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HOMOCLINIC SOLUTIONS FOR SUBQUADRATIC HAMILTONIAN SYSTEMS WITHOUT COERCIVE CONDITIONS

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Abstract. In this paper we investigate the existence and multiplicity of classical homoclinic solutions for the following second order Hamiltonian systems

$$\text{(HS)} \qquad \ddot{u} - L(t)u + \nabla W(t, u) = 0,$$

where $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $\nabla W(t, u)$ is the gradient of W at u. The novelty of this paper is that, assuming that L is bounded in the sense that there are constants $0 < \tau_1 < \tau_2$ such that $\tau_1|u|^2 \leq (L(t)u, u) \leq \tau_2|u|^2$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$ and $W(t, u)$ is of subquadratic growth at infinity, we are able to establish two new criteria to guarantee the existence and multiplicity of classical homoclinic solutions for (HS), respectively. Recent results in the literature are extended and significantly improved.

1. INTRODUCTION

The purpose of this work is to deal with the existence and multiplicity of homoclinic solutions for the following second order Hamiltonian systems

$$\text{(HS)} \qquad \ddot{u} - L(t)u + \nabla W(t, u) = 0,$$

where $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n)$ and $\nabla W(t, u)$ is the gradient of W at u. As usual, we say that a solution $u(t)$ of (HS) is classical homoclinic (to 0) if $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$. If $u(t) \not\equiv 0$, $u(t)$ is called one nontrivial homoclinic solution.

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It is well known that the existence of homoclinic solutions for Hamiltonian systems and their importance in the study of the behavior of dynamical systems have been recognized from Poincaré [14]. They may be "organizing centers" for the dynamics in their neighborhood. From their existence one may, under certain conditions, infer the existence of chaos nearby or the bifurcation behavior of periodic orbits. In the past two decades, with the works of [13] and [16] variational methods and critical point theory have been successfully applied for the search of the existence and multiplicity of homoclinic solutions of (HS). Assuming that $L(t)$ and $W(t,u)$ are either independent of t or periodic in t, many authors have studied the existence of homoclinic solutions of (HS), see for instance [3, 4, 6, 16, 25] and the references therein and some more general Hamiltonian systems are considered in the recent papers [8, 10, 21]. In this case, the existence of homoclinic solutions can be obtained by passing to the limit of periodic solutions of the approximating problems.

If $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t, the existence of homoclinic solutions of (HS) is quite different from the periodic systems, because of the lack of compactness of the Sobolev embedding, see for instance [1, 9, 13, 17] and the references therein. It is worthy of pointing out that to obtain the existence of homoclinic solutions of (HS), the following so-called global Ambrosetti-Rabinowitz condition ((AR) condition) on W due to Ambrosetti-Rabinowitz (e.g., [2]) is assumed in the works mentioned above. Explicitly, there is a constant $\theta > 2$ such that, for every $t \in \mathbb{R}$ and $u \in \mathbb{R}^n \setminus \{0\}$,

$$0 < \theta W(t,u) \leq (\nabla W(t,u), u), \tag{AR}$$

which implies that $W(t,u)$ is of superquadratic growth as $|u| \to +\infty$, where $(\cdot,\cdot): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^n$ and subsequently $|\cdot|$ is the induced norm. However, there are many potentials which are superquadratic as $|u| \to +\infty$ but do not satisfy (AR) condition. Therefore, many authors have been focusing their attention on obtaining the existence of homoclinic solutions under the conditions weaker than (AR) condition, see for instance [5, 11, 12, 22, 23, 30] and the references listed therein. In addition, to verify the (PS) condition for the corresponding functional of (HS), the following coercive assumption on L is often supposed. Specifically,

(L) $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there is a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) > 0$ for all $t \in \mathbb{R}$ and $(L(t)u, u) \geq \alpha(t)|u|^2$ and $\alpha(t) \to +\infty$ as $|t| \to +\infty$,

which indicates that the smallest eigenvalue $l(t)$ of $L(t)$ is coercive, i.e.,

$$l(t) \to +\infty \quad \text{as} \quad |t| \to +\infty, \tag{1.1}$$

where $l(t) := \inf_{|u|=1}(L(t)u, u)$. In [13], assuming (L) holds, Omana and Willem introduced some compact embedding theorem, see its Lemma 1, which has been utilizing from then on and plays an essential role in demonstrating that the corresponding functional verifies the (PS) condition. As is well-known, this coercive condition is a little demanding. For instance, for a simple choice like $L(t) = \tau Id_n$ the condition (1.1) is not satisfied, where $\tau > 0$ and Id_n is the $n \times n$ identity matrix.

Compared with the literature available for $W(t, u)$ being superquadratic as $|u| \to +\infty$, the study of the existence of homoclinic solutions for (HS) under the assumption that $W(t, u)$ is subquadratic at infinity is much more recent and the number of such references is considerably smaller, see for instance [6, 19, 21, 27, 28], where some other types of coercive conditions on L are utilized to obtain the corresponding compact embedding theorems. Besides, the existence of homoclinic solutions for the case that $W(t, u)$ is asymptotically quadratic at infinity has also been investigated by many researchers, see for instance [7, 24, 29, 30].

Inspired by the above papers, in the present paper we study the case when $L(t)$ is bounded in the sense that:

(H_1) $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there are constants $0 < \tau_1 < \tau_2$ such that

$$\tau_1 |u|^2 \leq (L(t)u, u) \leq \tau_2 |u|^2 \quad \text{for all } (t, u) \in \mathbb{R} \times \mathbb{R}^n,$$

and $W(t, u)$ is of subquadratic growth as $|u| \to +\infty$ and for convenience $W(t, 0) = 0$ for all $t \in \mathbb{R}$. For the statement of our main results, the potential $W(t, u)$ is supposed to satisfy the following conditions:

(H_2) there exist $t_0 \in \mathbb{R}$ and $\vartheta \in (1, 2)$ such that

$$\liminf_{(t,u)\to(t_0,0)} \frac{W(t, u)}{|u|^\vartheta} > 0;$$

(H_3) $|\nabla W(t, u)| \leq b(t)|u|^{\vartheta-1}$ for all $t \in \mathbb{R}$ and $u \in \mathbb{R}^n$, where $b : \mathbb{R} \to \mathbb{R}^+$ is a function such that $b \in L^\xi(\mathbb{R}, \mathbb{R}^+)$ for some $1 \leq \xi \leq 2$.

We are now ready to formulate our main results. The first one concerns the existence of at least one nontrivial homoclinic solution to (HS).

Theorem 1.1. *Under the assumptions of* (H_1)-(H_3), (HS) *has at least one nontrivial homoclinic solution.*

If, in addition, W has an even symmetry in u, i.e.,

(H_4) $W(t, u) = W(t, -u)$, $\forall (t, u) \in \mathbb{R} \times \mathbb{R}^n$,

then we get the existence of infinitely many nontrivial homoclinic solutions.

Theorem 1.2. *Under the assumptions of* (H_1)-(H_4), (HS) *possesses infinitely many nontrivial homoclinic solutions.*

Remark 1.3. From (H_3), it is easy to check that $W(t,u)$ is subquadratic as $|u| \to +\infty$. In fact, in view of $W(t,0)=0$ and (H_3), we have

$$|W(t,u)| = \Big|\int_0^1 (\nabla W(t,su),u)ds\Big| \le \frac{b(t)}{\vartheta}|u|^{\vartheta}. \tag{1.2}$$

Remark 1.4. To our best knowledge, for the case that L is bounded from below and unnecessary to satisfy the coercive condition such as (L) or some more demanding coercive condition, only the recent papers [20, 26] dealt with this case. In [26], the authors investigated the case that the potential $W(t,u)$ is superquadratic as $|u| \to +\infty$. In [20], assuming that L and W satisfy the following conditions:

(L)′ $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists a constant $\beta > 0$ such that

$$(L(t)u,u) \ge \beta|u|^2, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^n;$$

(W_1) $W \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ and there exist two constants $1 < \gamma_1 < \gamma_2 < 2$ and two functions a_1, $a_2 \in L^{2/(2-\gamma_1)}(\mathbb{R}, \mathbb{R}^+)$ such that

$$|W(t,u)| \le a_1(t)|u|^{\gamma_1}, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^n, |u| \le 1$$

and

$$|W(t,u)| \le a_2(t)|u|^{\gamma_2}, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^n, |u| \ge 1;$$

(W_2) there exist two functions $b \in L^{2/(2-\gamma_1)}(\mathbb{R}, \mathbb{R}^+)$ and $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$|\nabla W(t,u)| \le b(t)\varphi(|u|), \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^n,$$

where $\varphi(s) = O(s^{\gamma_1 - 1})$ as $s \to 0^+$;

(W_3) there exist an open set $\Omega \subset \mathbb{R}$ and two constants $\gamma_3 \in (1,2)$ and $\eta > 0$ such that

$$W(t,u) \ge \eta|u|^{\gamma_3}, \quad \forall (t,u) \in \Omega\times\mathbb{R}^n, |u| \le 1$$

and (H_4) is verified, then the authors showed that (HS) possesses infinitely many homoclinic solutions in the sense that $u(t) \to 0$ as $|t| \to +\infty$. In other words, they did not show that the homoclinic solutions verify that $\dot{u}(t) \to 0$ as $|t| \to +\infty$ (classical sense). Moreover, in (W_2), the function b is assumed to belong to the space $L^{2/(2-\gamma_1)}(\mathbb{R}, \mathbb{R}^+)$ with $2/(2-\gamma_1) > 2$ since $1 < \gamma_1 < 2$, whereas in our Theorem 1.2, we need only the assumption that $b \in L^{\xi}(\mathbb{R}, \mathbb{R}^+)$ for some $1 \le \xi \le 2$. Furthermore, it is straightforward to see that our condition (H_2) is weaker than (W_3). It turns out also that (H_2) is an essential condition to apply the genus properties for finding infinitely many homoclinic solutions of (HS). Comparing the results in [20] with our Theorems 1.1 and

1.2, we see that, if ∇W is uniformly sublinear (c.f. (H_3) vs (W_2)), then the integrability of b can vary, which can indeed be allowed to be either in $L^{2/(2-\vartheta)}(\mathbb{R}, \mathbb{R}^+)$ or in $L^{\xi}(\mathbb{R}, \mathbb{R}^+)$ for some $\xi \in [1, 2]$, also we offer a sufficient condition (the second part of (H_1)) to ensure that the obtained solutions are classical homoclinic ones. Furthermore, as Example 1.6 below shows there are cases which can be well covered by Theorem 1.2 but not by the results in [26]. Therefore, the results in [20] are extended and complemented.

Remark 1.5. In our present paper, we weaken (L) and (L)$'$ to (H_1). Therefore, one difficulty of this paper is to show that the (PS) condition is satisfied under the hypotheses of Theorem 1.1. Moreover, as we point out that the authors in [20] only investigated homoclinic solutions of (HS) in the sense that $u(t) \to 0$ as $t \to \pm\infty$. Therefore, in order to obtain the existence of homoclinic solutions of (HS) in the classical sense, another difficulty for us is to verify that, under the conditions of Theorem 1.1, $\dot{u}(t) \to 0$ as $t \to \pm\infty$ as well, which is one part of Lemma 3.1 below.

Example 1.6. Now, we present an interesting example to illustrate our main results. We first construct a nonnegative continuous function b on $\mathbb{R}$ such that $b \in L^1(\mathbb{R}, \mathbb{R}^+)$ but $b \notin L^p(\mathbb{R}, \mathbb{R}^+)$ for any $p > 1$; moreover, this function does not have a limit as $|t| \to \infty$. Hence, it is of interest for its own sake. Geometrically, this function consists of a series of "moving" trapezoids whose heights grow exponentially and widths shrink to zero as the trapezoids get far way from the origin; meantime, it maintains a finite area and an infinite "higher area". Mathematically, for any $n \in \mathbb{Z}^\circ := \mathbb{Z} - \{0\}$, set $\delta_n = e^{-|n|}/n^2$. Now, let $b : \mathbb{R} \to \mathbb{R}^+$ be the nonnegative continuous function defined as follows: $b(t) = e^{|n|}$ on $[n - \delta_n/2, n + \delta_n/2]$, $b(t) = 0$ outside $\cup_{n\in\mathbb{Z}^\circ}[n - \delta_n, n + \delta_n]$, and b is linear on the remaining intervals. Then $b \in L^1(\mathbb{R}, \mathbb{R}^+)$ because

$$\int_{\mathbb{R}} b(t)dt = \sum_{n\in\mathbb{Z}^\circ} \frac{3\delta_n}{2} \cdot e^{|n|} = \frac{3}{2} \sum_{n\in\mathbb{Z}^\circ} \frac{1}{n^2} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

On the other hand, for any $p > 1$, we have $b \notin L^p(\mathbb{R}, \mathbb{R}^+)$, since

$$\int_{\mathbb{R}} b^p(t)dt \geq \sum_{n\in\mathbb{Z}^\circ} \delta_n \cdot e^{p|n|} = \sum_{n\in\mathbb{Z}^\circ} \frac{e^{(p-1)|n|}}{n^2} = 2 \sum_{n=1}^{\infty} \frac{e^{(p-1)n}}{n^2} = \infty.$$

We next let

$$L(t) = (2 + \sin t)Id_n, \quad W(t, u) = \frac{2}{3} b(t)|u|^{3/2},$$

where Id_n is the $n \times n$ identity matrix. Then we have

$$|\nabla W(t, u)| = |b(t)|u|^{-1/2}u| \leq b(t)|u|^{1/2},$$

and it is easy to check that (H_1)-(H_4) are satisfied with $\tau_1 = 1$, $\tau_2 = 2$, $\xi = 1$ and t_0 can be any nonzero integer. By Theorem 1.2, (HS) has infinitely many classical homoclinic solutions. However, since $2/(2-\gamma_1) > 2$ for any $\gamma_1 \in (1,2)$, the above discussion shows that $b \notin L^{2/(2-\gamma_1)}(\mathbb{R}, \mathbb{R}^+)$. This means that ($W_2$) can not be satisfied, and so the result in [20] can not be applied.

The remaining part of this paper is structured as follows. Some preliminary results are presented in Section 2. In Section 3, we are devoted to accomplishing the proofs of Theorems 1.1 and 1.2.

2. Preliminary Results

In order to prove our main results, we firstly describe some properties of the space E on which the variational framework associated with (HS) is defined. Letting

$$E = \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^n) : \int_{\mathbb{R}} \Big[|\dot{u}(t)|^2 + (L(t)u(t), u(t)) \Big] dt < +\infty \right\},$$

then E is a Hilbert space with the inner product

$$(u,v)_E = \int_{\mathbb{R}} (\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t)) dt$$

and the corresponding norm $\|u\|^2 = (u,u)_E$. Note that

$$E \subset H^1(\mathbb{R}, \mathbb{R}^n) \subset L^p(\mathbb{R}, \mathbb{R}^n)$$

for all $p \in [2, +\infty]$ with the embedding being continuous. That is, for any $[2, +\infty]$, there is $C_p > 0$ such that

$$\|u\|_p \le C_p \|u\|, \quad \forall u \in E. \tag{2.1}$$

Here, $L^p(\mathbb{R}, \mathbb{R}^n)$ $(2 \le p < +\infty)$ and $H^1(\mathbb{R}, \mathbb{R}^n)$ denote the Banach space of functions on $\mathbb{R}$ with values into $\mathbb{R}^n$ under the norms

$$\|u\|_p := \left(\int_{\mathbb{R}} |u(t)|^p dt \right)^{1/p} \quad \text{and} \quad \|u\|_{H^1} := \left(\|u\|_2^2 + \|\dot{u}\|_2^2 \right)^{1/2},$$

respectively. $L^\infty(\mathbb{R}, \mathbb{R}^n)$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_\infty := \operatorname{ess\,sup} \{ |u(t)| : t \in \mathbb{R} \}.$$

To deal with the existence of homoclinic solutions of (HS), we appeal to the following well-known result, see for example [15].

Definition 2.1. $I \in C^1(\mathcal{B}, \mathbb{R})$ is said to satisfy the (PS) condition if any sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{B}$, for which $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$, possesses a convergent subsequence in $\mathcal{B}$.

Lemma 2.2. *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying the* (PS) *condition. If I is bounded from below, then $c = \inf_E I(u)$ is a critical value of I.*

To obtain the existence of infinitely many homoclinic solutions of (HS) under the assumptions of Theorem 1.2, we shall employ the "genus" properties in critical point theory, see [15, 18].

Let $\mathcal{B}$ be Banach space, $I \in C^1(\mathcal{B}, \mathbb{R})$ and $c \in \mathbb{R}$. We set

$$\Sigma = \{A \subset \mathcal{B} - \{0\} : A \text{ is closed in } \mathcal{B} \text{ and symmetric with respect to } 0\},$$
$$K_c = \{u \in \mathcal{B} : I(u) = c, I'(u) = 0\}, \quad I^c = \{u \in \mathcal{B} : I(u) \le c\}.$$

Definition 2.3. For $A \in \Sigma$, we say the genus of A is j (denoted by $\gamma(A) = j$) if there is an odd map $\psi \in C(A, \mathbb{R}^j \backslash \{0\})$ and j is the smallest integer with this property.

Lemma 2.4. *Let I be an even C^1 functional on $\mathcal{B}$ and satisfy the* (PS)*-condition. For any $j \in \mathbb{N}$, set*

$$\Sigma_j = \{A \in \Sigma : \gamma(A) \ge j\}, \quad c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} I(u).$$

(i) *If $\Sigma_j \neq \emptyset$ and $c_j \in \mathbb{R}$, then c_j is a critical value of I;*

(ii) *if there exists $r \in \mathbb{N}$ such that*

$$c_j = c_{j+1} = \cdots = c_{j+r} = c \in \mathbb{R},$$

and $c \neq I(0)$, then $\gamma(K_c) \ge r + 1$.

Remark 2.5. From Remark 7.3 in [15], we know that if $K_c \in \Sigma$ and $\gamma(K_c) > 1$, then K_c contains infinitely many distinct points, i.e., I has infinitely many distinct critical points in $\mathcal{B}$.

3. Proofs of the Main Results

Now we are going to establish the corresponding variational framework to obtain the existence and multiplicity of homoclinic solutions of (HS). To this end, define the functional $I : \mathcal{B} = E \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \Big[\frac{1}{2}|\dot{u}(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t))\Big] dt \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t, u(t)) dt. \end{aligned} \tag{3.1}$$

The purpose of this section is to prove Theorems 1.1 and 1.2. To this aim, we present some lemmas which will be used in the subsequent discussion.

Lemma 3.1. *Under the conditions of* (H_1)-(H_3)*,* $I \in C^1(E, \mathbb{R})$*, i.e.,* I *is a continuously Fréchet-differentiable functional defined on* E*. Moreover, we have*

$$I'(u)v = \int_{\mathbb{R}} \Big[(\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))\Big] dt \tag{3.2}$$

for all u*,* $v \in E$*, which gives*

$$I'(u)u = \|u\|^2 - \int_{\mathbb{R}} (\nabla W(t, u(t)), u(t)) dt.$$

In addition, any critical point of I *on* E *is a classical solution of* (HS) *with* $u(\pm\infty) = 0 = \dot{u}(\pm\infty)$.

Proof. We firstly show that $I : E \to \mathbb{R}$. By the Hölder inequality, (1.2) and the embedding (2.1), we have

$$\begin{aligned} 0 \le \int_{\mathbb{R}} |W(t, u(t))| dt &\le \frac{1}{\vartheta} \int_{\mathbb{R}} |b(t)||u(t)|^{\vartheta} dt \\ &\le \frac{1}{\vartheta} \|b\|_{\xi} \|u\|^{\vartheta}_{\vartheta\xi^*} \le \frac{C^{\vartheta}_{\vartheta\xi^*}}{\vartheta} \|b\|_{\xi} \|u\|^{\vartheta}, \end{aligned} \tag{3.3}$$

where ξ^* is the conjugate exponent of ξ, i.e., $1 = \frac{1}{\xi} + \frac{1}{\xi^*}$. Combining this with (3.1), we see that $I : E \to \mathbb{R}$.

Next we prove that $I \in C^1(E, \mathbb{R})$. To this end, we rewrite I as follows:

$$A(u) = \frac{1}{2}\|u\|^2, \quad B(u) = \int_{\mathbb{R}} W(t, u(t)) dt.$$

It is easy to check that $A \in C^1(E, \mathbb{R})$, and we have

$$A'(u)v = \int_{\mathbb{R}} [(\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t))] dt.$$

Therefore, it is sufficient to show that this is the case for B. In the process we shall see that $B \in C^1(E, \mathbb{R})$ and

$$B'(u)v = \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t)) dt, \tag{3.4}$$

which is defined for all u, $v \in E$. For any given $u \in E$, let us define $J(u) : E \to \mathbb{R}$ as follows

$$J(u)v = \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t)) dt, \quad \forall v \in E. \tag{3.5}$$

It is obvious that $J(u)$ is linear. Now we show that $J(u)$ is bounded. Indeed, for any given $u \in E$, in view of (2.1) and the Hölder inequality, we obtain that

$$\begin{aligned} |J(u)v| &= \Big|\int_{\mathbb{R}} (\nabla W(t,u(t)), v(t))dt\Big| \le \int_{\mathbb{R}} b(t)|u(t)|^{\vartheta-1}|v(t)|dt \\ &\le \|u\|_\infty^{\vartheta-1}\|b\|_\xi\|v\|_{\xi^*} \le C_\infty^{\vartheta-1}C_{\xi^*}\|u\|^{\vartheta-1}\|b\|_\xi\|v\|. \end{aligned} \tag{3.6}$$

Moreover, for $u, v \in E$, using the Mean Value Theorem, we have

$$\int_{\mathbb{R}} W(t,u(t)+v(t))dt - \int_{\mathbb{R}} W(t,u(t))dt = \int_{\mathbb{R}} (\nabla W(t,u(t)+h(t)v(t)), v(t))dt$$

for some $h(t) \in (0,1)$. On the other hand, $b \in L^\xi(\mathbb{R},\mathbb{R}^+)$ implies that, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\Big(\int_{|t|>T} b^\xi(t)dt\Big)^{1/\xi} < \varepsilon. \tag{3.7}$$

Therefore, on account of (2.1), the Sobolev compact theorem ($E|_{[-T,T]}$ is compactly embedded in $L^\infty([-T,T],\mathbb{R}^n)$) and the Hölder inequality, we have

$$\begin{aligned} &\frac{1}{\|v\|}\Big[\int_{\mathbb{R}} (\nabla W(t,u(t)+h(t)v(t)), v(t))dt - \int_{\mathbb{R}} (\nabla W(t,u(t)), v(t))dt\Big] \\ =&\frac{1}{\|v\|}\int_{\mathbb{R}} (\nabla W(t,u(t)+h(t)v(t)) - \nabla W(t,u(t)), v(t))dt \\ =&\frac{1}{\|v\|}\int_{|t|\le T} (\nabla W(t,u(t)+h(t)v(t)) - \nabla W(t,u(t)), v(t))dt \\ &+ \frac{1}{\|v\|}\int_{|t|>T} (\nabla W(t,u(t)+h(t)v(t)) - \nabla W(t,u(t)), v(t))dt \\ \le&\frac{1}{\|v\|}\Big(\int_{|t|\le T} |\nabla W(t,u(t)+h(t)v(t)) - \nabla W(t,u(t))|^2dt\Big)^{1/2}\Big(\int_{|t|\le T} |v(t)|^2dt\Big)^{1/2} \\ &+ \frac{1}{\|v\|}\int_{|t|>T} b(t)(2|u(t)|^{\vartheta-1} + |v(t)|^{\vartheta-1})|v(t)|dt \\ \le&\frac{1}{\|v\|}C_2\|v\|\Big(\int_{|t|\le T} |\nabla W(t,u(t)+h(t)v(t)) - \nabla W(t,u(t))|^2dt\Big)^{1/2} \\ &+ \frac{1}{\|v\|}\Big(\int_{|t|>T} b^\xi(t)dt\Big)^{1/\xi}\Big(2C_\infty^{\vartheta-1}\|u\|^{\vartheta-1} + C_\infty^{\vartheta-1}\|v\|^{\vartheta-1}\Big)\Big(\int_{|t|>T} |v(t)|^{\xi^*}dt\Big)^{1/\xi^*} \\ \le&C_2\Big(\int_{|t|\le T} |\nabla W(t,u(t)+h(t)v(t)) - \nabla W(t,u(t))|^2dt\Big)^{1/2} \\ &+ \varepsilon C_\infty^{\vartheta-1}C_{\xi^*}\Big(2\|u\|^{\vartheta-1} + \|v\|^{\vartheta-1}\Big) \to 0 \quad \text{as } v \to 0, \end{aligned}$$

which, together with (3.6), implies that (3.4) holds. It remains to prove that B' is continuous. Suppose that $u \to u_0$ in E, then, by (2.1) and (3.7), it deduces that

$$
\begin{aligned}
&\sup_{\|v\|=1} |B'(u)v - B'(u_0)v| \\
=& \sup_{\|v\|=1} \Big|\int_{\mathbb{R}} (\nabla W(t,u(t)) - \nabla W(t,u_0(t)), v(t))dt\Big| \\
\le& \sup_{\|v\|=1} \int_{|t|\le T} \Big|(\nabla W(t,u(t)) - \nabla W(t,u_0(t)), v(t))\Big| dt \\
&+ \sup_{\|v\|=1} \int_{|t|>T} b(t)(|u(t)|^{\vartheta-1} + |u_0(t)|^{\vartheta-1})|v(t)|dt \\
\le& \sup_{\|v\|=1} \Big(\int_{|t|\le T} |\nabla W(t,u(t)) - \nabla W(t,u_0(t))|^2 dt\Big)^{1/2} \Big(\int_{|t|\le T} |v(t)|^2 dt\Big)^{1/2} \\
&+ \sup_{\|v\|=1} (\|u\|_\infty^{\vartheta-1} + \|u_0\|_\infty^{\vartheta-1}) \Big(\int_{|t|>T} b^{\xi}(t)dt\Big)^{1/\xi} \Big(\int_{|t|>T} |v(t)|^{\xi^*} dt\Big)^{1/\xi^*} \\
\le& C_2 \Big(\int_{|t|\le T} |\nabla W(t,u(t)) - \nabla W(t,u_0(t))|^2 dt\Big)^{1/2} \\
&+ C_{\xi^*}(\|u\|_\infty^{\vartheta-1} + \|u_0\|_\infty^{\vartheta-1}) \Big(\int_{|t|>T} b^{\xi}(t)dt\Big)^{1/\xi} \\
&\le \varepsilon C_2 + \varepsilon C_{\xi^*}(\|u\|_\infty^{\vartheta-1} + \|u_0\|_\infty^{\vartheta-1}),
\end{aligned}
$$

which yields that $B'(u)v - B'(u_0)v \to 0$ as $u \to u_0$ uniformly with respect to v, which implies that B' is continuous. Therefore, we have shown that $I \in C^1(E, \mathbb{R})$.

Lastly, we check that critical points of I are classical solutions of (HS) satisfying $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to +\infty$. It is well known that $E \subset C^0(\mathbb{R}, \mathbb{R}^n)$ (the space of continuous functions u on $\mathbb{R}$ such that $u(t) \to 0$ as $|t| \to +\infty$). Now, if $u \in E$ is a critical point of I, we deduce from (3.2) that $L(t)u - \nabla W(t,u)$ is the weak derivative of $\dot{u}$. Recall that $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ and that $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, we thus have u is indeed in $C^2(\mathbb{R}, \mathbb{R}^n)$. Hence, u is a classical solution of (HS) with $u(t) \to 0$ as $|t| \to +\infty$. In what follows, we show that $\dot{u}(t) \to 0$ as $|t| \to +\infty$ as well. To do this, due to the fact that (see Fact 2.8 in [8])

$$|u(t)| \le \sqrt{2}\Big(\int_{t-1/2}^{t+1/2} (|u(s)|^2 + |\dot{u}(s)|^2) ds\Big)^{1/2},$$

we observe that

$$|\dot{u}(t)|^2 \le 2\int_{t-1/2}^{t+1/2} (|u(s)|^2 + |\dot{u}(s)|^2) ds + 2\int_{t-1/2}^{t+1/2} |\ddot{u}(s)|^2 ds.$$

On the other hand, since

$$\int_{t-1/2}^{t+1/2}(|u(s)|^2+|\dot{u}(s)|^2)ds \to 0,$$

as $t \to \pm\infty$, it suffices to prove that

$$\int_{t-1/2}^{t+1/2}|\ddot{u}(s)|^2 ds \to 0 \tag{3.8}$$

as $t \to \pm\infty$. In fact, in view of (HS), we obtain that

$$\begin{aligned}\int_{t-1/2}^{t+1/2}|\ddot{u}(s)|^2 ds &= \int_{t-1/2}^{t+1/2}\Big(|\nabla W(s,u(s))|^2+|L(s)u(s)|^2\Big)ds\\ &\quad - 2\int_{t-1/2}^{t+1/2}(L(s)u(s),\nabla W(s,u(s)))ds\\ &\le 2\int_{t-1/2}^{t+1/2}\Big(|\nabla W(s,u(s))|^2+|L(s)u(s)|^2\Big)ds.\end{aligned}$$

In addition, we note that $\nabla W(s,u(s))$ is continuous and that $\nabla W(t,0)=0$ for all $t\in\mathbb{R}$ (see (H_3)), $u(t)\to 0$ as $|t|\to+\infty$ and $L(t)$ is bounded, therefore (3.8) follows directly. ∎

Lemma 3.2. *If* (H_1) *and* (H_3) *hold, then* I *satisfies* (PS)-*condition.*

Proof. Assume that $\{u_j\}_{j\in\mathbb{N}}\subset E$ is a sequence such that $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'(u_j)\to 0$ as $j\to+\infty$. Then there exists a constant $C>0$ such that

$$|I(u_j)|\le C, \tag{3.9}$$

for every $j\in\mathbb{N}$. We firstly prove that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. From (3.1), (3.3) and (3.9), it is easy to deduce that

$$\begin{aligned}\|u_j\|^2 &= 2I(u_j)+2\int_{\mathbb{R}}W(t,u_j(t))dt\\ &\le 2C+\frac{2}{\vartheta}C_{\vartheta\xi^*}^{\vartheta}\|b\|_\xi\|u\|^{\vartheta}.\end{aligned} \tag{3.10}$$

Since $1<\vartheta<2$, the inequality (3.10) shows that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. Then the sequence $\{u_j\}_{j\in\mathbb{N}}$ has a subsequence, again denoted by $\{u_j\}_{j\in\mathbb{N}}$, and there exists $u\in E$ such that

$$u_j \rightharpoonup u \text{ weakly in } E,$$

which yields that

$$(I'(u_j)-I'(u))(u_j-u)\to 0 \quad \text{as } j\to+\infty, \tag{3.11}$$

and there exists a constant $M > 0$ such that

$$\|u_j\|_\infty \leq C_\infty \|u_j\| \leq M \quad \text{and} \quad \|u\|_\infty \leq C_\infty \|u\| \leq M. \tag{3.12}$$

On account of the continuity of $\nabla W(t,u)$ and $u_j \to u$ in $L^\infty_{loc}(\mathbb{R},\mathbb{R}^n)$, it follows that there exists $k_0 \in \mathbb{N}$ such that

$$\int_{|t|\leq T} (\nabla W(t,u_j(t)) - \nabla W(t,u(t)), u_j(t)-u(t))dt < \varepsilon \quad \text{for } k \geq k_0. \tag{3.13}$$

On the other hand, joining (H_3), (2.1), (3.7) and (3.12), we obtain that

$$\begin{aligned}
&\int_{|t|>T} (\nabla W(t,u_j(t)) - \nabla W(t,u(t)), u_j(t)-u(t))dt \\
&\leq \int_{|t|>T} |\nabla W(t,u_j(t)) - \nabla W(t,u(t)||u_j(t)-u(t)|dt \\
&\leq \int_{|t|>T} b(t)(|u_j(t)|^{\vartheta-1} + |u(t)|^{\vartheta-1})(|u_j(t)| + |u(t)|)dt \\
&\leq 2\int_{|t|>T} b(t)(|u_j(t)|^{\vartheta} + |u(t)|^{\vartheta})dt \\
&\leq 2\Big(\int_{|t|>T} b^\xi(t)dt\Big)^{1/\xi}(\|u_j\|^\vartheta_{\vartheta\xi^*} + \|u\|^\vartheta_{\vartheta\xi^*}) \\
&\leq 2\Big(\int_{|t|>T} b^\xi(t)dt\Big)^{1/\xi} C^\vartheta_{\vartheta\xi^*}(\|u_j\|^\vartheta + \|u\|^\vartheta) \\
&\leq 4\varepsilon C^\vartheta_{\vartheta\xi^*}\Big(\frac{M}{C_\infty}\Big)^\vartheta.
\end{aligned} \tag{3.14}$$

Since $\varepsilon > 0$ is arbitrary, combining (3.13) with (3.14), we get

$$\int_{\mathbb{R}} (\nabla W(t,u_j(t)) - \nabla W(t,u(t)), u_j(t)-u(t))dt \to 0 \tag{3.15}$$

as $j \to +\infty$. Consequently, in view of (3.11), (3.15) and the following equality

$$\begin{aligned}
(I'(u_j) - I'(u), u_j - u) =& \|u_j - u\|^2 \\
&- \int_{\mathbb{R}} (\nabla W(t,u_j(t)) - \nabla W(t,u(t)), u_j(t)-u(t))dt,
\end{aligned}$$

it is easy to conclude that $\|u_j - u\| \to 0$ as $j \to +\infty$. ■

Now we are in the position to complete the proof of Theorems 1.1 and 1.2. The attention will be given mainly to the proof of Theorem 1.2.

Proof of Theorem 1.1. It is clear that $I(0) = 0$, and by Lemma 3.2 we have known that I is a C^1 functional on E satisfying the (PS)-condition. On the other hand, in view of (3.1) and (3.3), we obtain that

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{C^{\vartheta}_{\vartheta\xi^*}}{\vartheta}\|b\|_\xi\|u\|^{\vartheta}, \tag{3.16}$$

which implies that I is bounded below on E. Hence by Lemma 2.2, $c = \inf_E I(u)$ is a critical value of I, namely, there is a critical point $u^* \in E$ such that $I(u^*) = c$ and $I'(u^*) = 0$. Moreover, this critical value c is a negative real number as the following argument will show, and so u^* is a nontrivial classical homoclinic solution by Lemma 3.1. ∎

Proof of Theorem 1.2. Now, we additionally have from (H_4) that I is even and $I(0) = 0$. In order to apply Lemma 2.4, we prove that

$$\text{for any } j \in \mathbb{N} \text{ there exists } \varepsilon > 0 \text{ such that } \gamma(I^{-\varepsilon}) \geq j. \tag{3.17}$$

By (H_2), there exist an open set $D \subset \mathbb{R}$ with $t_0 \in D$, $\sigma > 0$ and $\eta > 0$ such that

$$W(t,u) \geq \eta|u|^{\vartheta}, \quad \forall (t,u) \in D \times \mathbb{R}^n, |u| \leq \sigma. \tag{3.18}$$

For any $j \in \mathbb{N}$, we take j disjoint open sets D_i such that $\bigcup_{i=1}^{j} D_i \subset D$. For $i = 1, 2, \ldots, j$, let $u_i \in (W_0^{1,2}(D_i) \cap E)\backslash\{0\}$ with $\|u_i\| = 1$, and

$$E_j = \text{span}\{u_1, u_2, \ldots, u_j\}, \quad S_j = \{u \in E_j : \|u\| = 1\}.$$

Then, for any $u \in E_j$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, j$ such that

$$u(t) = \sum_{i=1}^{j} \lambda_i u_i(t) \quad \text{for } t \in \mathbb{R}. \tag{3.19}$$

From which it follows that

$$\|u\|_\vartheta = \Big(\int_{\mathbb{R}} |u(t)|^{\vartheta}\Big)^{1/\vartheta} = \Big(\sum_{i=1}^{j} |\lambda_i|^{\vartheta} \int_{D_i} |u_i(t)|^{\vartheta} dt\Big)^{1/\vartheta} \tag{3.20}$$

and

$$\begin{aligned} \|u\|^2 &= \int_{\mathbb{R}} [|\dot{u}(t)|^2 + (L(t)u(t), u(t))]dt \\ &= \sum_{i=1}^{j} \lambda_i^2 \int_{D_i} [|\dot{u}_i(t)|^2 + (L(t)u_i(t), u_i(t))]dt \\ &= \sum_{i=1}^{j} \lambda_i^2 \int_{\mathbb{R}} [|\dot{u}_i(t)|^2 + (L(t)u_i(t), u_i(t))]dt \\ &= \sum_{i=1}^{j} \lambda_i^2 \|u_i\|^2 = \sum_{i=1}^{j} \lambda_i^2. \end{aligned} \tag{3.21}$$

Since all norms of a finite dimensional norm space are equivalent, there is a constant $d = d(j) > 0$ such that

$$d\|u\| \leq \|u\|_{\vartheta}, \quad \forall u \in E_j. \tag{3.22}$$

Note that $W(t,0) = 0$, and so according to (3.18)-(3.22), we have

$$\begin{aligned} I(su) &= \frac{s^2}{2}\|u\|^2 - \int_{\mathbb{R}} W(t, su(t))dt \\ &= \frac{s^2}{2}\|u\|^2 - \sum_{i=1}^{j} \int_{D_i} W(t, s\lambda_i u_i(t))dt \\ &\leq \frac{s^2}{2}\|u\|^2 - \eta s^{\vartheta} \sum_{i=1}^{j} |\lambda_i|^{\vartheta} \int_{D_i} |u_i(t)|^{\vartheta} dt \\ &= \frac{s^2}{2}\|u\|^2 - \eta s^{\vartheta}\|u\|_{\vartheta}^{\vartheta} \\ &\leq \frac{s^2}{2}\|u\|^2 - \eta (ds)^{\vartheta}\|u\|^{\vartheta} \\ &= \frac{s^2}{2} - \eta (ds)^{\vartheta} \end{aligned} \tag{3.23}$$

for all $u \in S_j$ and sufficient small $s > 0$. In this case (3.18) is applicable, since u is continuous on $\overline{D}$ and so $|s\lambda_i u_i(t)| \leq \sigma, \forall t \in D, i = 1, 2, \cdots, j$ can be true for sufficiently small s. Therefore, it follows from (3.23) that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$I(\delta u) < -\varepsilon \quad \text{for } u \in S_j. \tag{3.24}$$

Let

$$S_j^{\delta} = \{\delta u : u \in S_j\}, \quad \Omega = \{(\lambda_1, \lambda_2, \ldots, \lambda_j) \in \mathbb{R}^j : \sum_{i=1}^{j} \lambda_i^2 < \delta^2\}.$$

Then it follows from (3.24) that

$$I(u) < -\varepsilon, \quad \forall u \in S_j^{\delta},$$

which, together with the fact that I is an even C^1 functional on E, yields that

$$S_j^{\delta} \subset I^{-\varepsilon} \in \Sigma,$$

where $I^{-\varepsilon}$ and Σ have been previously introduced in Section 2. On the other hand, it follows from (3.19) and (3.21) that there exists an odd homeomorphism $\psi \in C(S_j^{\delta}, \partial\Omega)$. By some properties of the genus (see 3° of Propositions 7.5 and 7.7 in [15]), we infer

$$\gamma(I^{-\varepsilon}) \geq \gamma(S_j^{\delta}) = j, \tag{3.25}$$

so (3.17) follows. Set

$$c_j = \inf_{A\in\Sigma_j} \sup_{u\in A} I(u),$$

where Σ_j is defined in Lemma 2.4. It follows from (3.25) and the fact that I is bounded from below on E (see (3.16)), we have $-\infty < c_j \leq -\varepsilon < 0$, which implies that, for any $j \in \mathbb{N}$, c_j is a real negative number. By lemma 2.4 and Remark 2.5, I has infinitely many nontrivial critical points, and consequently, (HS) possesses infinitely many classical homoclinic solutions. ■

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