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ON THE COMPRESSIBLE BOUSSINESQ EQUATIONS WITH PARTIAL DISSIPATION TERM

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Abstract. In this paper, we consider the strong solutions of the compressible Boussinesq equations in $\mathbb{R}^3$ and prove the existence of unique local strong solutions for all initial data satisfying some compatibility conditions. The initial density need not be positive and may vanish in an open set. We use the Lax-Milgram theorem and contraction mapping argument to get the result. Moreover, we establish a blow-up criterion for possible breakdown of strong solutions at a finite time in terms of the gradient of velocity.

1. INTRODUCTION

This paper is concerned with the Cauchy problem of the following compressible Boussinesq equations with partial dissipation term,

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + \rho \theta e_3, \tag{1.2}$$

$$\partial_t \theta + u \cdot \nabla \theta = 0 \tag{1.3}$$

with initial data

$$(\rho, u, \theta)\big|_{t=0} = \big(\rho_0(x), u_0(x), \theta_0(x)\big),$$

where the unknown functions $\rho = \rho(x,t)$, $u(x,t) = (u_1, u_2, u_3)$, $\theta(x,t)$ and e_3 denotes the density, the velocity of fluid, temperature and the vector $(0,0,1)$, respectively. The

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constant viscosity coefficient μ satisfies the physical restriction $\mu > 0$. The pressure P is a given state equation, we assume that $P(0) = 0$ and:

(1.4) $$P : [0, \infty) \to \mathbb{R} \text{ is a locally Lipschitz continuous function.}$$

Throughout this paper, we give some notations by reason of the convenience of discussions. We denote

$$\int f dx = \int_{\mathbb{R}^3} f dx, \ \int_0^t \int f dx ds = \int_0^t \int_{\mathbb{R}^3} f dx ds.$$

For $1 \leq r \leq \infty$ and integer $k \geq 0$, the standard homogeneous and inhomogeneous Sobolev spaces are denoted by

$$L^r = L^r(\mathbb{R}^3), \ W^{k,r} = W^{k,r}(\mathbb{R}^3), \ H^k = W^{k,2},$$

$$D^{k,r} = \{v \in L^1_{loc}(\mathbb{R}^3) : |\nabla^k v|_{L^r} < \infty\}, \ D^k = D^{k,2},$$

$$D^1_0 = \{v \in L^6 : |\nabla^k v|_{L^2} < \infty, \text{and } u \to 0 \text{ as } |x| \to \infty\}, \ H^1_0 = D^1_0 \cap L^2.$$

Boussinesq equations play an important role in the atmospheric science and applied mathematics. There is a huge literatures on the incompressible Boussinesq equations such as [1, 2, 5, 17, 21] and the references therein. More precisely, many authors devoted to studying the following form:

(1.5) $$\begin{cases} \partial_t u + u\nabla u + \nabla P = \mu\Delta u + \rho\theta e_3, \\ \partial_t \theta + u \cdot \nabla\theta = \kappa\Delta\theta, \\ \text{div} u = 0. \end{cases}$$

The global well-pose of (1.5) with $\mu > 0$ and $\kappa > 0$ is well-known ([4, 14]). While, as far as we know, the regularity and existence of the case of (1.5) with $\mu = \kappa = 0$ is an outstanding open problem. Therefore, some authors studied the global well-posedness of (1.5) with partial viscosity cases and obtained many results, such as, the zero diffusivity case $\mu = 0$ and $\kappa > 0$, and the zero viscosity case $\mu > 0$ and $\kappa = 0$ ([6, 7, 15, 16]). In [7], Chae established a blow up criterion with partial viscosity cases, analogous to the Beale-Kato-Majda blow-up criterion [3] for the incompressible flows. In order to understand the mechanism of long-term weather prediction and climate changes, some mathematicians begin to study the mathematical equations and models in compressible case as the atmosphere is a specific compressible fluid. Thus, it is interesting to consider the Boussinesq equations in the compressible case. Opposite with respect to the incompressible case, the mathematical analysis to compressible Boussinesq system is much more complicated, as the oscillation of the density. In

[29], under the assumption $\rho_0 > 0$, Xu had investigated the isentropic compressible Boussinesq system with $\mu = \kappa = 0$ and obtained the existence of classical solution and corresponding blow-up criterion.

The aim of this paper is to prove the existence of unique local strong solutions to (1.1)-(1.3) with $\inf \rho_0 = 0$, and investigate the blow-up mechanism. Before stating the main theorems, we first give the definition of strong solutions.

Definition 1.1. For $T > 0$, (ρ, u, θ) is called a strong solution to the compressible Boussinesq equations $(1.1) - (1.4)$ in $(0, T) \times \mathbb{R}^3$, if for some $q \in (3, 6]$,

$$\begin{aligned}
&0 \le \rho \in C([0,T]; H^1 \cap W^{1,q}), \ \ \rho_t \in C([0,T]; L^2 \cap L^q), \\
&\theta \in C([0,T]; H^1 \cap W^{1,q}), \ \ \theta_t \in C([0,T]; L^2 \cap L^q), \\
&u \in C([0,T]; D_0^1 \cap D^2) \cap L^2(0,T; D^{2,q}), \\
&u_t \in L^2([0,T]; D_0^1), \ \sqrt{\rho} u_t \in L^\infty([0,T]; L^2),
\end{aligned}$$

and (ρ, u, θ) satisfies $(1.1) - (1.4)$ a.e. in $(0, T) \times \mathbb{R}^3$.

The first main result is concerned with local existence of strong solutions:

Theorem 1.1. *Assume that P satisfies (1.4), for some $q \in (3, 6]$*

$$\begin{aligned}
&0 \le \rho_0 \in W^{1,q} \cap H^1, \ \theta_0 \in W^{1,q} \cap H^1, \\
&u_0 \in D_0^1 \cap D^2,
\end{aligned}$$

and for positive constant r_0

$$\|\rho_0\|_{H^1 \cap W^{1,q}} + \|\theta_0\|_{H^1 \cap W^{1,q}} + \|u_0\|_{D_0^1 \cap D^2} \le r_0.$$

If, in addition, the following compatibility condition holds

$$-\mu \Delta u_0 + \nabla P(\rho_0) - \rho_0 \theta_0 e_3 = \rho_0^{\frac{1}{2}} g \tag{1.6}$$

for some $g \in L^2$, then there exist a positive time T_0 and a unique strong solution (ρ, u, θ) for $(1.1) - (1.4)$ in $(0, T_0) \times \mathbb{R}^3$.

Motivated by these works on the blow-up criterion of local strong solutions to the Navier-Stokes equations and incompressible Boussinesq equations, we will establish the following blow-up criterion for the compressible Boussinesq equations.

Theorem 1.2. *Let (ρ, u, θ) be a strong solution to (1.1)-(1.4). Assume that P satisfies (1.4) and the initial data (ρ_0, u_0, θ_0) satisfies (1.6). If $0 < T_\star < +\infty$ is the maximum time of existence, then*

$$\lim_{T \to T_\star} \int_0^T ||\nabla u||_{L^\infty} dt = \infty.$$

Now, we briefly outline the main ideas of the proof, some of which are inspired by pioneer works on the compressible Naiver-Stokes equations. To obtain the existence of a unique local strong solution to (1.1)-(1.3), we employ the compatibility condition (1.6) to establish the key $W^{2,q}-$estimate. Then, we use the contraction mapping argument to obtain the existence result, in some sense simply and extend the result of [9] and [29].

This paper is written as follows. In Section 2, we introduce the working space which will be needed in later analysis. In Sections 3, we consider a linearized problem and derive some local estimates for the solutions independent of the lower bound of initial density. In Section 4, we use the contraction mapping principle to get the existence of local strong solutions. In Section 5, we give the proof of Theorem 1.2.

2. Function Space

In this section, we will introduce the working function space which plays an important role in the proof of Theorem 1.1,

$$\mathcal{W}' = \{v \in L^2(0,T;D^2), \partial_t v \in L^2(0,T;L^2)\}$$

with norm $||v||_{\mathcal{W}'}$, and for $q \in (3,6]$, we define

$$\mathcal{W} = \{v \in \mathcal{W}' \cap L^\infty(0,T;D_0^1 \cap D^2) \cap L^2(0,T;D^{2,q}), \partial_t v \in L^2(0,T;D_0^1)\}.$$

Remark 2.1. Before the proof, we point out that the approach of proving Theorem 1.1 is to apply the contraction mapping principle. Since the system (1.1)-(1.3) is of mixed hyperbolic-parabolic type and the initial density may vanish, we encounter a well-known difficulty in the theory of symmetric quasilinear hyperbolic systems. For these systems, contraction cannot be proved in the usual setting, that is, to consider self-mapping and contraction in the same regularity class $\mathcal{W}$. To resolve this problem, Kato [20] and Lax [22] offered an ingenious idea by studying contraction in a larger space. Taking up this idea, we establish the contraction in the space $\mathcal{L}$ (see in Section 4). Chu et. al. [11] adopted the same idea to tackle the compressible liquid crystal system.

3. Existence for the Linearized Equations

In this section, we reformulate the nonlinear equation (1.1)-(1.3) such that the left-hand becomes linear and the starting problem can be transferred to a fixed point equation:

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{3.1}$$

$$\rho \partial_t u - \mu \Delta u = -\rho v \cdot \nabla v - \nabla P(\rho) + \rho \theta e_3, \tag{3.2}$$

$$\partial_t \theta + v \cdot \nabla \theta = 0, \tag{3.3}$$

with the given $v \in \mathcal{W}$ and the initial conditions

$$(\rho, u, \theta)\big|_{t=0} = \big(\rho_0(x) + \delta, u_0(x), \theta_0(x)\big). \tag{3.4}$$

Here $\delta > 0$ is a constant and $\rho_0 \geq 0$.

If the initial density vanishes from below, we cannot expect the density ρ is bounded away from zero. As a result, the lack of a positive lower bound of ρ causes (3.1) to become a degenerate linear parabolic equation. This prevents us from using the standard argument to construct the local solutions. For this reason, we consider the linearized problem (3.1)-(3.3) with initial density bounded away form zero and derive some uniform bounds which are independent of the lower bounds of initial density. Firstly, we solve out the density, and obtain estimates for density.

Lemma 3.1. *For given v with $||v||_{\mathcal{W}} \leq A$, there exists a unique solution ρ to the linear transport problem (3.1) and (3.4) such that*

$$\|\rho\|_{L^\infty(0,T;H^1\cap W^{1,q})} \leq C\|\rho_0\|_{H^1\cap W^{1,q}}(1 + T^{\frac{1}{2}}A)\exp(CT^{\frac{1}{2}}A),$$

$$\|\rho_t\|_{L^\infty(0,T;L^2\cap L^q)} \leq C\|\rho_0\|_{H^1\cap W^{1,q}}A\exp(CT^{\frac{1}{2}}A).$$

Here and in what follows, the notation C stands for a generic positive constant.

Proof. *From the linear transport equation theory, we have*

$$\rho(x,t) = \rho_0(x)\exp\Big(\int_0^t -\mathit{div}v ds\Big). \tag{3.5}$$

As a consequence,

$$\nabla\rho = \nabla\rho_0\exp\Big(\int_0^t -\mathit{div}v ds\Big) - \rho_0\exp\Big(\int_0^t -\mathit{div}v ds\Big)\int_0^t \nabla\mathit{div}v ds,$$

$$\rho_t = -\rho_0\mathit{div}v\exp\Big(\int_0^t -\mathit{div}v ds\Big).$$

From the Minkowski inequality, we get ($r = 2, q$)

$$\begin{aligned}
||\nabla\rho||_{L^r} &\leq C\|\rho_0\|_{W^{1,r}}\Big(1 + ||\int_0^t \nabla^2 v||_{L^r}\Big)\exp\Big(\int_0^t ||\mathit{div}v||_{L^\infty}ds\Big)\\
&\leq C\|\rho_0\|_{W^{1,r}}\Big(1 + \int_0^t ||\nabla^2 v||_{L^r}\Big)\exp\Big(\int_0^t ||\mathit{div}v||_{L^\infty}ds\Big)\\
&\leq C\|\rho_0\|_{W^{1,r}}(1 + T^{\frac{1}{2}}||v||_{\mathcal{W}})\exp(CT^{\frac{1}{2}}||v||_{\mathcal{W}})\\
&\leq C\|\rho_0\|_{W^{1,r}}(1 + T^{\frac{1}{2}}A)\exp(CT^{\frac{1}{2}}A),
\end{aligned}$$

$$
\begin{aligned}
||\rho_t||_{L^r} &\le C\|\rho_0\|_{L^\infty}||\nabla v||_{L^r}\exp\Big(\int_0^t ||divv||_{L^\infty}ds\Big)\\
&\le C\|\rho_0\|_{W^{1,r}}||v||_{\mathcal{W}}\exp(CT^{\frac{1}{2}}||v||_{\mathcal{W}})\\
&\le C\|\rho_0\|_{W^{1,r}}A\exp(CT^{\frac{1}{2}}A).
\end{aligned}
$$

This completes the proof. ■

Combining the structure of (3.3) and the corresponding previous results, we can solve the temperature θ similarly as Lemma 3.1.

Lemma 3.2. *For given v with $||v||_{\mathcal{W}} \le A$, there exists a unique solution θ which satisfies (3.3) such that*

$$
\begin{aligned}
&\|\theta\|_{L^\infty(0,T;H^1\cap W^{1,q})} \le C\|\theta_0\|_{H^1\cap W^{1,q}}(1+T^{\frac{1}{2}}A)\exp(CT^{\frac{1}{2}}A),\\
&\|\theta_t\|_{L^\infty(0,T;L^2\cap L^q)} \le C\|\theta_0\|_{H^1\cap W^{1,q}}A\exp(CT^{\frac{1}{2}}A).
\end{aligned}
$$

Proof. *The existence and estimates can be obtained similarly as Lemma 3.1.* ■

The next lemma gives the estimates on the velocity.

Lemma 3.3. *Under the conditions $\rho|_{t=0} = \rho_0 + \delta$, suppose $||v||_{\mathcal{W}} \le A$, there exists a unique solution u which satisfies (3.2), such that for T small enough,*

$$
\|u\|_{L^\infty(0,T;D_0^1\cap D^2)} + \|u\|_{L^2(0,T;D^{2,q})} + \|u_t\|_{L^2(0,T;D_0^1)} \le C.
$$

Proof. Since (3.5) and the initial condition, we get

$$
\rho(x,t) \ge \delta\exp(\int_0^t -||\nabla v||_{L^\infty}ds) > 0. \tag{3.6}
$$

The standard theory of parabolic equations, such as a semidiscrete Galerkin method, implies the existence of the solution to (3.2). Inspired by [26], we use the Lax-Milgram theorem to achieve the existence. For reader's convenience, we will roughly recall the proof with some changes. For simplicity of the presentations, we assume $\mu = 1$.

We consider the bilinear form $E(u,\psi)$ and linear function $L(\psi)$ defined by

$$
\begin{aligned}
E(u,\psi) &= \int_0^T(\rho\partial_t u - \Delta u, \partial_t\psi - k\Delta\psi)dt - (u(0),\Delta\psi(0)),\\
L(\psi) &= -\int_0^T(\rho v\cdot\nabla v + \nabla P(\rho) - \rho\theta e_3, \partial_t\psi - k\Delta\psi)dt - (u_0,\Delta\psi(0)),
\end{aligned}
$$

with

$$
k = (2||\rho||_{L^\infty(0,T;L^\infty)})^{-1}
$$

for $\psi \in \mathcal{W}'$, where $(\cdot,\cdot)$ denotes the inner product in L^2.

Obviously, $L(\psi)$ is linear continuous on ψ, with respect to the norm $||\psi||_{\mathcal{W}'}$. Moreover, by the Cauchy inequality, we have

$$
\begin{aligned}
& E(\psi,\psi) \\
&= \int_0^T (\|\sqrt{\rho}\partial_t\psi\|_{L^2}^2 + k\|\Delta\psi\|_{L^2}^2 - k(\rho\partial_t\psi,\Delta\psi)dt + \frac{1}{2}(||\nabla\psi(T)||_{L^2}^2 + ||\nabla\psi(0)||_{L^2}^2) \\
&\geq \int_0^T (\|\sqrt{\rho}\partial_t\psi\|_{L^2}^2 + k\|\Delta\psi\|_{L^2}^2 - \frac{3}{4}\|\sqrt{\rho}\partial_t\psi\|_{L^2}^2 - \frac{2k}{3}\|\Delta\psi\|_{L^2}^2)dt \\
&\quad + \frac{1}{2}(||\nabla\psi(T)||_{L^2}^2 + ||\nabla\psi(0)||_{L^2}^2) \\
&\geq C||\psi||_{\mathcal{W}'}^2
\end{aligned}
$$

for some $C > 0$.

Therefore, by the Lax-Milgram theorem, there exists a $u \in \mathcal{W}'$ such that

$$E(u,\psi) = L(\psi) \tag{3.7}$$

for every $\psi \in \mathcal{W}'$.

If we assume that $\overline{\psi}$ is a solution of the following problem

$$\partial_t\overline{\psi} - k\Delta\overline{\psi} = 0,\ \overline{\psi}(0) = h(x),$$

with $h(x)$ smooth enough. Replacing in (3.7) ψ by $\overline{\psi}$, then we have

$$(u(0) - u_0, \Delta h) = 0,$$

which implies $u(0) = u_0$. Similarly, let $\widetilde{\psi}$ be a solution of the problem

$$\partial_t\widetilde{\psi} - k\Delta\widetilde{\psi} = \widetilde{g(x,t)},\ \widetilde{\psi}(0) = 0,$$

with $\widetilde{g(x,t)}$ smooth enough. Replacing in (3.7) ψ by $\widetilde{\psi}$, then we get

$$\int_0^T (\rho\partial_t u - \Delta u + \rho v\cdot\nabla v + \nabla P(\rho) - \rho\theta e_3, \widetilde{g})dt = 0. \tag{3.8}$$

This implies that (ρ,u,θ) satisfies (1.1)-(1.3) a.e. in $(0,T)\times\mathbb{R}^3$.

To ensure the higher regularity, specially as to the term u_t, we need some compatibility condition. In order to derive estimate for ∇u_t, we differentiate (3.2) with respect to t and get

$$
\begin{aligned}
& \rho\partial_{tt}^2 u - \mu\Delta\partial_t u \\
&= -\partial_t\rho\partial_t u - \partial_t\rho v\cdot\nabla v - \rho\partial_t v\cdot\nabla v - \rho v\cdot\nabla\partial_t v - \nabla\partial_t P + \partial_t(\rho\theta e_3).
\end{aligned} \tag{3.9}
$$

Multiplying the identity by u_t and using integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx+\int \mu|\nabla u_t|^2dx\\
&=-\frac{1}{2}\int \rho_t|u_t|^2dx-\int \rho_t v\nabla v u_t dx-\int \rho v_t\nabla v u_t dx\\
&\quad-\int \rho v\cdot\nabla v_t u_t dx+\int P_t\mathrm{div}u_t dx\\
&\quad+\int \rho_t\theta e_3 u_t dx+\int \theta_t\rho e_3 u_t dx=\sum_{k=1}^{7} I_k.
\end{aligned} \tag{3.10}$$

Using the continuity equation and the previous Gagliardo-Nirenberg inequality, we get

$$\begin{aligned}
|I_1|&=|-\frac{1}{2}\int \rho v\nabla|\partial_t u|^2dx|=|\int \rho v\partial_t u\nabla\partial_t u dx|\\
&\le \int \rho|u_t|^2(\rho|v|^2)dx+\epsilon\int|\nabla u_t|^2dx\\
&\le \epsilon||\nabla u_t||_{L^2}^2+CA^2\exp(CT^{\frac{1}{2}}A)||\sqrt{\rho}u_t||_{L^2}^2,\\
|I_2|&=|\int \mathrm{div}(\rho v)v\cdot\nabla v\partial_t u dx|=|\int \rho v\nabla(v\cdot\nabla v\partial_t u)dx|\\
&\le \int \rho|v||\nabla v|^2|\partial_t u|+\rho|v|^2|\nabla^2 v||\partial_t u|+\rho|v||\nabla v||\nabla\partial_t u|dx\\
&\le \epsilon||\nabla u_t||_{L^2}^2+A^6||\sqrt{\rho}u_t||_{L^2}^2+C\exp(CT^{\frac{1}{2}}A)+C(\epsilon)A^4\exp(CT^{\frac{1}{2}}A),\\
|I_3|&\le \int \rho|v_t||\nabla v||u_t|dx\\
&\le A^6\int \rho|u_t|^2dx+A^{-6}\int \rho|v_t|^2|\nabla v|^2dx\\
&\le A^6||\sqrt{\rho}u_t||_{L^2}^2+CA^{-4}\exp(CAT^{\frac{1}{2}})||v_t||_{L^2}^2,\\
|I_4|&\le \int \rho|v||\nabla v_t||u_t|dx\\
&\le A^6||\sqrt{\rho}u_t||_{L^2}^2+CA^{-4}\exp(CAT^{\frac{1}{2}})||\nabla v_t||_{L^2}^2,\\
|I_5|&\le ||P_t||_{L^2}||\nabla u_t||_{L^2}\le ||P'||_{L^\infty}||\rho_t||_{L^2}||\nabla u_t||_{L^2}\\
&\le \epsilon||\nabla u_t||_{L^2}^2+C(\epsilon)A^2\exp(CT^{\frac{1}{2}}A),\\
|I_6|&=|\int \rho v\nabla\theta u_t\cdot e_3dx+\int \rho v\theta\nabla u_t e_3dx|\\
&\le (||\rho||_{L^3}||v||_{L^\infty}||\nabla\theta||_{L^2}+||\rho||_{L^\infty}||v||_{L^\infty}||\theta||_{L^2})||\nabla u_t||_{L^2}\\
&\le \epsilon||\nabla u_t||_{L^2}^2+C(\epsilon)\exp(CT^{\frac{1}{2}}A),\\
|I_7|&\le \epsilon||\nabla u_t||_{L^2}^2+C(\epsilon)\exp(CT^{\frac{1}{2}}A).
\end{aligned}$$

Substituting $I_1 - I_7$ into (3.10), choosing ϵ sufficiently small and integrating with respect to t, we have

$$\begin{aligned}\int \rho|u_t|^2dx+\int_0^T\int|\nabla u_t|^2dxdt &\leq \int\rho_0|u_{0t}|^2dx+CA^4T\exp(CT^{\frac{1}{2}}A)\\ &+CA^6\exp(CT^{\frac{1}{2}}A)\int_0^T\int\rho|u_t|^2dxdt.\end{aligned} \tag{3.11}$$

To estimate $||\sqrt{\rho}u_{0t}||_{L^2}^2$, we observe from (3.2) and the compatibility condition (1.6)

$$\int|\sqrt{\rho_0}u_{0t}|^2dx\leq C\int\rho_0|v_0|^2|\nabla v_0|^2+\frac{1}{\rho_0}|\mu\Delta u_0-\nabla P(\rho_0)+\rho_0\theta_0e_3|^2dx\leq C. \tag{3.12}$$

Adding (3.11) and (3.12), we obtain the following estimates by Growall's inequality:

$$\int\rho|u_t|^2dx+\int_0^T\int|\nabla u_t|^2dxdt\leq\big(C+CA^4T\exp(CT^{\frac{1}{2}}A)\big)A^6T\exp(CT^{\frac{1}{2}}A). \tag{3.13}$$

Finally, we have to estimate

$$u\in L^\infty([0,T];D^2)\cap L^2([0,T];D^{2,q}),\ q\in(3,6].$$

To obtain futher estimates, we rewrite (3.2) as

$$\mu\Delta u=\rho\partial_tu+\rho v\cdot\nabla v+\nabla P(\rho)-\rho\theta e_3,$$

which is a strongly elliptic system. By the classical elliptic regularity theory, we deduce

$$||u||_{D^2}\leq C(||\rho u_t||_{L^2}+||\nabla P||_{L^2}+||\rho v\nabla v||_{L^2}+||\rho\theta e_3||_{L^2})+C. \tag{3.14}$$

From the previous lemmas, we get

$$\begin{aligned}||\rho u_t||_{L^2} &\leq C\exp(CT^{\frac{1}{2}}A)||\sqrt{\rho}u_t||_{L^2},\\ ||\nabla P||_{L^2} &\leq C\exp(CT^{\frac{1}{2}}A)+CT^{\frac{1}{2}}A\exp(CT^{\frac{1}{2}}A),\\ ||\rho\theta e_3||_{L^2} &\leq C\exp(CT^{\frac{1}{2}}A)+CT^{\frac{1}{2}}A\exp(CT^{\frac{1}{2}}A),\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}&||\rho v\nabla v||_{L^2}\\ &\leq||\rho||_{L^\infty}^2\int|v|^2|\nabla v|^2dx\\ &\leq C\exp(CT^{\frac{1}{2}}A)\Big(\int|v-u_0|^2|\nabla v|^2dx+||u_0||_{L^\infty}^2\int|\nabla v-\nabla u_0|^2dx+C\Big)\\ &\leq C\exp(CT^{\frac{1}{2}}A)\Big(\int|\int_0^t v_t|^2|\nabla v|^2\,dx+C\int|\int_0^t\nabla v_t|^2\,dx+C\Big)\\ &\leq C\exp(CT^{\frac{1}{2}}A)(A^4T+CA^2T+C).\end{aligned} \tag{3.16}$$

In a similar way, we can obtain

$$||u||_{L^\infty(0,T;D^2)} + ||u||_{L^2(0,T;D^{2,q})} \le C + T^\alpha \ \ (\alpha > 0). \tag{3.17}$$

Gathering (3.13)-(3.17) and choosing T small enough, we get $u \in \mathcal{W}$.

Hence, the proof is finished. ∎

Combining all the lemmas, we get the existence for the linearized equations $(3.1)-(3.4)$.

Lemma 3.4. *There exists a unique strong solution* (ρ, u, θ) *to the linearized system* $(3.1) - (3.4)$ *in* $[0, T_0] \times \Omega$ *with the regularity*

$$\rho \in C([0,T_0]; H^1 \cap W^{1,q}), \ \ \rho_t \in C([0,T_0]; L^2 \cap L^q),$$

$$\theta \in C([0,T_0]; H^1 \cap W^{1,q}), \ \ \theta_t \in C([0,T_0]; L^2 \cap L^q),$$

$$u \in C([0,T_0]; D_0^1 \cap D^2) \cap L^2(0,T_0; D^{2,q}),$$

$$u_t \in L^2([0,T_0]; D_0^1), \ \sqrt{\rho} u_t \in L^\infty([0,T_0]; L^2),$$

where $T_0 \in (0, T)$.

4. Proof of Theorem 1.1

This section is devoted to proving the existence of a unique local solution of (1.1)-(1.3) via the contraction mapping principle.

By virtue of Lemma 3.4, there exist a time $T_0 \in (0, T)$ and a unique solution $(\rho^\delta, u^\delta, \theta^\delta)$ of $(3.1) - (3.3)$ with initial data $\rho(x, 0) = \rho_0 + \delta$. Let $\delta \to 0$, we obtain a unique solution u of the linearized system $(3.1) - (3.3)$ with $\rho(x, 0) = \rho_0$ such that $||u||_{\mathcal{W}} \le C$. So we can define a map

$$\mathcal{J} : \mathcal{M} \to \mathcal{M}, \mathcal{J}(v) = u,$$

where $\mathcal{M} = \mathcal{W} \cap \mathcal{L} = \mathcal{W}$, with

$$\mathcal{L} = \{u : \|u\|_{L^2(0,T;H^1(\Omega))} < \infty\}.$$

Thus, there are essentially two main tasks we have to prove, the self-mapping and contraction. The former has been done due to Lemma 3.3, which guarantees the self-mapping. As mentioned in Remark 2.1, we need to prove a contraction estimate in the lager space $\mathcal{L}$. The following lemma implies that the map $\mathcal{J}$ is contracted in the sense of weaker norm for $v \in \mathcal{M}$.

Lemma 4.1. *There exists a constant* $0 < \lambda < 1$ *such that for any* $v_i \in \mathcal{M}$, $i = 1, 2$,

$$\|\mathcal{J}(v_1) - \mathcal{J}(v_2)\|_{\mathcal{L}} \leq \lambda \|v_1 - v_2\|_{\mathcal{L}}$$

for some small $T > 0$. *Proof.* *Suppose* (ρ_i, u_i, F_i) *are the solutions to* $(3.1) - (3.3)$ *corresponding to given* $v_i \in \mathcal{M}$. *Define* $\rho = \rho_2 - \rho_1$, $v = v_2 - v_1$, $\theta = \theta_2 - \theta_1$ *and* $u = u_2 - u_1$. *Then*

$$\partial_t \rho + div(\rho_2 v) = -div(\rho_1 v), \tag{4.1}$$

$$\partial_t \theta + v \cdot \nabla \theta_2 + v_1 \cdot \nabla \theta = 0, \tag{4.2}$$

$$\begin{aligned} \rho_2 \partial_t u - \mu \Delta u = {} & (\rho_1 - \rho_2) u_{1t} + \rho_1 v_1 \cdot \nabla v_1 - \rho_2 v_2 \cdot \nabla v_2 \\ & + \nabla P(\rho_1) - \nabla P(\rho_2) + \rho_2 \theta_2 e_3 - \rho_1 \theta_1 e_3. \end{aligned} \tag{4.3}$$

Multiplying (4.1) *by* ρ *and integrating over* $\mathbb{R}^3$, *we get*

$$\begin{aligned} & \frac{d}{dt} \int \frac{1}{2} |\rho|^2 dx \\ & = -\frac{1}{2} \int |\rho|^2 div v_2 dx - \int \rho (\nabla \rho_1 \cdot v + \rho_1 div v) dx \\ & \leq C \|\nabla v_2\|_{L^\infty} \|\rho\|_{L^2}^2 + C \|\rho\|_{L^2} \|\nabla \rho_1\|_{L^3} \|v\|_{L^6} + C \|\rho\|_{L^2} \|\rho_1\|_{L^\infty} \|\nabla v\|_{L^2} \\ & \leq E_1(t) \|\rho\|_{L^2}^2 + \epsilon \|\nabla v\|_{L^2}^2, \end{aligned} \tag{4.4}$$

where $E_1(t) = C(\|\nabla v_2\|_{L^\infty} + \|\rho_1\|_{L^\infty}^2 + \|\nabla \rho_1\|_{L^3}^2)$.

Similarly, multiplying (4.1) by $\text{sgn}\rho |\rho|^{\frac{1}{2}}$ and integrating, we have

$$\begin{aligned} \frac{d}{dt} \int |\rho|^{\frac{3}{2}} dx & \leq C \int |\nabla v_2| |\rho|^{\frac{3}{2}} + (|\nabla \rho_1| |v| + |\rho_1| |\nabla v|) |\rho|^{\frac{1}{2}} dx \\ & \leq \|\nabla v_2\|_{L^\infty} \|\rho\|_{\frac{3}{2}}^{\frac{3}{2}} + C \|\rho_1\|_{H^1} \|\nabla v\|_{L^2} \|\rho\|_{\frac{3}{2}}^{\frac{1}{2}}. \end{aligned} \tag{4.5}$$

Multiplying (4.5) by $||\rho||_{L^{\frac{3}{2}}}^{\frac{1}{2}}$, and using Cauchy's inequality, one has

$$\frac{d}{dt} \|\rho\|_{\frac{3}{2}}^2 \leq E_2(t) \|\rho\|_{\frac{3}{2}}^2 + \epsilon \|\nabla v\|_{L^2}^2, \tag{4.6}$$

where $E_2(t) = C(\|\nabla v_2\|_{L^\infty} + \|\rho\|_{H^1}^2)$.

In a similar way, we obtain

$$\frac{d}{dt} \int \frac{1}{2} |\theta|^2 dx \leq E_3(t) \|\theta\|_{L^2}^2 + \epsilon \|\nabla v\|_{L^2}^2, \tag{4.7}$$

where $E_3(t) = C(\|\nabla v_2\|_{L^\infty} + \|\theta_1\|_{L^\infty}^2 + \|\nabla\theta_1\|_{L^3}^2)$.

Multiplying (4.3) by u and integrating, we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho_2|u|^2dx + \mu\int|\nabla u|^2dx \\
&\le \int \rho_2 v_2 u\nabla u + (\rho_1-\rho_2)u_{1t}\cdot u + (p_2-p_1)\mathrm{div}u \\
&\quad +(\rho_1 v_1\nabla v_1 - \rho_2 v_2\nabla v_2)\cdot u + \rho_2\theta_2 e_3 u - \rho_1\theta_1 e_3 u dx \\
&= \int \rho_2 v_2 u\nabla u + (\rho_1-\rho_2)(u_{1t}+v_1\nabla v_1)u - \rho_2(v\nabla v_2 + v_1\nabla v)u \\
&\quad +(p_2-p_1)\mathrm{div}u + \rho\theta_2 e_3 u + \rho_1\theta e_3 u dx \qquad (4.8)\\
&\le \|\sqrt{\rho_2}\|_{L^\infty}\|\sqrt{\rho_2}u\|_{L^2}\|v_2\|_{L^\infty}\|\nabla u\|_{L^2} + \|\rho\|_{L^{\frac{3}{2}}}\|u_{1t}+v_1\nabla v_1\|_{L^6}\|u\|_{L^6} \\
&\quad +\|\sqrt{\rho_2}\|_{L^\infty}\|\sqrt{\rho_2}u\|_{L^2}\|v\|_{L^6}\|\nabla v_2\|_{L^3} + \|\sqrt{\rho_2}\|_{L^\infty}\|\sqrt{\rho_2}u\|_{L^2}\|v_1\|_{L^\infty}\|\nabla v\|_{L^2} \\
&\quad +|P'|\|\rho\|_{L^2}\|\nabla u\|_{L^2} + \|\rho\|_{L^2}\|\theta_2\|_{L^\infty}\|u\|_{L^2} + \|\rho_1\|_{L^\infty}\|\theta\|_{L^2}\|u\|_{L^2} \\
&\le \epsilon\|\nabla u\|_{L^2}^2 + \epsilon\|\nabla v\|_{L^2}^2 + E_4(t)(\|\sqrt{\rho_2}u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\rho\|_{L^{\frac{3}{2}}}^2 + \|\theta\|_{L^2}^2),
\end{aligned}$$

where

$$\begin{aligned}
E_4(t) = C(&\|\sqrt{\rho_2}u\|_{L^\infty}^2\|v_2\|_{L^\infty}^2 + \|\sqrt{\rho_2}\|_{L^\infty}^2\|\nabla v_2\|_{L^3}^2 + \|\nabla u_{1t}\|_{L^2}^2 + \|v_1\|_{D^2}^2 \\
&+ \|\theta_2\|_{L^\infty}^2 + \|\rho_1\|_{L^\infty}^2).
\end{aligned}$$

Summing inequalities $(4.4)-(4.8)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho_2}u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\rho\|_{L^{\frac{3}{2}}}^2 + \|\theta\|_{L^2}^2) + \int|\nabla u|^2dx \\
&\le \epsilon\int|\nabla v|^2dx + E(t)(\|\sqrt{\rho_2}u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\rho\|_{L^{\frac{3}{2}}}^2 + \|\theta\|_{L^2}^2),
\end{aligned} \qquad (4.9)$$

where $E(t) = E_1(t) + E_2(t) + E_3(t) + E_4(t)$ satisfies

$$\int_0^T E(s)ds \le K,$$

where K is a constant dependent on initial data, thanks to Lemma 3.1-3.3.

Let T small enough, we obtain the following by Gronwall's inequality

$$\|\rho\|_{L^\infty(0,T;L^2)} + \|\theta\|_{L^\infty(0,T;L^2)} + \|\sqrt{\rho_2}u\|_{L^\infty(0,T;L^2)} \le C$$

and

$$\int_0^T\int|\nabla u|^2dxdt\le\lambda\int_0^T\int|\nabla v|^2dxdt,\ \text{with}\ 0<\lambda<1.$$

So we finish the proof. ■

Proof of Theorem 1.1. By the contractibility of $\mathcal{J}$ and utilizing the iteration methods used in [22] and [24], we can obtain a unique fixed point u. This proves the existence of a strong solution.

Thus, we complete the proof of Theorem 1.1.

5. Proof of Theorem 1.2

Let $0<T_\star<\infty$ be the maximum time for the existence of strong solution (ρ,u,θ) to $(1.1)-(1.3)$. In other words, (ρ,u,θ) is a strong solution to $(1.1)-(1.3)$ in $\mathbb{R}^3\times(0,T]$ for any $0<T<T_\star$, but not a strong solution in $\mathbb{R}^3\times(0,T_\star]$. Motivated by work of Huang et al. [19], Huang, Wang and Wen [18], we will prove Theorem 1.2 by a contradiction argument. To this end, we suppose that for any $0<T<T_\star$, there is a positive constant M such that

$$\int_0^T||\nabla u||_{L^\infty}dt\le M<+\infty. \tag{5.1}$$

The goal is to show under assumption (5.1), there is a bound $C>0$ depending only on initial data and T, such that ($r=2,q$)

$$\sup_{0\le t\le T}(||\rho||_{W^{1,r}}+||\theta||_{W^{1,r}}+||\rho_t||_{L^r}+||\theta_t||_{L^r}+||\sqrt{\rho}u_t||_{L^2}+||\nabla u||_{H^1})\le C, \tag{5.2}$$

and

$$\int_0^T(||u_t||^2_{D^1}+||u||^2_{D^{2,r}})\le C. \tag{5.3}$$

With (5.2) and (5.3), we will deduce a contradiction to the maximality of $T_\star$.

It is well-known that the bound of ∇u yields that ρ is bounded from the mass equation (1.1). More precisely, we have

Lemma 5.1. *Under the assumption (5.1), for any* $0<T<T_\star$, *we have*

$$\sup_{0\le t\le T}(||\rho||_{L^\infty}+||\theta||_{L^\infty})\le C. \tag{5.4}$$

Proof. We first show that the density ρ is bounded due to the assumption (5.1), which was proved in [19]. Multiplying (1.1) by $p|\rho|^{p-2}\rho$ in $L^2(2\le p\le\infty)$ and using integration by parts, we obtain that

$$\partial_t ||\rho||_{L^p} \leq \frac{p-1}{p} ||\nabla u||_{L^\infty} ||\rho||_{L^p},$$

which, together with Gronwall's inequality, leads to

$$\sup_{0\leq t\leq T} ||\rho||_{L^p} \leq C.$$

Letting $p \to \infty$, we get

$$\sup_{0\leq t\leq T} ||\rho||_{L^\infty} \leq C.$$

In a similar way, we can obtain the estimate for θ. This completes the proof. ∎

By virtue of Lemma 5.1, we establish the global energy inequality for strong solutions.

Lemma 5.2. *Under the assumption (5.1), for any* $0 < T < T_\star$ *, we have*

$$\sup_{0\leq t\leq T} ||\sqrt{\rho} u||_{L^2}^2 + \mu \int_0^T ||\nabla u||_{L^2}^2 dt \leq C. \tag{5.5}$$

Proof. Since P is locally Lipschitz by (1.4) and Lemma 5.1, we obtain the following

$$|P(\rho)| \leq ||P'(\rho)||_{L^\infty} \rho \leq C\rho \leq C,$$

$$|\nabla P(\rho)| \leq ||P'(\rho)||_{L^\infty} |\nabla \rho| \leq C|\nabla \rho|.$$

Multiplying (1.2) by u and integrating by parts, we have

$$\frac{d}{dt} \int \rho |u|^2 dx + \int \mu |\nabla u|^2 dx = \int P \text{div} u dx + \int \rho\theta e_3 \cdot u dx.$$

Utilizing (5.4) and the conservation of mass equation, we get

$$\int \rho^2 dx \leq ||\rho||_{L^1} ||\rho||_{L^\infty} \leq C.$$

By Cauchy's inequality, we have

$$|\int P \text{div} u dx| \leq \int |P(\rho)||\text{div} u| dx \leq C \int \rho |\nabla u| dx \leq \epsilon ||\nabla u||_{L^2}^2 + C$$

and

$$|\int \rho\theta e_3 \cdot u dx| \leq ||\theta||_{L^3} ||\rho||_{L^2} ||u||_{L^6} \leq \epsilon ||\nabla u||_{L^2}^2 + C.$$

Putting these inequalities into the energy identity, we complete the proof. ∎

The following lemma will play a key role in Lemma 5.4 and Lemma 5.7.

Lemma 5.3. *Under the assumption (5.1), for any* $0 < T < T_\star$ *and* $p \in [2, 6]$, *we have*

$$\sup_{0\le t\le T}(||\nabla\rho||_{L^p} + ||\nabla\theta||_{L^p}) \le C(1 + \int_0^T ||\nabla^2 u||_{L^p} dt). \tag{5.6}$$

Proof. By a straightforward computation of (1.1), we get

$$\begin{aligned}&\partial_t(|\nabla\rho|^p) + \mathrm{div}(|\nabla\rho|^p u) + (p-1)|\nabla\rho|^p \mathrm{div} u\\&+ p|\nabla\rho|^{p-2}\nabla^t\rho\nabla u\cdot\nabla\rho + p\rho|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\mathrm{div}u = 0,\end{aligned}$$

which yields that

$$\frac{d}{dt}||\nabla\rho||_{L^p} \le C||\nabla u||_{L^\infty}||\nabla\rho||_{L^p} + C||\nabla^2 u||_{L^p}.$$

Hence, we get

$$||\nabla\rho||_{L^p} \le C(1 + \int_0^t ||\nabla^2 u||_{L^p} ds).$$

In a similar way, we can get the estimate for θ. ■

By virtue of Lemma 5.3, we are ready to obtain the L^2-estimate of the first derivatives (ρ, u, θ), which are important for estimating other quantities.

Lemma 5.4. *Under the assumption (5.1), for any* $0 < T < T_\star$, *we have*

$$\sup\nolimits_{0\le t\le T}(||\nabla\rho||_{L^2}^2 + ||\nabla\theta||_{L^2}^2 + ||\nabla u||_{L^2}^2) + \int_0^T ||\nabla u||_{H^1}^2 dt \le C. \tag{5.7}$$

Proof. Multiplying (1.2) by $\rho^{-1}(\mu\Delta u + \rho\theta e_3 - \nabla P)$ in L^2 and integrating the result over $\mathbb{R}^3$, one has after integration by parts

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\int |\nabla u|^2 dx + \int \rho^{-1}|\mu\Delta u - \nabla P + \rho\theta e_3|^2 dx\\&= \mu\int u\nabla u\Delta u dx - \int u\nabla u\cdot\nabla P dx - \int u_t\cdot\nabla P dx\\&+ \int u_t\cdot\rho\theta e_3 dx + \int u\nabla u\cdot\rho\theta e_3 dx.\end{aligned} \tag{5.8}$$

The first term in the right hand of (5.8) can be estimated as follows,

$$
\begin{aligned}
&\mu|\int u\nabla u\Delta u dx|\\
&=\mu|\int u^i\partial_i u^j\partial_{kk}u^j dx| = \mu|\int \partial_k u^j\partial_i u^j\partial_k u^j + u^j\partial_{ik}u^j\partial_k u^j dx|\\
&=\mu|\int \partial_k u^j\partial_i u^j\partial_k u^j dx - \frac{1}{2}\int(\partial_k u^j)^2\text{div}u dx|\\
&\le C||\nabla u||_{L^\infty}||\nabla u||_{L^2}^2.
\end{aligned} \tag{5.9}
$$

In order to estimate the second term in the right hand of (5.8), we utilize the Sobolev inequality

$$
\begin{aligned}
|\textstyle\int u\nabla u\cdot\nabla P dx| &\le C||u||_{L^6}||\text{div}u||_{L^3}||\nabla P||_{L^2}\\
&\le C||\nabla u||_{L^2}||\text{div}u||_{L^3}||\nabla\rho||_{L^2}\\
&\le C||\nabla u||_{L^2}^{\frac{5}{3}}||\nabla u||_{L^\infty}^{\frac{1}{3}}||\nabla\rho||_{L^2}\\
&\le C||\nabla\rho||_{L^2}^2||\nabla u||_{L^2}^2 + C||\nabla u||_{L^\infty}||\nabla u||_{L^2}^2 + C.
\end{aligned} \tag{5.10}
$$

The remaining terms can be estimated as follows. Since (1.5) and (1.1) imply

$$
|P(\rho)_t| \le ||P'(\rho)||_{L^\infty}|\rho_t| \le ||P'(\rho)||_{L^\infty}(\rho|\nabla u| + |\nabla\rho||u|) \le C(|\nabla u| + |u||\nabla\rho|),
$$

we have

$$
\begin{aligned}
&-\int u_t\cdot\nabla P dx\\
&=\frac{d}{dt}\int P(\rho)\text{div}u dx - \int(P(\rho))_t\text{div}u dx\\
&\le \frac{d}{dt}\int P(\rho)\text{div}u dx + C\int(|u||\nabla\rho||\text{div}u| + |\nabla u|^2)dx\\
&\le \frac{d}{dt}\int P(\rho)\text{div}u dx + C||\nabla u||_{L^2}||\text{div}u||_{L^3}||\nabla\rho||_{L^2} + C||\nabla u||_{L^2}^2\\
&\le \frac{d}{dt}\int P(\rho)\text{div}u dx + C||\nabla\rho||_{L^2}^2||\nabla u||_{L^2}^2 + C(||\nabla u||_{L^\infty}+1)||\nabla u||_{L^2}^2 + C.
\end{aligned} \tag{5.11}
$$

Similarly, we get

$$
\begin{aligned}
&\int u_t\cdot\rho\theta e_3 dx\\
&=\frac{d}{dt}\int\rho\theta u e_3 dx - \int\rho_t\theta u e_3 dx - \int\theta_t\rho u e_3 dx\\
&=\frac{d}{dt}\int\rho\theta u e_3 dx - \int\rho u\nabla\theta u e_3 dx - \int\rho u\theta\nabla u e_3 dx + \int\rho u\nabla\theta u e_3 dx\\
&\le \frac{d}{dt}\int\rho\theta u e_3 dx + C||\nabla u||_{L^2}^2
\end{aligned} \tag{5.12}
$$

and

$$\int u\nabla u\rho\theta e_3 dx \le C||\rho||_{L^3}||\theta||_{L^\infty}||u||_{L^6}||\nabla u||_{L^2} \le C||\nabla u||_{L^2}^2. \tag{5.13}$$

On the other hand, by the theory of elliptic system and due to $\rho^{-1} \ge C > 0$, we obtain

$$\begin{aligned} ||\nabla^2 u||_{L^2}^2 &\le C||\mu\Delta u||_{L^2}^2 \\ &\le C\int \rho^{-1}(\mu\Delta u - \nabla P + \rho\theta e_3)^2 dx + C||\nabla\rho||_{L^2}^2 + C||\theta||_{L^2}^2. \end{aligned} \tag{5.14}$$

Putting $(5.8)-(5.14)$ together, we have

$$\begin{aligned} &\frac{d}{dt}\int (\frac{\mu}{2}|\nabla u|^2 + \rho\theta u e_3 - P\text{div}u)dx + C^{-1}||\nabla^2 u||_{L^2}^2 \\ &\le C||\nabla\rho||_{L^2}^2||\nabla u||_{L^2}^2 + C(1+||\nabla u||_{L^\infty})||\nabla u||_{L^2}^2 + C. \end{aligned} \tag{5.15}$$

By Young's inequality, we get

$$\begin{aligned} \int \rho\theta u e_3 - P\text{div}u dx &\le \epsilon||\nabla u||_{L^2}^2 + C(\epsilon)(||\rho||_{L^3}^2||\theta||_{L^2}^2 + ||\rho||_{L^2}^2) \\ &\le \epsilon||\nabla u||_{L^2}^2 + C. \end{aligned} \tag{5.16}$$

Integrating (5.15) over $(0,t)$ and substituting (5.16) into the result, we deduce the following after choosing ϵ sufficiently small

$$\begin{aligned} &||\nabla u||_{L^2}^2 + \int_0^t ||\nabla u||_{H^1}^2 ds \\ &\le C\int_0^t (1+||\nabla u||_{L^\infty} + ||\nabla u||_{L^2}^2)(||\nabla u||_{L^2}^2 + ||\nabla\rho||_{L^2}^2)ds + C. \end{aligned}$$

Since

$$[1+||\nabla u||_{L^\infty} + ||\nabla u||_{L^2}^2] \in L^1(0,T),$$

the Gronwall's inequality and the mass equation imply that

$$||\nabla\rho||_{L^2}^2 + ||\nabla\theta||_{L^2}^2 + ||\nabla u||_{L^2}^2 \le C.$$

The proof of Lemma 5.4 is completed. ■

As an immediate consequence of Lemma 5.4, we get

Lemma 5.5. *Under the assumption (5.1), for any* $0 < T < T_\star$, *we have*

$$\int_0^T ||\rho_t||_{L^2}^2 + ||\theta_t||_{L^2}^2 dt \le C. \tag{5.17}$$

Proof. It is easy to see that $L_t^2 L_x^2-$estimate of $\nabla^2 u$ follows from (5.14). The mass equation (1.1) and Lemma 5.1 imply that $L_t^2 L_x^2-$estimate of ρ_t

$$|\rho_t| \le |\nabla\rho||u| + \rho|\text{div}u| \le |\nabla\rho||u| + C|\nabla u|.$$

By Sobolev's embedding, we obtain $u \in L^2(0,T;L^\infty)$. Thus,

$$|||u|\nabla\rho||_{L^2(0,T;L^2)} \le ||u||_{L^2(0,T;L^\infty)}||\nabla\rho||_{L^\infty(0,T;L^2)} \le C.$$

This clearly implies $\rho_t \in L^2(0,T;L^2)$. By a similar method, we can obtain the estimate of θ_t. So we complete the proof of Lemma 5.5. ■

Next, we improve the regularity of the density ρ and the velocity u, using the compatibility condition (1.6).

Lemma 5.6. *Under the assumption (5.1), for any* $0 < T < T_\star$, *we have*

$$\sup_{0\le t\le T}(||\sqrt{\rho}u_t||_{L^2}^2 + ||\nabla u||_{H^1}) + \int_0^T ||\nabla u_t||_{L^2}^2 dt \le C. \tag{5.18}$$

Proof. Differentiating (1.2) with respect to t, we get

$$\rho\partial_{tt}^2 u + \rho u\cdot\nabla u_t - \mu\Delta\partial_t u = -\nabla\partial_t P - \partial_t\rho\partial_t u - \rho\partial_t u\cdot\nabla u - \rho_t u\cdot\nabla u + \partial_t(\rho\theta e_3).$$

Taking the inner product of the above equation with u_t in L^2 and integrating by parts, one gets

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2 dx + \int \mu|\nabla u_t|^2 dx \\ &= \int \partial_t P \text{div}u_t dx - \int \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)dx - \int \rho u\cdot\nabla|u_t|^2 dx \\ &\quad - \int \rho u_t\cdot\nabla u\cdot u_t dx + \int(\rho\theta e_3)_t\cdot u_t dx = \sum_{k=1}^5 J_k. \end{aligned} \tag{5.19}$$

We estimate the terms one by one

$$
\begin{aligned}
|J_1| &\le C||\rho_t||_{L^2}||\mathrm{div} u_t||_{L^2} \le \epsilon||\nabla u_t||_{L^2}^2 + C(\epsilon)||\rho_t||_{L^2}^2,\\
|J_2| &\le \int \rho|u||\nabla u|^2|u_t|dx + \int \rho|u|^2|\nabla u||\nabla u_t|dx + \int \rho|u|^2|\nabla^2 u||u_t|dx\\
&\le C||u||_{L^6}||\nabla u||_{L^3}^2||u_t||_{L^6} + C||u||_{L^6}^2||\nabla u||_{L^6}||\nabla u_t||_{L^2}\\
&\quad + C||u||_{L^6}^2||\nabla^2 u||_{L^2}||u_t||_{L^6}\\
&\le C||\nabla u||_{H^1}||\nabla u_t||_{L^2}\\
&\le \epsilon||\nabla u_t||_{L^2}^2 + C(\epsilon)||\nabla u||_{H^1}^2,\\
|J_3| &\le C||u||_{L^\infty}||\sqrt{\rho}u_t||_{L^2}||\nabla u_t||_{L^2} \le C||\nabla u||_{H^1}||\sqrt{\rho}u_t||_{L^2}||\nabla u_t||_{L^2}\\
&\le \epsilon||\nabla u_t||_{L^2}^2 + C(\epsilon)||\nabla u||_{H^1}^2||\sqrt{\rho}u_t||_{L^2}^2,\\
|J_4| &\le C||\nabla u||_{L^2}||\sqrt{\rho}u_t||_{L^4}^2 \le C||\sqrt{\rho}u_t||_{L^6}^{\frac{3}{2}}||\sqrt{\rho}u_t||_{L^2}^{\frac{1}{2}}\\
&\le C||\nabla u_t||_{L^2}^{\frac{3}{2}}||\sqrt{\rho}u_t||_{L^2}^{\frac{1}{2}}\\
&\le \epsilon||\nabla u_t||_{L^2}^2 + C(\epsilon)||\sqrt{\rho}u_t||_{L^2}^2,\\
|J_5| &\le C\int |\rho_t||u_t||\theta| + |\theta_t||u_t||\rho|\\
&\le \epsilon||\nabla u_t||_{L^2}^2 + C(\epsilon)(||\rho_t||_{L^2}^2||\theta||_{L^3}^2 + ||\theta_t||_{L^2}^2||\rho||_{L^3}^2).
\end{aligned}
$$

Substituting $J_1 - J_5$ into (5.19), we get

$$
\frac{d}{dt}\int \rho|u_t|^2dx + \int |\nabla u_t|^2dx \le C||\sqrt{\rho}u_t||_{L^2}^2(1 + ||\nabla u||_{H^1}^2) + C||\nabla u||_{H^1}^2 + C.
$$

Employing Gronwall's inequality and using (1.6) and Lemma 5.4, we obtain

$$
\sup_{0\le t\le T}\int \rho|u_t|^2dx + \int_0^T\int |\nabla u_t|^2dxdt \le C.
$$

To see $\nabla^2 u \in L^\infty(0,T;L^2)$, we rewrite (1.2) as an elliptic system as follows

$$
\mu\Delta u = \rho u_t + \rho u\cdot\nabla u + \nabla P - \rho\theta e_3. \tag{5.20}
$$

From the standard L^2-estimates for the elliptic system, we get

$$
\begin{aligned}
||\nabla^2 u||_{L^2} &\le C(||\rho u_t||_{L^2} + ||\rho u\nabla u||_{L^2} + ||\nabla P||_{L^2} + ||\rho\theta e_3||_{L^2})\\
&\le C + C||u||_{L^\infty}||\nabla u||_{L^2} + C||\nabla\rho||_{L^2}\\
&\le C + \frac{1}{2}||\nabla^2 u||_{L^2},
\end{aligned}
$$

which immediately implies $\sup_{0\leq t\leq T}||\nabla u||_{H^1}\leq C$.

The proof of Lemma 5.6 is completed. ■

The final step is to obtain the L^p-estimates of the first derivatives of (ρ,θ) and the second derivatives of the velocity u.

Lemma 5.7. *Under the assumption (5.1), for any* $0<T<T_\star$ *and some* $q\in(3,6]$, *we have*

$$\sup_{0\leq t\leq T}(||\rho||_{W^{1,q}}+||\theta||_{W^{1,q}}+||\rho_t||_{L^q}+||\theta_t||_{L^q})+\int_0^T||\nabla^2u||_{L^q}^2dt\leq C. \tag{5.21}$$

Proof. From Lemma 5.3, we have

$$||\nabla\rho||_{L^q}+||\nabla\theta||_{L^q}\leq C(1+\int_0^t||\nabla^2u||_{L^q}ds). \tag{5.22}$$

Applying the standard L^q-estimates to elliptic system (5.20) and Gagliardo-Nirenberg inequality, we find

$$\begin{aligned}||\nabla^2u||_{L^q}&\leq C(||\rho u_t||_{L^q}+||\rho u\nabla u||_{L^q}+||\nabla P||_{L^q}+||\rho\theta e_3||_{L^p})\\&\leq C||\sqrt{\rho}u_t||_{L^2}^{\frac{6-q}{2q}}||u_t||_{L^6}^{\frac{3q-6}{2q}}+C||u||_{L^\infty}||\nabla u||_{L^q}+C||\nabla\rho||_{L^q}\\&\leq C||\nabla\rho||_{L^q}+C||\nabla u_t||_{L^2}+C.\end{aligned} \tag{5.23}$$

Substituting (5.23) into (5.22), and using the Gronwall's inequality, we get

$$||\nabla\rho||_{L^q}+||\nabla\theta||_{L^q}\leq C.$$

For $r=2$ or q, (1.1) implies that

$$\begin{aligned}||\rho_t||_{L^r}&\leq||u||_{L^\infty}||\nabla\rho||_{L^r}+||\rho||_{L^\infty}||\mathrm{div}u||_{L^r}\\&\leq||\nabla u||_{H^1}||\nabla\rho||_{L^r}+||\rho||_{L^\infty}||\nabla u||_{H^1}\leq C.\end{aligned}$$

In a similar way, we can get estimates for θ_t. ■

Proof of Theorem 1.2. All the estimates in Lemma 5.1-Lemma 5.7 will be enough to extend the strong solution (ρ,u,θ) beyond the maximal time of existence $T_\star$, which contradicts the definition of $T_\star$. Therefore, (5.1) is false. The proof of Theorem 1.2 is now complete.

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