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THE HIERARCHICAL MINIMAX THEOREMS

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Abstract. We study the minimax theorems for set-valued mappings with several hierarchical process, and propose three versions for minimax theorems in topological vector spaces setting. These problems arise from some minimax theorems in the vector settings. As applications, we discuss the existences of two kinds of saddle points. Our results are new or include as special cases recent existing results.

1. Introduction Preliminaries

Let X, Y be two nonempty sets in two Hausdorff topological vector spaces, respectively, Z be a Hausdorff topological vector space, $C \subset Z$ a closed convex and pointed cone with apex at the origin and $intC \neq \emptyset$, that is, C is proper closed with $intC \neq \emptyset$ and satisfies $\lambda C \subseteq C$, $\forall \lambda > 0$; $C + C \subseteq C$; and $C \cap (-C) = \{0\}$. The scalar hierarchical minimax theorems stated as follows: for given three mappings $F, G, H : X \times Y \rightrightarrows \mathbb{R}$, under some suitable conditions so that the following inequality holds:

$$(s-H) \qquad \min \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x,y) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} H(x,y).$$

For given three mappings $F, G, H : X \times Y \rightrightarrows Z$, the first version of hierarchical minimax theorems stated that under some suitable conditions so that the following inequality holds:

$$(H_1) \qquad Max \bigcup_{y \in Y} Min_w \bigcup_{x \in X} F(x,y) \subset Min(co \bigcup_{x \in X} Max_w \bigcup_{y \in Y} F(x,y)) + C.$$

The second version of hierarchical minimax theorems stated that under some suitable conditions so that the following inequality holds:

$$(H_2) \qquad Max \bigcup_{y \in Y} Min_w \bigcup_{x \in X} F(x,y) \subset Min \bigcup_{x \in X} Max_w \bigcup_{y \in Y} F(x,y) + C.$$

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The third version of hierarchical minimax theorems stated that under some suitable conditions so that the following inequality holds:

$$(H_3)\ \ Min \bigcup_{x\in X} Max_w \bigcup_{y\in Y} F(x,y) \subset Max \bigcup_{y\in Y} Min_w \bigcup_{x\in X} H(x,y) + Z \setminus (C\setminus\{0\}).$$

These versions ($H_1 - H_3$) arise naturally from some minimax theorems in the vector settings. We refer to [1, 2, 3].

We present some fundamental concepts which will be used in the sequel.

Definition 1.1. [1, 2]. Let A be a nonempty subset of Z. A point $z\in A$ is called a

(a) *minimal point* of A if $A\cap(z-C)=\{z\}$; $MinA$ denotes the set of all minimal points of A;

(b) *maximal point* of A if $A\cap(z+C)=\{z\}$; $MaxA$ denotes the set of all maximal points of A;

(c) *weakly minimal point* of A if $A\cap(z-\text{int}C)=\emptyset$; Min_wA denotes the set of all weakly minimal points of A;

(d) *weakly maximal point* of A if $A\cap(z+\text{int}C)=\emptyset$; Max_wA denotes the set of all weakly maximal points of A.

We note that , for a nonempty compact set A, the both sets $MaxA$ and $MinA$ are nonempty. Furthermore, $MinA\subset \text{Min}_wA$, $MaxA\subset Max_wA$, $A\subset MinA+C$, and $A\subset MaxA-C$. Following [2], we denote both Max and Max_w by $\max$ (both Min and Min_w by $\min$) in $\mathbb{R}$ since both Max and Max_w (both Min and Min_w) are same in $\mathbb{R}$.

Definition 1.2. Let U, V be Hausdorff topological spaces. A set-valued map $F: U \rightrightarrows V$ with nonempty values is said to be

(a) *upper semi-continuous at* $x_0\in U$ if for every $x_0\in U$ and for every open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M)\subset N$;

(b) *lower semi-continuous at* $x_0\in U$ if for any sequence $\{x_n\}\subset U$ such that $x_n\to x_0$ and any $y_0\in F(x_0)$, there exists a sequence $y_n\in F(x_n)$ such that $y_n\to y_0$;

(c) *continuous at* $x_0\in U$ if F is upper semi-continuous as well as lower semi-continuous at x_0.

We note that T is upper semicontinuous at x_0 and $T(x_0)$ is compact, then for any net $\{x_\nu\}\subset U$, $x_\nu\to x_0$, and for any net $y_\nu\in T(x_\nu)$ for each ν, there exists $y_0\in T(x_0)$ and a subnet $\{y_{\nu_\alpha}\}$ such that $y_{\nu_\alpha}\to y_0$. We can refer to [4] for more details. We also note that T is lower semicontinuous at x_0 if for any net $\{x_\nu\}\subset U$,

$x_\nu \to x_0$, $y_0 \in T(x_0)$ implies that there exists net $y_\nu \in T(x_\nu)$ such that $y_\nu \to y_0$. For more details, we refer the reader to [4] or [5].

Definition 1.3. [2, 6]. Let $k \in \text{int}C$ and $v \in Z$. The *Gerstewitz function* $\xi_{kv} : Z \to \mathbb{R}$ is defined by

$$\xi_{kv}(u) = \min\{t \in \mathbb{R} : u \in v + tk - C\}.$$

We present some fundamental properties of the scalarization function.

Proposition 1.1. [2, 6]. *Let $k \in \text{int}C$ and $v \in Z$. The Gerstewitz function $\xi_{kv} : Z \to \mathbb{R}$ has the following properties :*

(a) $\xi_{kv}(u) > r \Leftrightarrow u \notin v + rk - C$;

(b) $\xi_{kv}(u) \geq r \Leftrightarrow u \notin v + rk - \text{int}C$;

(c) $\xi_{kv}(\cdot)$ *is a convex function;*

(d) $\xi_{kv}(\cdot)$ *is an increasing function, that is,* $u_2 - u_1 \in \text{int}S \Rightarrow \xi_{kv}(u_1) < \xi_{kv}(u_2)$;

(e) $\xi_{kv}(\cdot)$ *is a continuous function.*

We also need the following different kinds of cone-convexities for set-valued mappings.

Definition 1.4. [1]. Let X be a nonempty convex subset of a topological vector space. A set-valued mapping $F : X \rightrightarrows Z$ is said to be

(a) *above-C-convex* (respectively, *above-C-concave*) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$,

$$F(\lambda x_1 + (1 - \lambda)x_2) \subset \lambda F(x_1) + (1 - \lambda)F(x_2) - C$$

$$(\text{respectively, } \lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) - C)\,;$$

(b) *above-naturally C-quasi-convex* on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$,

$$F(\lambda x_1 + (1 - \lambda)x_2) \subset \text{co}\{F(x_1) \cup F(x_2)\} - C,$$

where $\text{co}A$ denotes the convex hull of a set A;

The following whole intersection theorem is given by Ha [7].

Lemma 1.1. [7]. *Let X be a nonempty compact convex subset of real Hausdorff topological vector space, Y be a nonempty convex subset of real Hausdorff topological vector space. Let the three mappings $A, B, D : X \rightrightarrows Y$ with $A(x) \subset B(x) \subset D(x)$ for all $x \in X$ and satisfy*

(a) *$A(x), D(x)$ are convex in Y for each $x \in X$, $A^{-1}(y), D^{-1}(y)$ are open in X for each $y \in Y$; and*

(b) $B(x)$ *is open for each* $x \in X$*, and* $X \setminus B^{-1}(y)$ *is convex in* X *for each* $y \in Y$*.*

Then either there is an $x_0 \in X$ *such that* $A(x_0)$ *is a empty set, or the whole intersection* $\bigcap_{x\in X} D(x)$ *is nonempty.*

2. Scalar Hierarchical Minimax Theorems

We first establish the following scalar hierarchical minimax theorem.

Theorem 2.1. *Let* X *be a nonempty compact convex subset of real Hausdorff topological vector space,* Y *be a nonempty convex subset of real Hausdorff topological vector space. Suppose that the set-valued mappings* $F, G, H : X \times Y \rightrightarrows \mathbb{R}$ *with* $F(x, y) \subset G(x, y) \subset H(x, y)$ *for all* $(x, y) \in X \times Y$ *and satisfy the following conditions:*

(i) $x \mapsto F(x, y)$ *is lower semi-continuous on* X *for each* $y \in Y$ *and* $y \mapsto F(x, y)$ *is above-*$\mathbb{R}_+$*-concave on* Y *for each* $x \in X$*;*

(ii) $x \mapsto G(x, y)$ *is above-naturally* $\mathbb{R}_+$*-quasi-convex for each* $y \in Y$*, and* $y \mapsto G(x, y)$ *is lower semi-continuous on* Y *for each* $x \in X$*; and*

(iii) $x \mapsto H(x, y)$ *is lower semi-continuous on* X *for each* $y \in Y$*, and* $y \mapsto H(x, y)$ *is above-*$\mathbb{R}_+$*-concave on* Y *for each* $x \in X$*.*

Then, for each $t \in \mathbb{R}$*, either there is* $x_0 \in X$ *such that*

$$F(x_0, y) \in t - \mathbb{R}_+$$

for all $y \in Y$ *or there is* $y_0 \in Y$ *such that*

$$H(x, y_0) \bigcap (t + \mathbb{R}_+) \neq \emptyset$$

for all $x \in X$*. Furthermore, in additional, if* Y *is compact,* $\bigcup_{y\in Y} F(x, y)$ *is compact for each* $x \in X$*,* $y \mapsto H(x, y)$ *is lower semi-continuous for each* $x \in X$ *and* $\bigcup_{x\in X} H(x, y)$ *is compact for each* $y \in Y$*, then we have the relation* $(s - H)$ *hold.*

Proof. For any given $t \in \mathbb{R}$. Let three sets $A, B, D \subset X \times Y$ be defined by

$$A = \{(x, y) \in X \times Y : \exists f \in F(x, y), f > t\},$$

$$B = \{(x, y) \in X \times Y : \exists g \in G(x, y), g > t\},$$

and

$$D = \{(x, y) \in X \times Y : \exists h \in H(x, y), h > t\}.$$

Since $F(x, y) \subset G(x, y) \subset H(x, y)$ for all $(x, y) \in X \times Y$,

$$A \subset B \subset D.$$

(a) Choose any $y_1, y_2 \in A(x) = \{y \in Y : \exists f \in F(x,y), f > t\}$. There exist $f_1 \in F(x, y_1)$ with $f_1 > t$ and $f_2 \in F(x, y_2)$ with $f_2 > t$. Then, for any $\lambda \in [0,1]$, $t \in \lambda F(x, y_1) + (1-\lambda)F(x, y_2) - \mathbb{R}_+$. By the above-$\mathbb{R}_+$-concavity of F, we have there is $f_\lambda \in F(x, \lambda y_1 + (1-\lambda)y_2)$ such that $f_\lambda > t$. Thus, $\lambda y_1 + (1-\lambda)y_2 \in A(x)$, and hence $A(x)$ is convex for each $x \in X$. Similarly, by the above-$\mathbb{R}_+$-concavity of H, the set $D(x)$ is convex for each $x \in X$.

Next, we claim that the set $X \setminus A^{-1}(y) = \{x \in X : \text{for all } f \in F(x,y),\ f \leq t\}$ is closed for each $y \in Y$. From this, the set $A^{-1}(y)$ will be open for each $y \in Y$. Let the net $\{x_n\} \subset \{x \in X : \text{for all } f \in F(x,y),\ f \leq t\}$ that converges to some point $x_0 \in X$. By the lower semi-continuity of F, for any $f_0 \in F(x_0, y)$ there exists $f_n \in F(x_n, y)$ such that $f_n \to f_0$. Since $f_n \leq t$, we have $f_0 \leq t$ and hence $x_0 \in \{x \in X : \text{for all } f \in F(x,y),\ f \leq t\}$. This proves that the set $\{x \in X : \text{for all } f \in F(x,y),\ f \leq t\}$ is closed. Similarly, by the lower semi-continuity of G and H, the sets $D^{-1}(y)$ and $B(x)$ are open for each $x \in X$ and $y \in Y$.

We now claim that the set $X \setminus B^{-1}(y)$ is convex in X for each $y \in Y$. Fixed any $y \in Y$, choosing any $x_1, x_2 \in X \setminus B^{-1}(y)$ and any $\tau \in [0,1]$. For any $g_1 \in G(x_1, y)$ with $g_1 \leq t$ and any $g_2 \in G(x_2, y)$ with $g_2 \leq t$, we have $\tau g_1 + (1-\tau)g_2 \leq t$. That is,

$$co\{G(x_1) \cup G(x_2)\} \subset t - \mathbb{R}_+.$$

By the above-naturally quasi-$\mathbb{R}_+$-convexity of G, we have

$$G(\tau x_1 + (1-\tau)x_2) \subset t - \mathbb{R}_+.$$

Thus, $\tau x_1 + (1-\tau)x_2 \in X \setminus B^{-1}(y)$ and the set $X \setminus B^{-1}(y)$ is convex in X for each $y \in Y$.

Since all conditions of Lemma 1.1 hold, by Lemma 1.1, either there is an $x_0 \in X$ such that $A(x_0)$ is a empty set, or the whole intersection $\bigcap_{x \in X} D(x)$ is nonempty. That is, for each $t \in \mathbb{R}$, either there is $x_0 \in X$ such that $F(x_0, y) \in t - \mathbb{R}_+$ for all $y \in Y$ or there is $y_0 \in Y$ such that $H(x, y_0) \bigcap (t + \mathbb{R}_+) \neq \emptyset$ for all $x \in X$.

Furthermore, if we combine the additional conditions, we have the relation $(s - H)$ is valid. ∎

We note that Theorem 1 includes some special cases such as $F(x,y) = G(x,y) \subset H(x,y)$, $F(x,y) \subset G(x,y) = H(x,y)$ and $F(x,y) = G(x,y) = H(x,y)$. We state the last one as follows.

Corollary 2.1. *Let X be a nonempty compact convex subset of real Hausdorff topological vector space, Y be a nonempty convex subset of real Hausdorff topological vector space. Let the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ satisfy the following conditions:*

(i) *$x \mapsto F(x,y)$ is lower semi-continuous on X for each $y \in Y$ and $y \mapsto F(x,y)$ is above-$\mathbb{R}_+$-concave on Y for each $x \in X$; and*

(ii) $x \mapsto F(x,y)$ *is above-naturally* $\mathbb{R}_+$*-quasi-convex for each* $y \in Y$*, and* $y \mapsto F(x,y)$ *is lower semi-continuous on* Y *for each* $x \in X$*;*

Then, for each $t \in \mathbb{R}$*, either there is* $x_0 \in X$ *such that*

$$F(x_0, y) \in t - \mathbb{R}_+$$

for all $y \in Y$ *or there is* $y_0 \in Y$ *such that*

$$F(x, y_0) \bigcap (t + \mathbb{R}_+) \neq \emptyset$$

for all $x \in X$*. Furthermore, in additional, if* Y *is compact,* $\bigcup_{y\in Y} F(x,y)$ *and* $\bigcup_{x\in X} F(x,y)$ *are compact for each* $x \in X$ *and for each* $y \in Y$*, then we have*

$$\min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y) = \max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y).$$

Remark 2.1. We can not compare Corollary 2.1 with Fan's famous minimax theorem[8] even when the mapping F is single-valued function since we have no upper semi-continuity for the mapping F.

3. Hierarchical Minimax Theorems

In this section, we will present three versions of hierarchical minimax theorems. The first one so that the relation (H_1) is true as follows.

Theorem 3.1. *Let* X*,* Y *be nonempty compact convex subsets of real Hausdorff topological vector spaces, respectively. Let the set-valued mappings* $F, G, H : X \times Y \rightrightarrows Z$ *such that* $F(x,y) \subset G(x,y) \subset H(x,y)$ *for all* $(x,y) \in X \times Y$*, and satisfy the following conditions:*

(i) $(x,y) \mapsto F(x,y)$ *is continuous with nonempty compact values, and* $y \mapsto F(x,y)$ *is above-*C*-concave on* Y *for each* $x \in X$*;*

(ii) $x \mapsto G(x,y)$ *is above-naturally* C*-quasi-convex for each* $y \in Y$*, and* $y \mapsto G(x,y)$ *is lower semi-continuous on* Y *for each* $x \in X$*;*

(iii) $(x,y) \mapsto H(x,y)$ *is continuous with nonempty compact values,* $y \mapsto H(x,y)$ *is above-*C*-concave on* Y *for each* $x \in X$*; and*

(iv) *for each* $y \in Y$*,*

$$Max \bigcup_{y\in Y} Min_w \bigcup_{x\in X} F(x,y) \subset Min_w \bigcup_{x\in X} H(x,y) + C.$$

Then the relation (H_1) *is valid.*

Proof. Let $\Gamma(x) := Max_w \bigcup_{y\in Y} F(x,y)$ for all $x \in X$. From Lemma 2.4 and Proposition 3.5 in [1], the mapping $x \mapsto \Gamma(x)$ is upper semi-continuous with nonempty compact values on X. Hence $\bigcup_{x\in X}\Gamma(x)$ is compact, and so is $co(\bigcup_{x\in X}\Gamma(x))$. Then $co(\bigcup_{x\in X}\Gamma(x)) + C$ is closed convex set with nonempty interior. Suppose that $v \notin co(\bigcup_{x\in X}\Gamma(x)) + C$. By separation theorem, there is a $k \in \mathbb{R}$, $\epsilon > 0$, and a nonzero continuous linear functional $\xi : Z \mapsto \mathbb{R}$ such that

$$\xi(v) \le k - \epsilon < k \le \xi(u + c) \tag{1}$$

for all $u \in co(\bigcup_{x\in X}\Gamma(x))$ and $c \in C$. From this we can see that $\xi \in C^\star$, where

$$C^\star = \{g \in Z^\star : g(c) \ge 0 \text{ for all } c \in C\},$$

$Z^\star$ is the set of all nonzero continuous linear functional on Z, and $\xi(v) < \xi(u)$ for all $u \in co(\bigcup_{x\in X}\Gamma(x))$. By Proposition 3.14 of [1], for any $x \in X$, there is a $y_x^\star \in Y$ and $f(x, y_x^\star) \in F(x, y_x^\star)$ with $f(x, y_x^\star) \in \Gamma(x)$ such that

$$\xi f(x, y_x^\star) = \max \bigcup_{y\in Y} \xi F(x,y).$$

Let us choose $c = 0$ and $u = f(x, y_x^\star)$ in equation (1), we have

$$\xi(v) < \xi(f(x, y_x^\star)) = \max \bigcup_{y\in Y} \xi F(x,y)$$

for all $x \in X$. Therefore,

$$\xi(v) < \min \bigcup_{x\in X} \max \bigcup_{y\in Y} \xi F(x,y).$$

From conditions (i)-(iii), applying Proposition 3.9 and Proposition 3.13 in [1], all conditions of Theorem 2.1 hold. Hence, we have

$$\xi(v) < \max \bigcup_{y\in Y} \min \bigcup_{x\in X} \xi H(x,y).$$

Since Y is compact, there is a $y' \in Y$ such that

$$\xi(v) < \min \bigcup_{x\in X} \xi H(x,y').$$

Thus,

$$v \notin \bigcup_{x\in X} H(x,y') + C,$$

and hence,

$$v \notin Min_w \bigcup_{x\in X} H(x,y') + C. \tag{2}$$

If $v \in Max\bigcup_{y\in Y} Min_w \bigcup_{x\in X} F(x,y)$, then, by (iv), we have $v \in Min_w \bigcup_{x\in X} H(x,y) + C$ which contradicts (2). Hence, for every $v \in Max\bigcup_{y\in Y} Min_w \bigcup_{x\in X} F(x,y)$, $v \in co(\bigcup_{x\in X} \Gamma(x)) + C$. That is, the relation (H_1) is valid. ■

Remark 3.1. We note that Theorem 3.1 includes the following three versions as special cases, such as $F = G \subset H$, $F \subset G = H$, $F = G = H$.

In the following result, we apply the Gerstewitz function $\xi_{kv} : Z \mapsto \mathbb{R}$ to introduce the second version of hierarchical minimax theorems, where $k \in intC$ and $v \in Z$.

Theorem 3.2. *Under the framework of Theorem 3.1 except the convexities of F and H, if, in additional, the mappings $y \mapsto \xi_{kv}F(x,y)$ and $y \mapsto \xi_{kv}H(x,y)$ are above-$\mathbb{R}_+$-concave on Y for each $x \in X$. Then the relation (H_2) is valid.*

Proof. Let $\Gamma(x)$ be defined the same as in Theorem 3.1 for all $x \in X$. Using the same process in the proof of Theorem 3.1, we know that the set $\bigcup_{x\in X}\Gamma(x)$ is nonempty compact. Suppose that $v \notin \bigcup_{x\in X}\Gamma(x) + C$. For any $k \in intC$, there is a Gerstewitz function $\xi_{kv} : Z \mapsto \mathbb{R}$ such that

$$\xi_{kv}(u) > 0 \tag{3}$$

for all $u \in \bigcup_{x\in X}\Gamma(x)$. Then, for each $x \in X$, there is $y_x^\star \in Y$ and $f(x,y_x^\star) \in F(x,y_x^\star)$ with $f(x,y_x^\star) \in Max_w \bigcup_{y\in Y} F(x,y)$ such that

$$\xi_{kv}(f(x,y_x^\star)) = \max \bigcup_{y\in Y} \xi_{kv}F(x,y).$$

Choosing $u = f(x,y_x^\star)$ in equation (3), we have

$$\max \bigcup_{y\in Y} \xi_{kv}F(x,y) > 0$$

for all $x \in X$. Therefore,

$$\min \bigcup_{x\in X} \max \bigcup_{y\in Y} \xi_{kv}F(x,y) > 0.$$

By conditions (i)-(iii), we know that all conditions of Theorem 2.1 hold, and hence, by relation $(s-H)$, we have

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} \xi_{kv}H(x,y) > 0.$$

Since Y is compact, there is a $y' \in Y$ such that

$$\min \bigcup_{x\in X} \xi_{kv} H(x, y') > 0.$$

Thus,

$$v \notin \bigcup_{x\in X} H(x, y') + C,$$

and hence,

$$v \notin Min_w \bigcup_{x\in X} H(x, y') + C. \tag{4}$$

If $v \in Max \bigcup_{y\in Y} Min_w \bigcup_{x\in X} F(x, y)$, then, by (iv), we have

$$v \in Min_w \bigcup_{x\in X} H(x, y) + C$$

which contradicts (4). From this, we can deduce the relation (H_2) is valid. ■

The third version of hierarchical minimax theorems is as follows. We remove the condition (iv) in Theorem 3.2 to deduce the relation (H_3) is valid.

Theorem 3.3. *Under the framework of Theorem 3.1 except the convexities of F and H, and condition (iv). If, in additional, the mappings $y \mapsto \xi_{kv} F(x, y)$ and $y \mapsto \xi_{kv} H(x, y)$ are above-$\mathbb{R}_+$-concave on Y for each $x \in X$. Then the relation (H_3) is valid.*

Proof. Let $\Gamma(x)$, for all $x \in X$, be defined the same as in Theorem 3.1. Fixed any $v \in Min \bigcup_{x\in X} \Gamma(x)$. Then

$$\Big(\bigcup_{x\in X} \Gamma(x) \setminus \{v\}\Big) \bigcap (v - C) = \emptyset.$$

For any $k \in intC$, there is a Gerstewitz function $\xi_{kv} : Z \mapsto \mathbb{R}$ such that

$$\xi_{kv}(u) > 0$$

and

$$\xi_{kv}(v) = 0.$$

Since ξ_{kv} is continuous, by the compactness of $\bigcup_{y\in Y} F(x, y)$, for each $x \in X$, there exist $y_1 \in Y$ and $f_1 \in F(x, y_1)$ such that

$$\xi_{kv}(f_1) = \max \bigcup_{y\in Y} \xi_{kv} F(x, y).$$

By Proposition 3.14 [1], $f_1 \in Max_w \bigcup_{y\in Y} F(x, y)$. Thus, we have, for each $x \in X$,

$$\max \bigcup_{y \in Y} \xi_{kv} F(x, y) \geq 0,$$

or

$$\min \bigcup_{x \in X} \max \bigcup_{y \in Y} \xi_{kv} F(x, y) \geq 0.$$

From the condition (i)-(iii) and according to a similar argument in Theorem 3.2, we know that all conditions of Theorem 2.1 hold for the mappings $\xi_{kv}F$, $\xi_{kv}G$ and $\xi_{kv}H$. Hence, by Theorem 2.1, we have

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} \xi_{kv} H(x, y) \geq 0.$$

Since X and Y are compact, there are $x_0 \in X$, $y_0 \in Y$ and $h_0 \in H(x_0, y_0)$ such that

$$\xi_{kv}(h_0) = \min \bigcup_{x \in X} \xi_{kv} H(x, y_0) \geq 0.$$

Applying Proposition 3.14 in [1], we have $h_0 \in Min_w \bigcup_{x \in X} H(x, y_0)$. If $h_0 = v$, we have $v \notin h_0 + (C \setminus \{0\})$. If $h_0 \neq v$, we have $\xi_{kv}(h_0) > 0$, and hence $h_0 \notin v - C$. Therefore, $v \notin h_0 + (C \setminus \{0\})$. Thus, in any case, we have $v \in h_0 + Z \setminus (C \setminus \{0\})$. This implies that the relation (H_3) is valid. ■

4. Saddle Points

In this section, we discuss the existence of saddle points for set-valued mappings including the weakly C-saddle points and $\mathbb{R}_+$-saddle points of F on $X \times Y$. For more detail we refer to [1, 9].

Definition 4.1. Let $F : X \times Y \rightrightarrows Z$ be a set-valued mapping. A point $(\bar{x}, \bar{y}) \in X \times Y$ is said to be a

(a) *weakly C-saddle point* [1] of F on $X \times Y$ if

$$F(\bar{x}, \bar{y}) \bigcap \left(\mathrm{Max}_w \bigcup_{y \in Y} F(\bar{x}, y) \right) \bigcap \left(\mathrm{Min}_w \bigcup_{x \in X} F(x, \bar{y}) \right) \neq \emptyset;$$

(b) *$\mathbb{R}_+$-saddle point* [1] of F on $X \times Y$ if $Z = \mathbb{R}$ and

$$\max \bigcup_{y \in Y} F(\bar{x}, y) = \min \bigcup_{x \in X} F(x, \bar{y}) = F(\bar{x}, \bar{y}).$$

Theorem 4.1. *Under the framework of Corollary 2.1, F has $\mathbb{R}_+$-saddle point.*

Proof. Since Y is compact, there is a $\bar{y} \in Y$ such that

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) = \min \bigcup_{x \in X} F(x, \bar{y}).$$

Similarly, by the compactness of X, there is an $\bar{x}$ such that

$$\min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y) = \max \bigcup_{y\in Y} F(\bar{x},y).$$

By Corollary 2.1, we have

$$\min \bigcup_{x\in X} F(x,\bar{y}) = \max \bigcup_{y\in Y} F(\bar{x},y) = F(\bar{x},\bar{y}).$$

The last equality holds for the following reasons:

$$F(\bar{x},\bar{y}) \subset \bigcup_{x\in X} F(x,\bar{y}) \subset \min \bigcup_{x\in X} F(x,\bar{y}) + \mathbb{R}_+,$$

and

$$F(\bar{x},\bar{y}) \subset \bigcup_{y\in Y} F(\bar{x},y) \subset \max \bigcup_{y\in Y} F(\bar{x},y) - \mathbb{R}_+.$$

Hence $(\bar{x},\bar{y})$ is a $\mathbb{R}_+$-saddle point of F on $X \times Y$. ■

Theorem 4.2. *Let X, Y be nonempty compact convex subsets of real Hausdorff topological vector spaces, respectively. Let the set-valued mapping $F : X \times Y \rightrightarrows Z$ such that $\bigcup_{x\in X} F(x,y)$ and $\bigcup_{y\in Y} F(x,y)$ are compact for all $x \in X$ and $y \in Y$, and satisfy the following conditions:*

(i) *$x \mapsto F(x,y)$ is lower semi-continuous and above-naturally C-quasi-convex for each $y \in Y$; and*

(ii) *$y \mapsto F(x,y)$ is lower semi-continuous and above-C-concave for each $x \in X$;*

Then F has a weakly C-saddle point.

Proof. For any $\xi \in C^\star$. Then, by Proposition 3.9 and Proposition 3.13 in [1], we have all conditions of Corollary 2.1 hold for the mapping ξF. By Corollary 2.1, we have

$$\min \bigcup_{x\in X} \max \bigcup_{y\in Y} \xi F(x,y) = \max \bigcup_{y\in Y} \min \bigcup_{x\in X} \xi F(x,y).$$

Then there is $(\bar{x},\bar{y}) \in X \times Y$ such that

$$\min \bigcup_{x\in X} \xi F(x,\bar{y}) = \max \bigcup_{y\in Y} \xi F(\bar{x},y) = \xi F(\bar{x},\bar{y}).$$

For any $f \in F(\bar{x},\bar{y})$, $\xi f = \max \bigcup_{y\in Y} \xi F(\bar{x},y)$. By Proposition 3.14 [1], we have

$$f \in Max_w \bigcup_{y\in Y} F(\bar{x},y).$$

Similarly, since $\xi f = \min \bigcup_{x \in X} \xi F(x, \bar{y})$, we have

$$f \in Min_w \bigcup_{x \in X} F(x, \bar{y}).$$

Thus,

$$F(\bar{x}, \bar{y}) \subset (Max_w \bigcup_{y \in Y} F(\bar{x}, y)) \bigcap (Min_w \bigcup_{x \in X} F(x, \bar{y})).$$

Hence, F has a weakly C-saddle point. ■

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