\eta; q)_\infty}{(a\xi, b\eta, s\xi\eta; q)_\infty}\right\} = \mathrm{E}\left\{\frac{1}{(a\xi, b\eta, s\xi\eta; q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} s^n\xi^n\eta^n}{(q;q)_n}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} s^n}{(q;q)_n}\mathrm{E}\left\{\frac{\xi^n\eta^n}{(a\xi, b\eta, s\xi\eta; q)_\infty}\right\}
\end{aligned}
$$

and

$$\begin{aligned}(3.12)\qquad & \mathrm{E}\left\{\frac{\xi^n\eta^n}{(a\xi, b\eta, s\xi\eta; q)_\infty}\right\} \\ &= \frac{1}{(a, ax, b, by; q)_\infty}\sum_{k=0}^{\infty}\varphi_{k+n}^{(a)}(x|q)\varphi_{k+n}^{(b)}(y|q)\frac{s^k}{(q;q)_k}.\end{aligned}$$

Substituting (3.12) into (3.11) gives

$$(3.13)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} s^{n+k}}{(q;q)_n(q;q)_k}\varphi_{k+n}^{(a)}(x|q)\varphi_{k+n}^{(b)}(y|q) = 1,$$

where $-1 < x < 0$, $-1 < y < 0$. By analytic continuation, we may replace the assumption $-1 < x < 0$ and $-1 < y < 0$ by $|x| < 1$ and $|y| < 1$, so we get (3.10). ■

When we consider the limiting behavior of the formula (2.1) as $m \to \infty$, we get the generating function of the Al-Salam-Carlitz polynomials $\varphi_n^{(a)}(x|q)$.

Theorem 3.4. *Let* $|y| < 1$, $|s| < 1$, *then*

$$(3.14)\qquad \textstyle\sum_{n=0}^{\infty}\varphi_n^{(b)}(y|q)\frac{s^n}{(q;q)_n} = \frac{(bsy;q)_\infty}{(s,sy;q)_\infty}.$$

Proof. Let $n = 0, m \to \infty$ in (2.1) gives

$$\begin{aligned}(3.15)\qquad & \lim_{m\to\infty}\mathrm{E}\left\{\frac{\xi^m}{(a\xi, b\eta, s\xi\eta; q)_\infty}\right\} \\ &= \lim_{m\to\infty}\frac{1}{(a, ax, b, by; q)_\infty}\sum_{k=0}^{\infty}\varphi_{k+m}^{(a)}(x|q)\varphi_k^{(b)}(y|q)\frac{s^k}{(q;q)_k}.\end{aligned}$$

In order to calculate the both side of (3.15), let ξ and η denote two independent random variables having distributions $W(x;q)$ and $W(y;q)$, respectively. Where, we set $-1 < x, y < 0$. For any positive integer m, we consider the following sequence of random variables (on a probability space):

$$(3.16)\qquad \left\{\frac{\xi^m}{(a\xi, b\eta, s\xi\eta; q)_\infty}\right\}_{m=1}^{\infty}, \quad |a| < 1.$$

It is easy to see

$$(3.17)\qquad \lim_{m\to\infty}\frac{\xi^m}{(a\xi, b\eta, s\xi\eta; q)_\infty} = \frac{I_{(\xi=1)}}{(a, b\eta, s\eta; q)_\infty},$$

where I_Ω is the indicator function defined by

$$I_\Omega(x)=\begin{cases}1, & if\ x\in\Omega\\ 0, & if\ x\notin\Omega.\end{cases}\tag{3.18}$$

Since,

$$\left|\frac{\xi^m}{(a\xi,b\eta,s\xi\eta;q)_\infty}\right|\le\frac{1}{(|a|,|b|,|s|;q)_\infty},\tag{3.19}$$

using Lebesgue's dominated convergence theorem and (1.7), we get the left hand side of (3.15)

$$\begin{aligned}&\lim_{m\to\infty}\mathrm{E}\left\{\frac{\xi^m}{(a\xi,b\eta,s\xi\eta;q)_\infty}\right\}=\mathrm{E}\left\{\frac{I_{(\xi=1)}}{(a,b\eta,s\eta;q)_\infty}\right\}\\ &=\frac{p(\xi=1)}{(a;q)_\infty}\mathrm{E}\left\{\frac{1}{(b\eta,s\eta;q)_\infty}\right\}=\frac{(bsy;q)_\infty}{(a,x,b,by,s,sy;q)_\infty}.\end{aligned}\tag{3.20}$$

On the other hand, by Tannery's theorem (1.13) and the q-binomial theorem (1.4) gives

$$\lim_{m\to\infty}\varphi_m^{(a)}(x|q)=\lim_{m\to\infty}\sum_{k=0}^{m}\begin{bmatrix}m\\k\end{bmatrix}x^k(a;q)_k=\sum_{k=0}^{\infty}\frac{(a;q)_kx^k}{(q;q)_k}=\frac{(ax;q)_\infty}{(x;q)_\infty}.\tag{3.21}$$

So, we get the right hand side of (3.15)

$$\begin{aligned}&\lim_{m\to\infty}\frac{1}{(a,ax,b,by;q)_\infty}\sum_{n=0}^{\infty}\varphi_{n+m}^{(a)}(x|q)\varphi_n^{(b)}(y|q)\frac{s^n}{(q;q)_n}\\ &=\frac{1}{(a,x,b,by;q)_\infty}\sum_{n=0}^{\infty}\varphi_n^{(b)}(y|q)\frac{s^n}{(q;q)_n}.\end{aligned}\tag{3.22}$$

Substituting (3.20) and (3.22) into (3.15) and using analytic continuation gives (3.14). ∎

Furthermore, by (2.1) we give an extension of the generating function of the Al-Salam-Carlitz polynomials $\varphi_n^{(a)}(x|q)$.

Theorem 3.5. *Let* $|x|<1$,$|y|<1$, $|a|<1$, $|b|<1$, $|t|<1$, *then*

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a,ay;q)_k}{(q;q)_k(q;q)_l}\varphi_{l+k}^{(b)}(x|q)\varphi_{l+n}^{(aq^k)}(y|q)a^lt^{k+l}=\frac{(tbx;q)_\infty}{(t,tx;q)_\infty}\varphi_n^{(a)}(y|q),\tag{3.23}$$

where n *is a nonnegative integer.*

Proof. Rewrite the q-binomial theorem (1.4) as

$$\sum_{k=0}^{\infty} \frac{1}{(q;q)_k} \frac{x^k}{(aq^k, ax;q)_\infty} = \frac{1}{(a,x;q)_\infty}. \tag{3.24}$$

First, substitute $t\xi$ and $a\eta$ for x and a, and then multiply $\frac{\eta^n}{(b\xi;q)_\infty}$, one obtains

$$\sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \frac{\xi^k \eta^n}{(b\xi, aq^k\eta, at\xi\eta;q)_\infty} = \frac{\eta^n}{(a\eta, t\xi, b\xi;q)_\infty}, \tag{3.25}$$

where $-1 < x, y < 0$, $|a| < 1$, $|b| < 1$, $|t| < 1$ and ξ and η denote two independent random variables having distributions $W(x;q)$ and $W(y;q)$, respectively. Applying the expectation operator E on both sides of (3.25), and using Lebesgue dominated convergence theorem, it follows that,

$$\sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \mathrm{E}\left\{ \frac{\xi^k \eta^n}{(b\xi, aq^k\eta, at\xi\eta;q)_\infty} \right\} = \mathrm{E}\left\{ \frac{\eta^n}{(a\eta, t\xi, b\xi;q)_\infty} \right\}. \tag{3.26}$$

Employing (2.1), (1.7) and (1.8) gives

$$\begin{aligned} &\mathrm{E}\left\{ \frac{\xi^k \eta^n}{(b\xi, aq^k\eta, at\xi\eta;q)_\infty} \right\} \\ &= \frac{1}{(b, bx, aq^k, aq^k y;q)_\infty} \sum_{l=0}^{\infty} \varphi_{l+k}^{(b)}(x|q) \varphi_{l+n}^{(aq^k)}(y|q) \frac{(at)^l}{(q;q)_l}, \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} &\mathrm{E}\left\{ \frac{\eta^n}{(a\eta, t\xi, b\xi;q)_\infty} \right\} = \mathrm{E}\left\{ \frac{\eta^n}{(a\eta;q)_\infty} \right\} \mathrm{E}\left\{ \frac{1}{(t\xi, b\xi;q)_\infty} \right\} \\ &= \frac{\varphi_n^{(a)}(y|q)}{(a, ay;q)_\infty} \frac{(tbx, ;q)_\infty}{(t, b, tx, bx;q)_\infty}. \end{aligned} \tag{3.28}$$

Substituting (3.27) and (3.28) into (3.26) gives

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a, ay;q)_k}{(q;q)_k (q;q)_l} \varphi_{l+k}^{(b)}(x|q) \varphi_{l+n}^{(aq^k)}(y|q) a^l t^{k+l} = \frac{(tbx;q)_\infty}{(t, tx;q)_\infty} \varphi_n^{(a)}(y|q), \tag{3.29}$$

where $-1 < x < 0$, $-1 < y < 0$ and $|a| < 1$, $|t| < 1$. By analytic continuation, we may replace the assumption $-1 < x < 0$ and $-1 < y < 0$ by $|x| < 1$ and $|y| < 1$, so we get (3.23). ∎

The special case of $a = 0, y = 0$ in (3.23) gives the generating function of the Al-Salam-Carlitz polynomials.

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