TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 2, pp. 337-355, April 2014
DOI: 10.11650/tjm.18.2014.3495
This paper is available online at http://journal.taiwanmathsoc.org.tw

SYSTEMS OF PARAMETRIC STRONG QUASI-EQUILIBRIUM PROBLEMS: EXISTENCE AND WELL-POSEDNESS ASPECTS

Jia-wei Chen* and Yeong-Cheng Liou

Abstract. In this article, we investigate the existence of solutions and Levitin-Polyak well-posedness for a class of system of parametric strong quasi-equilibrium problems (SPSQEP) involving set-valued mappings in Hausdorff topological vector spaces. The existence of solutions to the problem (SPSQEP) are presented, and then the notions of Levitin-Polyak well-posedness and generalized Levitin-Polyak well-posedness for (SPSQEP) are introduced. Moreover, some metric characterizations of these well-posedness are derived under quite mild conditions. The relationships between these well-posedness of (SPSQEP) and the existence and uniqueness of its solutions are established. Finally, some examples are given to illustrate the presented results.

1. INTRODUCTION

The equilibrium problem, which was first introduced by Blum and Oettli [1], provides a unified model of many problems such as optimization problems, variational inequality problems, complementarity problems, fixed point problems and so on. For the past decades, many authors intensively studied and generalized different types of equilibrium problems and obtained a lot of existence results (see, e.g., [2, 3, 4, 5, 6] and the references therein). It is well-known that well-posedness is an important topic in optimization theory and applications, because it ensures that, for approximating solution sequences, there exists a subsequence which converges to a solution. Since any

Received July 6, 2013, accepted August 26, 2013.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: 49J40, 49K40, 90C33.
Key words and phrases: System of parametric strong set-valued quasi-equilibrium problem, Levitin-Polyak well-posedness, Generalized Levitin-Polyak well-posedness, Hausdorff metric, Kuratowski measure of noncompactness.
This work was supported by the Natural Science Foundation of China (71171150, 11201039, 11226226), the Doctor Fund of Southwest University (SWU113037) and the Fundamental Research Fund for the Central Universities. It was also partially supported by the grant NSC 101-2628-E-230-001-MY3 and NSC 101-2622-E-230-005-CC3.
*Corresponding author.

algorithm can generate only an approximating solution sequence, which is applicable only if the problem is well-posed under consideration. For parametric problems, well-posedness is closely related to stability. The concept of well-posedness for minimization problems (MP) was first introduced by Levitin and Polyak[7] and Tykhonov [8], respectively, which are so-called Levitin-Polyak and Tykhonov well-posedness, respectively. The well-posedness of (MP) implies the existence and uniqueness of solutions of (MP). In many practical situations, the solutions of (MP) are more than one. In this case, the notion of well-posedness in the generalized sense was introduced, which implies the existence of solutions of (MP). The study of Levitin-Polyak well-posedness for scalar convex optimization problems with functional constraints was initiated by Konsulova and Revalski [9]. Since then, many authors proposed various well-posedness for (MP) (see, e.g., [2, 10, 11] and the references therein). In 1981, Lucchetti and Patrone [12] introduced the first notion of well-posedness for variational inequalities, which is a generalization of the Tykhonov well-posedness of (MP). Lignola and Morgan [13] also introduced another notion of well-posedness for variational inequalities, which is distinct from that of Lucchetti and Patrone [12]. In the following years, the well-posedness and generalized well-posedness have attracted much attention because of the importance in the stability theory and existence of solution for variational inequalities and equilibrium problems (see, e.g., [14, 15, 16, 17, 18, 19] and the references therein). In [20], Hu, Fang, Huang and Wong investigated the well-posedness and generalized well-posedness for a system of equilibrium problems, derived some metric characterizations for these well-posedness, and proved that the well-posedness of system of equilibrium problems is equivalent to the existence and uniqueness of its solution. Li and Li [21] also introduced two type of Levitin-Polyak well-posedness for equilibrium problems with abstract set constraints. Thereafter, Peng, Wang and Zhao [22] introduced four type of Levitin-Polyak well-posedness for vector equilibrium problems with abstract set and functional constraints. Peng and Wu [23] also explored the generalized Tykhonov well-posedness for system of vector quasi-equilibrium problems, and gave some metric characterizations for these well-posedness in locally convex Hausdorff topological vector spaces. Stability and existence of solutions to the system problem are very important topics. Note that well-posedness plays an important role in studying the stability and the existence of solutions. In the objective reality, many practical problems always appear in the systems. However, there are very few results on the Levitin-Polyak well-posedness for systems of parametric quasi-equilibrium problems.

Inspired and motivated by the works of [20, 23], the aim of this paper is to investigate the existence of solutions and Levitin-Polyak well-posedness for a class of system of parametric strong quasi-equilibrium problems involving set-valued mappings (SPSQEP) in Hausdorff topological vector spaces. The existence of solutions to the problem (SPSQEP) are presented, and then the notions of Levitin-Polyak well-posedness and generalized Levitin-Polyak well-posedness for (SPSQEP) are introduced. Moreover, some metric characterizations of these well-posedness are derived under quite mild conditions. We also establish the relationships between these well-posedness of (SPSQEP) and the existence and uniqueness of its solution.

2. PRELIMINARIES

Throughout this paper, without other specifications, let I be an index set, R be the set of real numbers, and let $\bigwedge$ (the space of parameters) be a metric space, Z_i be topological vector space, X_i be a normed linear space and Y_i be Hausdorff topological vector spaces for $i \in I$, H_i and K_i be nonempty convex subsets of X_i and Y_i, respectively. Let $X = \prod_{i\in I} X_i, Y = \prod_{i\in I} Y_i, H = \prod_{i\in I} H_i, K = \prod_{i\in I} K_i$ and $X^{-i} = \prod_{j\in I, j\neq i} X_j$. Denote the element of X^{-i} by x^{-i}, and so $x \in X$ denote by $x = (x^{-i}, x_i) \in X^{-i} \times X_i$. We always denote 2^X by the family of all nonempty subsets of X. Let $\Gamma_i : H \times \bigwedge \to 2^{H_i}, T_i : H \times \bigwedge \to 2^{K_i}$, $\Psi_i : H \times H_i \to 2^{Z_i}$, $F_i : H \times K \times H_i \to 2^{Z_i}$ and $C_i : H \to 2^{Z_i}$ be set-valued mappings such that, for each $x \in H, C_i(x)$ is a proper closed convex and pointed cone in Z_i with int$C_i(x) \neq \emptyset$ for $i \in I$.

For each $p \in \bigwedge$, we consider the following system of parametric strong quasi-equilibrium problems involving set-valued mappings (in short, (SPSQEP)): find $x^* \in H$ such that for each $i \in I$, $x_i^* \in \Gamma_i(x^*, p)$, and there exists $y_i^* \in T_i(x^*, p)$ satisfying

$$F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \subseteq C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*, p).$$

We denote the solution set of (SPSQEP) by $S(p)$.

Special cases are as follows:

(1) If I is singled, (SPSQEP) is equivalent to the following parametric strong set-valued quasi-equilibrium problems (in short, (PSSQEP)): find $x^* \in H$ such that $x^* \in \Gamma(x^*, p)$, and there exists $y^* \in T(x^*, p)$ satisfying

$$F(x^*, y^*, x) + \Psi(x^*, x) \subseteq C(x^*), \quad \forall x \in \Gamma(x^*, p),$$

where $\Gamma : H \times \bigwedge \to 2^H, T : H \times \bigwedge \to 2^K$, $\Psi : H \times H \to 2^Z$, $F : H \times K \times H \to 2^Z$ and $C : H \to 2^Z$ be set-valued mappings such that, for each $x \in H, C(x)$ is a proper closed convex and pointed cone in Z with int$C(x) \neq \emptyset$.

(2) If, for each $i \in I, C_i(x) = -D_i(x)$, then (SPSQEP) is equivalent to find $x^* \in H$ such that for each $i \in I$, $x_i^* \in \Gamma_i(x^*, p)$, and there exists $y_i^* \in T_i(x^*, p)$ satisfying

$$F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \subseteq -D_i(x^*), \quad \forall x_i \in \Gamma_i(x^*, p).$$

(3) If, for each $i \in I$ and some $p \in \bigwedge$, $\Gamma_i(x, p) = \Gamma_i(x)$ and $T_i(x, p) = T_i(x)$ for $x \in H$, then (SPSQEP) is equivalent to find $x^* \in H$ such that for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$, and there exists $y_i^* \in T_i(x^*)$ satisfying

$$F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \subseteq -D_i(x^*), \quad \forall x_i \in \Gamma_i(x^*), \tag{2.1}$$

which has been studied by Xie and He[24].

(4) If the index set I is a singleton, $F(x, y, z) \equiv 0$ for all $(x, y, z) \in H \times K \times H$, then (SPSQEP) is equivalent to find $x^* \in H$ such that $x^* \in \Gamma(x^*)$ and

$$\Psi(x^*, x) \subseteq -D(x^*), \quad \forall x \in \Gamma(x^*),$$

which has been studied by Fu [25] and references therein.

We first recall some definitions and facts which are needed in the sequel.

Definition 2.1. ([26]). Let A, B be nonempty subsets of a normed linear space $(X, \|\cdot\|)$. The *Hausdorff metric* $\mathscr{H}(\cdot,\cdot)$ between A and B is defined by

$$\mathscr{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a\in A} d(a, B)$ is the *excess* of set A to set B, and $d(a, B) = \inf_{b\in B} \|a - b\|$.

Definition 2.2. ([26]). Let A be a nonempty subset of a normed linear space $(X, \|\cdot\|)$ X. The *Kuratowski measure of noncompactness* $\mathscr{M}$ of the set A is defined by

$$\mathscr{M}(A) = \inf\{\epsilon > 0 : A \subset \cup_{i=1}^{n} A_i, \mathrm{diam} A_i < \epsilon, i = 1, 2, \cdots, n\},$$

where diam stands for the diameter of a set.

Facts 2.3. ([19, 26]). If Δ, ∇ are nonempty closed subset of a normed linear space $(X, \|\cdot\|)$, Δ is compact and $\Delta \subseteq \nabla$, then the following hold:

(i) $\mathscr{M}(\Delta) = 0$;

(ii) $\mathscr{M}(\nabla) \leq 2\,\mathscr{H}(\nabla, \Delta) = 2e(\nabla, \Delta)$.

Definition 2.4. ([27, 28]). Let $\bigvee$ and E be Hausdorff topological vector spaces. A set-valued mapping $\psi : \bigvee \to 2^E$ is said to be

(i) *upper semicontinuous* (in short, u.s.c.) at $\upsilon_0 \in \bigvee$ iff, for each open set V with $\psi(\upsilon_0) \subset V$, there exists $\delta > 0$ such that

$$\psi(\upsilon) \subset V, \quad \forall \upsilon \in B(\upsilon_0, \delta),$$

where $B(\upsilon_0, \delta)$ denotes the closed ball centered at υ_0 with radius δ;

(ii) *lower semicontinuous* (in short, l.s.c.) at $\upsilon_0 \in \bigvee$ iff, for each open set V with $\psi(\upsilon_0) \bigcap V \neq \emptyset$, there exists $\delta > 0$ such that

$$\psi(\upsilon) \cap V \neq \emptyset, \quad \forall \upsilon \in B(\upsilon_0, \delta);$$

(iii) *closed* iff, the graph of ψ is closed, i.e., the set $Gr(\psi) = \{(\zeta, \upsilon) \in \bigvee \times E : \zeta \in \psi(\upsilon)\}$ is closed in $\bigvee \times E$.

We say ψ is l.s.c. (resp. u.s.c.) on $\bigvee$ iff it is l.s.c. (resp. u.s.c.) at each $\upsilon \in \bigvee$. ψ is called continuous on $\bigvee$ iff it is both l.s.c. and u.s.c. on $\bigvee$.

Facts 2.5. ([27, 28]). (i) ψ is l.s.c. at $\upsilon_0 \in \bigvee$ if and only if, for any net $\{\upsilon_\alpha\} \subseteq \bigvee$ with $\upsilon_\alpha \to \upsilon_0$ and $\zeta_0 \in \psi(\upsilon_0)$, there exists a net $\{\zeta_\alpha\} \subseteq E$ with $\zeta_\alpha \in \psi(\upsilon_\alpha)$ for all α, such that $\zeta_\alpha \to \zeta_0$.

(ii) If ψ is compact-valued, then ψ is u.s.c. at $\upsilon_0 \in \bigvee$ if and only if, for any net $\{\upsilon_\alpha\} \subseteq \bigvee$ with $\upsilon_\alpha \to \upsilon_0$ and for any net $\{\zeta_\alpha\} \subseteq E$ with $\zeta_\alpha \in \psi(\upsilon_\alpha)$ for all α, there exists $\zeta_0 \in \psi(\upsilon_0)$ and a subnet $\{\zeta_\beta\}$ of $\{\zeta_\alpha\}$ such that $\zeta_\beta \to \zeta_0$.

(iii) If ψ is u.s.c. with closed values, then ψ is closed; Conversely, if ψ is closed and E is compact, then ψ is u.s.c.

Definition 2.6. ([29]). Let E_1, E_2 be topological vector spaces, $g : E_1 \times E_1 \to 2^{E_2}$ and $C : E_1 \to 2^{E_2}$ be set-valued mappings such that, for each $x \in E_1$, $C(x)$ is a proper closed convex and pointed cone in E_2 with int$C(x) \neq \emptyset$. g is said to be *above* $C(x)$-*convex* with respect to the second argument if, for any $x, y, z \in E_1$ and $t \in [0, 1]$,

$$g(x, ty + (1-t)z) \subseteq tg(x, y) + (1-t)g(x, z) - C(x).$$

Facts 2.7. ([24, Theorem 3.2]). For each $i \in I$, let Z_i be topological vector space, H_i and K_i be nonempty convex subsets of Hausdorff topological vector spaces X_i and Y_i, respectively, and let $F_i : H \times K \times H_i \to 2^{Z_i}$, $C_i : H \to 2^{Z_i}$ be two set-valued mappings such that C_i is l.s.c. and, for each $x \in H$, $C_i(x)$ is a proper closed convex and pointed cone in Z_i with int$C_i(x) \neq \emptyset$, $\Gamma_i : H \to 2^{H_i}$, $T_i : H \to 2^{K_i}$ be two closed convex-valued mappings and $\Psi_i : H \times H_i \to 2^{Z_i}$ be l.s.c. with respect to the first argument and above $C_i(x)$-convex with respect to the second argument. Assume that the following conditions hold:

(i) for each $i \in I$, $x_i \in H_i, y_i \in K_i$, $\Gamma_i^{-1}(x_i)$ and $T_i^{-1}(y_i)$ are open sets of H;

(ii) for each $i \in I$, $x'_i \in H_i$, the mapping $(x, y) \mapsto F_i(x, y, x'_i)$ is l.s.c. and, for each $x \in H$ and $y_i \in T_i(x)$, the mapping $x'_i \mapsto F_i(x, y, x'_i)$ is above $C_i(x)$-convex;

(iii) there exist nonempty compact sets $\Omega \subseteq H, \Xi \subseteq K$ and nonempty compact convex sets $U_i \subseteq H_i$, $L_i \subseteq K_i$ for each $i \in I$ such that, for any $(x, y) \in H \times K \setminus (\Omega \times \Xi)$, there exists $i' \in I$ with $x_{i'} \in U_{i'} \bigcap \Gamma_{i'}(x)$ and $y_{i'} \in L_{i'} \bigcap T_{i'}(x)$ satisfying

$$F_{i'}(x, y, x_{i'}) + \Psi_{i'}(x, x_{i'}) \nsubseteq -C_{i'}(x).$$

Then the problem (2.1) has a solution.

3. Existence of Solutions to (SPSQEP)

In this section, we shall study the existence and closedness of the approximation solutions set of the problem (SPSQEP) under some suitable conditions. Let $e_i : H \to Z_i$

be continuous such that $e_i(x) \in \text{int}C_i(x)$ for all $x \in H$. For $\epsilon \geq 0$ and $p \in \bigwedge$, we firstly consider the following approximation for (SPSQEP): find $x^* \in H$ such that for each $i \in I$, $x_i^* \in \Gamma_i(x^*, p)$, and there exists $y_i^* \in T_i(x^*, p)$ satisfying

$$(3.1) \quad F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) + \epsilon e_i(x^*) \subseteq C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*, p).$$

Theorem 3.1. *Let $\bigwedge$ be a metric space and, for each $i \in I$, Z_i, K_i and Y_i be the same as Facts 2.7, H_i be a nonempty convex subset of a normed linear space X_i, and let $F_i : H \times K \times H_i \to 2^{Z_i}$, $C_i : H \to 2^{Z_i}$ be two set-valued mappings such that C_i is u.s.c., and for each $x \in H$, $C_i(x)$ is a proper closed convex and pointed cone in Z_i with int$C_i(x) \neq \emptyset$, $e_i : H \to Z_i$ be continuous such that $e_i(x) \in$ int$C_i(x)$ for all $x \in H$, $\Gamma_i : H \times \bigwedge \to 2^{H_i}$, $T_i : H \times \bigwedge \to 2^{K_i}$ be two closed convex-valued mappings and $\Psi_i : H \times H_i \to 2^{Z_i}$ be l.s.c. with respect to the first argument and above $-C_i(x)$-convex with respect to the second argument. Assume that the following conditions hold:*

(i) *for each $i \in I$, $x_i \in H_i, y_i \in K_i$, $\Gamma_i^{-1}(x_i)$ and $T_i^{-1}(y_i)$ are open sets of $H \times \bigwedge$;*

(ii) *for each $i \in I$, $x_i' \in H_i$, the mapping $(x, y) \mapsto F_i(x, y, x_i')$ is l.s.c. and, for each $x \in H$ and $y_i \in T_i(x)$, the mapping $x_i' \mapsto F_i(x, y, x_i')$ is above $-C_i(x)$-convex;*

(iii) *there exist nonempty compact sets $\Omega \subseteq H, \Xi \subseteq K$ and nonempty compact convex sets $U_i \subseteq H_i$, $L_i \subseteq K_i$ for each $i \in I$ such that, for any $(x, y) \in H \times K \setminus (\Omega \times \Xi)$, there exists $i' \in I$ with $x_{i'} \in U_{i'} \bigcap \Gamma_{i'}(x, p)$ and $y_{i'} \in L_{i'} \bigcap T_{i'}(x, p)$ for each $p \in \bigwedge$ satisfying*

$$(3.2) \qquad F_{i'}(x, y, x_{i'}) + \Psi_{i'}(x, x_{i'}) \not\subseteq C_{i'}(x).$$

Then, for each $\epsilon \geq 0, p \in \bigwedge$, the problem (3.1) has a solution.

Proof. Let $\epsilon \geq 0, p \in \bigwedge$ and let $D_i(x) = -C_i(x)$ for all $x \in H$. Then, for each $x \in H$ and $y_i \in T_i(x)$, the mapping $x_i' \mapsto F_i(x, y, x_i')$ is above $D_i(x)$-convex and $\Psi : H \times H_i \to 2^{Z_i}$ is above $D_i(x)$-convex with respect to the second argument. It follows from (3.2) that

$$F_{i'}(x, y, x_{i'}) + \Psi_{i'}(x, x_{i'}) \not\subseteq -D_{i'}(x).$$

Since C_i is u.s.c. for each $i \in I$, D_i is l.s.c. Therefore, by Facts 2.7, there exists $x^* \in H$ such that for each $i \in I$, $x_i^* \in \Gamma_i(x^*, p)$, and there exists $y_i^* \in T_i(x^*, p)$ satisfying

$$F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \subseteq -D_i(x^*), \quad \forall x_i \in \Gamma_i(x^*, p).$$

By virtue of $e_i(x) \in \text{int}C_i(x)$ for each $i \in I$, one has $\epsilon e_i(x) \in -D_i(x)$ for each $i \in I, x \in H$ and so,

$$\begin{aligned} &F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) + \epsilon e_i(x^*) \\ \subseteq &-D_i(x^*) - D_i(x^*) \subseteq -D_i(x^*), \quad \forall x_i \in \Gamma_i(x^*, p). \end{aligned}$$

This implies that there exists $x^* \in H$ such that for each $i \in I$, $x_i^* \in \Gamma_i(x^*, p)$, and there exists $y_i^* \in T_i(x^*, p)$ satisfying

$$F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) + \epsilon e_i(x^*) \subseteq C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*, p).$$

Therefore, for each $\epsilon \geq 0, p \in \bigwedge$, the problem (3.1) has a solution. This completes the proof.

Remark 3.2. It is easy to see that for each $p \in \bigwedge$, the solutions sets of the problem (SPSQEP) is nonempty under the conditions of Theorem 3.1. Moreover, if I is a singleton, the problem (PSSQEP) is also solvable.

For $p^* \in \bigwedge$, for any $\delta, \epsilon > 0$, we introduce the following approximating solution set for (SPSQEP):

$$\begin{aligned}\Omega_{p^*}(\delta, \epsilon) = \bigcup_{p \in B(p^*, \delta)} & \{x \in H : \forall i \in I, d_i(x_i, \Gamma_i(x, p)) \leq \epsilon, \exists y_i \in T_i(x, p), \text{s.t.} \\ & F_i(x, y, \omega_i) + \Psi_i(x, \omega_i) + \epsilon e_i(x) \subseteq C_i(x), \quad \forall \omega_i \in \Gamma_i(x, p)\}.\end{aligned}$$

where $B(p^*, \delta)$ means the closed ball centered at p^* with radius δ in $\bigwedge$, $d_i(x_i, \Gamma_i(x, p)) = \inf_{w \in \Gamma_i(x,p)} \|x_i - w\|$.

It is easy to see that if $0 \leq \delta_1 \leq \delta_2, 0 \leq \epsilon_1 \leq \epsilon_2$, then $S(p^*) \subseteq \Omega_{p^*}(\delta_1, \epsilon_1) \subseteq \Omega_{p^*}(\delta_2, \epsilon_2)$.

The following result shows the closedness of the approximating solution set $\Omega_{p^*}(\delta, \epsilon)$, and the relationship between $\Omega_{p^*}(\delta, \epsilon)$ and the solution set $S(p^*)$ of (SPSQEP) for $p^* \in \bigwedge$.

Theorem 3.3. *Let $\bigwedge$ be finite dimensional. For each $i \in I$, let $C_i : H \to 2^{Z_i}$ be a set-valued mappings such that C_i is u.s.c., and for each $x \in H, C_i(x)$ is a proper closed convex and pointed cone in Z_i with int$C(x) \neq \emptyset$, $e_i : H \to Z_i$ be continuous with $e_i(x) \in$ int$C_i(x)$ for $x \in H$, and let the set-valued mappings $F_i : H \times K \times H_i \to 2^{Z_i}$ and $\Psi_i : H \times H_i \to 2^{Z_i}$ be continuous, $T_i : H \times \bigwedge \to 2^{K_i}$ be u.s.c. with compact values, and $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ be l.s.c. and closed. Then the following statements hold:*

(*i*) *for each $\delta, \epsilon \geq 0, \Omega_{p^*}(\delta, \epsilon)$ is closed;*

(*ii*) $S(p^*) = \bigcap_{\delta, \epsilon > 0} \Omega_{p^*}(\delta, \epsilon)$.

Proof. (i) Let us show that for each $\delta, \epsilon \geq 0, \Omega_{p^*}(\delta, \epsilon)$ is closed. Let any sequence $\{x^n\} \subset \Omega_{p^*}(\delta, \epsilon)$ and $x^n \to \hat{x}$. Then there exists $p^n \in B(p^*, \delta)$, for each $i \in I$,

$$d_i(x_i^n, \Gamma_i(x^n, p^n)) \leq \epsilon \tag{3.3}$$

and there exists $y_i^n \in T_i(x^n, p^n)$ such that

(3.4) $\quad F_i(x^n, y^n, \omega_i) + \Psi_i(x^n, \omega_i) + \epsilon e_i(x^n) \subseteq C_i(x^n), \quad \forall \omega_i \in \Gamma_i(x^n, p^n), n \in N.$

Without loss of generality, let $p^n \to \hat{p} \in B(p^*, \delta)$, since $\bigwedge$ is finite dimensional. Since $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ is l.s.c. and closed, and from (3.3), one has

$$d_i(\hat{x}_i, \Gamma_i(\hat{x}, \hat{p})) \leq \epsilon. \tag{3.5}$$

Again from the u.s.c. and compactness of $T_i, i \in I$, there exist a subsequence $\{y_i^{n_k}\}$ of $\{y_i^n\}$ and $\hat{y}_i \in T_i(\hat{x}, \hat{p})$ such that $y_i^{n_k} \to \hat{y}_i$. Since for each $i \in I$, e_i, F_i and Ψ_i are continuous, C_i is u.s.c., and for each $x \in H, C_i(x)$ is a proper closed convex and pointed cone, we obtain, from (3.4),

$$F_i(\hat{x}, \hat{y}, \omega_i) + \Psi_i(\hat{x}, \omega_i) + \epsilon e_i(\hat{x}) \subseteq C_i(\hat{x}), \quad \forall \omega_i \in \Gamma_i(\hat{x}, \hat{p}),$$

and therefore, $\hat{x} \in \Omega_{p^*}(\delta, \epsilon)$, which implies that $\Omega_{p^*}(\delta, \epsilon)$ is closed for all $\delta, \epsilon \geq 0$.

(ii) Let us prove that $S(p^*) = \bigcap_{\delta,\epsilon>0} \Omega_{p^*}(\delta, \epsilon)$. Clearly, $S(p^*) \subseteq \bigcap_{\delta,\epsilon>0} \Omega_{p^*}(\delta, \epsilon)$. We only need to prove that $S(p^*) \supseteq \bigcap_{\delta,\epsilon>0} \Omega_{p^*}(\delta, \epsilon)$. Let $\bar{x} \in \bigcap_{\delta,\epsilon>0} \Omega_{p^*}(\delta, \epsilon)$. Then $\bar{x} \in \Omega_{p^*}(\delta, \epsilon)$ for all $\delta, \epsilon > 0$. Without loss of generality, let two sequences $\{\delta_n\}$ and $\{\epsilon_n\}$ with $\delta_n, \epsilon_n > 0$ and $(\delta_n, \epsilon_n) \to (0, 0)$. So, $\bar{x} \in \Omega(\delta_n, \epsilon_n)$ and there exists $p^n \in B(p^*, \delta_n)$, for each $i \in I$,

$$d_i(\bar{x}_i, \Gamma_i(\bar{x}, p^n)) \leq \epsilon_n, \tag{3.6}$$

and there exists $\bar{y}_i^n \in T_i(\bar{x}, p^n)$ such that

$$F_i(\bar{x}, \bar{y}^n, \omega_i) + \Psi_i(\bar{x}, \omega_i) + \epsilon_n e_i(\bar{x}) \subseteq C_i(\bar{x}), \quad \forall \omega_i \in \Gamma_i(\bar{x}, p^n).$$

Since e_i, F_i, Ψ_i are continuous, T_i is u.s.c. with compact values, and $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ is l.s.c. with closed, taking the limit in (3.6), one can conclude that

$$d_i(\bar{x}_i, \Gamma_i(\bar{x}, p^*)) = 0, \tag{3.7}$$

and there exist a subsequence $\{\bar{y}_i^{n_k}\}$ of $\{\bar{y}_i^n\}$ and $\bar{y}_i \in T_i(\bar{x}, p^*)$ such that $\bar{y}_i^{n_k} \to \bar{y}_i$, and so,

$$F_i(\bar{x}, \bar{y}, \omega_i) + \Psi_i(\bar{x}, \omega_i) \subseteq C_i(\bar{x}), \quad \forall \omega_i \in \Gamma_i(\bar{x}, p^*).$$

Therefore $\bar{x} \in S(p^*)$, i.e., $\bigcap_{\delta,\epsilon>0} \Omega_{p^*}(\delta, \epsilon) \subseteq S(p^*)$. This completes the proof.

Remark 3.4. According to Theorem 3.3, the solution set $S(p)$ of (SPSQEP) is closed under the conditions of Theorem 3.3. Furthermore, the approximating solution set for (SPSQEP) corresponding to the parameter $p^* \in \bigwedge$, for any $\delta, \epsilon > 0$, $\Omega_{p^*}(\delta, \epsilon) \neq \emptyset$ under the assumptions of Theorems 3.1 and 3.3. We also show that $\Omega_{p^*}(\delta, \epsilon) \neq \emptyset$ for all $\delta, \epsilon > 0$ in Theorem 4.19.

4. Levitin-polyak Well-posedness for (SPSQEP)

In this section, we introduce the notions of Levitin-Polyak well-posedness and generalized Levitin-Polyak well-posedness for (SPSQEP), explore the necessary and sufficient conditions of these well-posedness, and establish the relationships between these well-posedness and the existence and uniqueness of solution to (SPSQEP).

Definition 4.1. Let $\bigwedge$ be a metric space and, a sequence $\{p^n\} \subset \bigwedge$ such that $p^n \to p^*$. A sequence $\{x^n\} \subset H$ is said to be Levitin-Polyak (in short, L-P) approximating solution sequence corresponding to $\{p^n\}$ for (SPSQEP) if there exist a sequence $\{\epsilon_n\}$ of positive real numbers with $\epsilon_n \to 0$ and for each $i \in I$, $y_i^n \in T_i(x^n, p^n)$ such that

$$d_i(x_i^n, \Gamma_i(x^n, p^n)) \leq \epsilon_n$$

and

$$F_i(x^n, y^n, \omega_i) + \Psi_i(x^n, \omega_i) + \epsilon_n e_i(x^n) \subseteq C_i(x^n), \quad \forall \omega_i \in \Gamma_i(x^n, p^n), n \in N.$$

Definition 4.2. (i) (SPSQEP) is said to be L-P well-posed if for each $p \in \bigwedge$, (SPSQEP) has a unique solution $x(p)$, and for any sequence $\{p^n\} \subset \bigwedge$ with $p^n \to p$, every L-P approximating solution sequence corresponding to $\{p^n\}$ of (SPSQEP) converges strongly to $x(p)$.

(ii) (SPSQEP) is said to be generalized L-P well-posed if for each $p \in \bigwedge$, the solution set $S(p) \neq \emptyset$, and for any sequence $\{p^n\} \subset \bigwedge$ with $p^n \to p$, every L-P approximating solution sequence corresponding to $\{p^n\}$ of (SPSQEP) has a subsequence which converges strongly to some point of $S(p)$.

Remark 4.3. It is easy to see that L-P well-posedness and generalized L-P well-posedness for (SPSQEP) imply that the solution set $S(p)$ of (SPSQEP) is nonempty and compact; Moreover, any L-P well-posedness for (SPSQEP) is also generalized L-P well-posedness for (SPSQEP).

Now we discuss the sufficient and necessary conditions for the (generalized) L-P well-posedness of (SPSQEP).

Theorem 4.4. *(SPSQEP) is L-P well-posed if and only if for each* $p \in \bigwedge$,

$$\Omega_p(\delta, \epsilon) \neq \emptyset, \quad \forall \delta, \epsilon > 0, \quad \text{and} \quad \text{diam}[\Omega_p(\delta, \epsilon)] \to 0, \quad \text{as} \quad (\delta, \epsilon) \to (0, 0).$$

Proof. For the necessity. Suppose that (SPSQEP) is L-P well-posed. It follows from Definition 4.2 that for each $p \in \bigwedge$, $S(p) = \{x(p)\}$ and so, $x(p) \in \Omega_p(\delta, \epsilon) \neq \emptyset$ for all $\delta, \epsilon > 0$. Suppose to the contrary that diam$[\Omega_p(\delta, \epsilon)] \not\to 0$ as $(\delta, \epsilon) \to (0, 0)$. Then there exist $\sigma > 0$ and two sequences $\{\delta_n\}$ and $\{\epsilon_n\}$ of positive real numbers with

$(\delta_n, \epsilon_n) \to (0,0)$ such that diam$[\Omega_p(\delta_n, \epsilon_n)] > \sigma$. Thus, there exists $x^n \in \Omega_p(\delta_n, \epsilon_n)$ such that

$$d(x(p), x^n) = \|x^n - x(p)\| \geq \sigma. \tag{4.1}$$

In view of $x^n \in \Omega_p(\delta_n, \epsilon_n)$, there exists $p^n \in B(p, \delta_n)$, for each $i \in I$,

$$d_i(x_i^n, \Gamma_i(x^n, p^n)) \leq \epsilon_n$$

and there exists $y_i^n \in T_i(x^n, p^n)$ such that

$$F_i(x^n, y^n, \omega_i) + \Psi_i(x^n, \omega_i) + \epsilon_n e_i(x^n) \subseteq C_i(x^n), \quad \forall \omega_i \in \Gamma_i(x^n, p^n), n \in N,$$

and $p^n \to p$, since $\delta_n \to 0$. Therefore, $\{x^n\}$ is a L-P approximating solution sequence corresponding to $\{p^n\}$ for (SPSQEP) and so, $\|x^n - x(p)\| \to 0$, which contradicts (4.1).

For the sufficiency. Suppose that for each $p \in \bigwedge, \Omega_p(\delta, \epsilon) \neq \emptyset$ for all $\delta, \epsilon > 0$, and diam$[\Omega_p(\delta, \epsilon)] \to 0$ as $(\delta, \epsilon) \to (0,0)$. Clearly, $S(p) = \{x(p)\}$. If not, take $\hat{x} \in S(p)$ arbitrarily and, $\hat{x} \neq x(p)$. Then $\hat{x} \in \Omega_p(\delta, \epsilon)$. Moreover, one has

$$\text{diam}[\Omega_p(\delta, \epsilon)] \geq \|\hat{x} - x(p)\| > 0,$$

which is a contradiction.

For any sequence $\{p^n\} \subset \bigwedge$ with $p^n \to p$. Let $\{x^n\}$ be a L-P approximating solution sequence corresponding to $\{p^n\}$ for (SPSQEP). Then there exist a sequence $\{\epsilon_n\}$ of positive real numbers with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$d_i(x_i^n, \Gamma_i(x^n, p^n)) \leq \epsilon_n$$

and

$$F_i(x^n, y^n, \omega_i) + \Psi_i(x^n, \omega_i) + \epsilon_n e_i(x^n) \subseteq C_i(x^n), \quad \forall \omega_i \in \Gamma_i(x^n, p^n), n \in N.$$

Put $\delta_n = \|p^n - p\|$. Then $\delta_n \to 0$ and $x^n \in \Omega_p(\delta_n, \epsilon_n)$. Consequently, we have

$$\|x^n - x(p)\| \leq \text{diam}\Omega_p(\delta_n, \epsilon_n) \to 0,$$

namely, $x^n \to x(p)$. Therefore, (SPSQEP) is L-P well-posed. This completes the proof.

It is well known that if (SPSQEP) has more than one solutions, then for each $p \in \bigwedge$, the diameters of the approximating solution sets $\Omega_p(\delta, \epsilon)$ do not tend to zero. For this reason, we consider the Furi-Vignoli type characterization (see, e.g., [10]) of the generalized L-P well-posedness for (SPSQEP) by using Kuratowski measure of noncompactness (see, e.g., [26]) instead of the diameter.

Theorem 4.5. *Assume that all conditions of Theorem 3.3 are satisfied. Then (SPSQEP) is generalized L-P well-posed if and only if for each* $p \in \bigwedge$,

$$\Omega_p(\delta,\epsilon) \neq \emptyset, \quad \forall \delta,\epsilon > 0 \quad \text{and} \quad \lim_{(\delta,\epsilon)\to(0,0)} \mathscr{M}(\Omega_p(\delta,\epsilon)) = 0.$$

Proof. For each $p \in \bigwedge$, by Theorem 3.3, $\Omega_p(\delta,\epsilon)$ is closed for all $\delta,\epsilon \geq 0$ and

$$S(p) = \bigcap_{\delta,\epsilon>0} \Omega_p(\delta,\epsilon).$$

Suppose that (SPSQEP) is generalized L-P well-posed. It follows from Remark 4.3 that $S(p)$ is nonempty compact. This, together with $S(p) \subseteq \Omega_p(\delta,\epsilon)$, yields that $\Omega_p(\delta,\epsilon) \neq \emptyset$ for all $\delta,\epsilon \geq 0$. According to Facts 2.3, one has

$$\mathscr{M}(\Omega_p(\delta,\epsilon)) \leq 2\mathscr{H}(\Omega_p(\delta,\epsilon), S(p)) = 2e(\Omega_p(\delta,\epsilon), S(p)). \tag{4.2}$$

So, to show that $\lim_{(\delta,\epsilon)\to(0,0)} \mathscr{M}(\Omega_p(\delta,\epsilon)) = 0$, we only to prove that $e(\Omega_p(\delta,\epsilon), S(p)) \to 0$ as $(\delta,\epsilon) \to (0,0)$. Suppose to the contrary that $e(\Omega_p(\delta,\epsilon), S(p)) \not\to 0$ as $(\delta,\epsilon) \to (0,0)$. Therefore there exist $\sigma > 0, \delta_n > 0$ and $\epsilon_n > 0$ with $(\delta_n,\epsilon_n) \to (0,0)$ and $x^n \in \Omega_p(\delta_n,\epsilon_n)$ such that

$$d(x^n, S(p)) > \sigma. \tag{4.3}$$

Since $x^n \in \Omega_p(\delta_n,\epsilon_n)$, there exists $p^n \in B(p,\delta_n)$ such that, for each $i \in I$, $x_i^n \in \Gamma_i(x^n,p^n)$, and there exists $y_i^n \in T_i(x^n,p^n)$ satisfy

$$F_i(x^n,y^n,\omega_i) + \Psi_i(x^n,\omega_i) + \epsilon_n e_i(x^n) \subseteq C_i(x^n), \quad \forall \omega_i \in \Gamma_i(x^n,p^n).$$

This implies that $p^n \to p^*$ and $\{x^n\}$ is a L-P approximating solution sequence corresponding to $\{p^n\}$ of (SPSQEP). By the generalized L-P well-posedness of (SPSQEP), there exists a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ which converges strongly to some point of $S(p)$. Furthermore,

$$d(x^{n_k}, S(p)) \to 0 \quad \text{as} \quad k \to \infty,$$

which contradicts (4.3). Consequently, $e(\Omega_p(\delta,\epsilon), S(p)) \to 0$ as $(\delta,\epsilon) \to (0,0)$. Hence, from (4.2), $\mathscr{M}(\Omega_p(\delta,\epsilon)) \to 0$ as $(\delta,\epsilon) \to (0,0)$, that is, $\lim_{(\delta,\epsilon)\to(0,0)} \mathscr{M}(\Omega_p(\delta,\epsilon)) = 0$.

Conversely, let $p \in \bigwedge$, $\Omega_p(\delta,\epsilon) \neq \emptyset$ for all $\delta,\epsilon > 0$ and $\lim_{(\delta,\epsilon)\to(0,0)} \mathscr{M}(\Omega_p(\delta,\epsilon)) = 0$. By the definition of $\Omega_p(\delta,\epsilon)$, we get

$$\Omega_p(\tilde{\delta},\tilde{\epsilon}) \subseteq \Omega_p(\hat{\delta},\hat{\epsilon}), \quad \forall \tilde{\epsilon},\tilde{\delta},\hat{\delta},\hat{\epsilon} \in R^+ \setminus \{0\}, \tilde{\delta} \leq \hat{\delta}, \tilde{\epsilon} \leq \hat{\epsilon}. \tag{4.4}$$

Since $\Omega_p(\delta,\epsilon)$ is nonempty and closed and $S(p) = \bigcap_{\delta,\epsilon>0} \Omega_p(\delta,\epsilon)$, from (4.4) and the Kuratowski theorem (see, e.g. [30] or [10, Theorem 2.1]), one has

$$\mathscr{H}(\Omega_p(\delta,\epsilon), S(p)) \to 0 \quad \text{as} \quad (\delta,\epsilon) \to (0,0) \tag{4.5}$$

and $S(p)$ is nonempty and compact. For any $\{p^n\} \subseteq \bigwedge$ with $p^n \to p$, let $\{x^n\}$ be any L-P approximating solution sequence corresponding to $\{p^n\}$ of (SPSQEP). Then for each $i \in I$, $x_i^n \in \Gamma_i(x^n, p^n)$ and there exist a sequence positive real numbers $\{\epsilon_n\}$ with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$F_i(x^n, y^n, \omega_i) + \Psi_i(x^n, \omega_i) + \epsilon_n e_i(x^n) \subseteq C_i(x^n), \quad \forall \omega_i \in \Gamma_i(x^n, p^n), n \in N.$$

Set $\delta_n = \|p^n - p\|$. Then $\delta_n \to 0$ and $x^n \in \Omega_p(\delta_n, \epsilon_n)$ for $n \in N$. By (4.5), one has

$$d(x^n, S(p)) \leq \mathscr{H}(\Omega_p(\delta, \epsilon), S(p)) \to 0 \quad \text{as} \quad (\delta, \epsilon) \to (0, 0).$$

Since $S(p)$ is nonempty, then there exists a sequence $\{\bar{x}^n\} \subseteq S(p)$ such that

$$d(x^n, \bar{x}^n) \to 0.$$

From the compactness of $S(p)$, it follows that there exists a subsequence $\{\bar{x}^{n_k}\}$ of $\{\bar{x}^n\}$ which converges strongly to some point $\bar{x} \in S(p)$. Then the corresponding subsequence $\{x^{n_k}\}$ of $\{x^n\}$ such that $x^{n_k} \to \bar{x}$. Therefore (SPSQEP) is generalized L-P well-posed. This completes the proof.

Similar to Fang etc. [19], we define a function $q : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ for (SPSQEP) corresponding to parameter $p \in \bigwedge$ by

$$q(\delta, \epsilon) = e(\Omega_p(\delta, \epsilon), S(p)), \quad \forall (\delta, \epsilon) \in [0, +\infty) \times [0, +\infty).$$

We now present the relationship between the noncompactness measure $\mathscr{M}(\Omega_p(\delta, \epsilon)) \to 0$ as $(\delta, \epsilon) \to (0, 0)$ and $q(\delta, \epsilon) \to 0$ as $(\delta, \epsilon) \to (0, 0)$.

Assume that $S(p)$ is nonempty and compact. It follows from $S(p) \subseteq \Omega_p(\delta, \epsilon)$ for each $(\delta, \epsilon) \in [0, +\infty) \times [0, +\infty)$, Definition 2.1 and Facts 2.3 that

$$\mathscr{M}(\Omega_p(\delta, \epsilon)) \leq 2q(\delta, \epsilon) = 2\mathcal{H}(\Omega_p(\delta, \epsilon), S(p)).$$

Therefore, $q(\delta, \epsilon) \to 0$ as $(\delta, \epsilon) \to (0, 0)$ which implies that $\mathcal{M}(\Omega_p(\delta, \epsilon)) \to 0$ as $(\delta, \epsilon) \to (0, 0)$.

From the proof of Theorem 4.5, we can obtain the following characterization by considering Hausdorff metric of approximate solution set.

Theorem 4.6. (*SPSQEP*) *is generalized L-P well-posed if and only if for each* $p \in \bigwedge$, $S(p)$ *is nonempty compact, and* $\mathscr{H}(\Omega_p(\delta, \epsilon), S(p)) \to 0$ *as* $(\delta, \epsilon) \to (0, 0)$ *[or,* $q(\delta, \epsilon) \to 0$ *as* $(\delta, \epsilon) \to (0, 0)$*].*

Proof. It directly follows from the proof of Theorem 4.5 and so omitted it here. This completes the proof.

The following theorems show that under some suitable conditions, the L-P well-posed and generalized L-P well-posed of (SPSQEP) is equivalent to the uniqueness and existence of its solutions.

Theorem 4.7. *Assume that all conditions of Theorem 3.3 are satisfied and, $\bigwedge$ is finite dimensional. If for each $p \in \bigwedge$, there exist some $\tilde{\delta} > 0$ and $\tilde{\epsilon} > 0$ such that $\Omega_p(\tilde{\delta}, \tilde{\epsilon})$ is nonempty bounded. Then (SPSQEP) is L-P well-posed if and only if for each $p \in \bigwedge$, $S(p)$ is a singleton.*

Proof. The necessity is obvious. For the sufficiency. Suppose that for each $p \in \bigwedge$, $S(p) = \{x(p)\}$. For any sequence $\{p^n\} \subseteq \bigwedge$ with $p^n \to p$, let $\{x^n\}$ be any L-P approximating solution sequence corresponding to $\{p^n\}$ of (SPSQEP). Then for each $i \in I$, $x_i^n \in \Gamma_i(x^n, p^n)$ and there exist a sequence $\{\epsilon_n\}$ of positive real numbers with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$F_i(x^n, y^n, \omega_i) + \Psi_i(x^n, \omega_i) + \epsilon_n e_i(x^n) \subseteq C_i(x^n), \quad \forall \omega_i \in \Gamma_i(x^n, p^n), n \in N.$$

Set $\delta_n = \|p^n - p\|$. Then, $x^n \in \Omega_p(\delta_n, \epsilon_n)$ and $\delta_n \to 0$. Since $\Omega_p(\tilde{\delta}, \tilde{\epsilon})$ is nonempty bounded, there exists $\tilde{n} \in N$ such that $\{x^n\} \subseteq \Omega_p(\delta_n, \epsilon_n) \subseteq \Omega_p(\tilde{\delta}, \tilde{\epsilon})$ for all $n \geq \tilde{n}$. So, $\{x^n\}$ is bounded. Let $\{x^{n_k}\}$ be any subsequence of $\{x^n\}$ with $x^{n_k} \to \bar{x}$. Then there exists $p^{n_k} \in B(p, \delta_n)$, for each $i \in I$,

$$d_i(x_i^{n_k}, \Gamma_i(x^{n_k}, p^{n_k})) \leq \epsilon_n \tag{4.6}$$

and there exists $y_i^{n_k} \in T_i(x^{n_k}, p^{n_k})$ such that

$$F_i(x^{n_k}, y^{n_k}, \omega_i) + \Psi_i(x^{n_k}, \omega_i) + \epsilon_n e_i(x^{n_k}) \subseteq C_i(x^{n_k}), \quad \forall \omega_i \in \Gamma_i(x^{n_k}, p^{n_k}). \tag{4.7}$$

It follows from $\delta_n \to 0$ that $p^{n_k} \to p \in B(p, \delta_n)$, since $\bigwedge$ is finite dimensional. Since $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ is l.s.c. and closed, by (4.6), we derive that

$$0 \leq d_i(\bar{x}_i, \Gamma_i(\bar{x}, p)) \leq 0,$$

that is, $d_i(\bar{x}_i, \Gamma_i(\bar{x}, p)) = 0$ and thus, $\bar{x}_i \in \Gamma_i(\bar{x}, p)$. By the u.s.c. and compactness of $T_i, i \in I$, there exist a subsequence $\{y_i^{n_k}\}$ of $\{y_i^n\}$ and $\bar{y}_i \in T_i(\bar{x}, p)$ such that $y_i^{n_k} \to \bar{y}_i$. Since for each $i \in I$, e_i, F_i and Ψ_i are continuous, C_i is u.s.c. such that for each $x \in H, C_i(x)$ is a proper closed convex and pointed cone, take the limit in (4.7), we have, by Facts 2.5 (iii),

$$F_i(\bar{x}, \bar{y}, \omega_i) + \Psi_i(\bar{x}, \omega_i) \subseteq C_i(\bar{x}), \quad \forall \omega_i \in \Gamma_i(\bar{x}, p)$$

and so, $\bar{x} \in S(p)$. Take into account $S(p) = \{x(p)\}$, we conclude that $\bar{x} = x(p)$. It follows from the arbitrariness of $\{x^{n_k}\}$ that x^n converges strongly to $x(p)$. Therefore (SPSQEP) is L-P well-posed. This proof is completed.

Theorem 4.8. *Assume that all conditions of Theorem 3.3 are satisfied and,* $\bigwedge$ *is finite dimensional. If for each* $p \in \bigwedge$, *there exist some* $\tilde{\delta} > 0$ *and* $\tilde{\epsilon} > 0$ *such that* $\Omega_p(\tilde{\delta}, \tilde{\epsilon})$ *is nonempty bounded. Then (SPSQEP) is generalized L-P well-posed.*

Proof. For each $p \in \bigwedge$, let $\{p^n\} \subseteq \bigwedge$ such that $p^n \to p$, and $\{x^n\}$ be any L-P approximating solution sequence corresponding to $\{p^n\}$ of (SPSQEP). Then for each $i \in I$, $x_i^n \in \Gamma_i(x^n, p^n)$ and there exist a sequence $\{\epsilon_n\}$ of positive real numbers with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$F_i(x^n, y^n, \omega_i) + \Psi_i(x^n, \omega_i) + \epsilon_n e_i(x^n) \subseteq C_i(x^n), \quad \forall \omega_i \in \Gamma_i(x^n, p^n), n \in N.$$

Put $\delta_n = \|p^n - p\|$. Then $\delta_n \to 0$ and $x^n \in \Omega_p(\delta_n, \epsilon_n)$ for $n \in N$. Furthermore, we have $x^n \in \Omega_p(\tilde{\delta}, \tilde{\epsilon})$ for all sufficiently large n. By the boundness of $\Omega_p(\tilde{\delta}, \tilde{\epsilon})$, there exists a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ with $x^{n_k} \to \bar{x}$. By the similar proof of Theorem 4.7, we obtain that $\bar{x} \in S(p)$ and so, $S(p) \neq \emptyset$. Therefore (SPSQEP) is generalized L-P well-posed. This proof is completed.

The following result says that the problem (SPSQEP) is solvable under the conditions of Theorem 4.8.

Theorem 4.9. *Let* $\bigwedge$ *be finite dimensional. Assume that all conditions of Theorem 3.3 are satisfied and for each* $p \in \bigwedge$, *there exist some* $\tilde{\delta} > 0$ *and* $\tilde{\epsilon} > 0$ *such that* $\Omega_p(\tilde{\delta}, \tilde{\epsilon})$ *is nonempty bounded. Then, for each* $p \in \bigwedge$, $S(p)$ *is nonempty and compact.*

Proof. It directly follows from Definition 4.2 and Theorems 4.6 and 4.8. This proof is completed.

The following theorem shows that the solvability of the problem (SPSQEP) is equivalent to its generalized L-P well-posedness under the conditions of Theorem 4.9.

Theorem 4.10. *Let* $\bigwedge$ *be finite dimensional. Assume that all conditions of Theorem 3.3 are satisfied and for each* $p \in \bigwedge$, *there exist some* $\tilde{\delta} > 0$ *and* $\tilde{\epsilon} > 0$ *such that* $\Omega_p(\tilde{\delta}, \tilde{\epsilon})$ *is nonempty bounded. Then the generalized L-P well-posedness of (SPSQEP) is equivalent to the nonemptiness of its solutions set.*

Proof. The necessity is obvious. For the sufficiency. Assume that the solutions set of the problem (SPSQEP) is nonempty. Then $S(p) \neq \emptyset$ for $p \in \bigwedge$. Let $\{p^n\} \subseteq \bigwedge$ such that $p^n \to p$. Suppose that (SPSQEP) is not generalized L-P well-posed. That is, there exists a L-P approximating solution sequence $\{x^n\}$ corresponding to $\{p^n\}$ of (SPSQEP) such that for any subsequence $\{x^{n_k}\}$ of $\{x^n\}$ with $x^{n_k} \to \hat{x}$, $d(x^{n_k}, S(p)) \not\to 0$. So, there exists $\rho > 0$ such that

$$\lim_k d(x^{n_k}, S(p)) = d(\hat{x}, S(p)) \geq \rho. \tag{4.8}$$

Since $\{x^n\}$ is a L-P approximating solution sequence corresponding to $\{p^n\}$ of (SP-SQEP). Then for each $i \in I$, $x_i^{n_k} \in \Gamma_i(x^{n_k}, p^{n_k})$ and there exist a sequence $\{\epsilon_{n_k}\}$ of positive real numbers with $\epsilon_{n_k} \to 0$ and $y_i^{n_k} \in T_i(x^{n_k}, p^{n_k})$ such that

$$F_i(x^{n_k}, y^{n_k}, \omega_i) + \Psi_i(x^{n_k}, \omega_i) + \epsilon_{n_k} e_i(x^{n_k}) \subseteq C_i(x^{n_k}), \quad \forall \omega_i \in \Gamma_i(x^{n_k}, p^{n_k}).$$

Set $\delta_{n_k} = \|p^{n_k} - p\|$. Then $\delta_{n_k} \to 0$ and $x^{n_k} \in \Omega_p(\delta_{n_k}, \epsilon_{n_k})$ for $k \in N$. By the similar proof of Theorem 4.8, $\hat{x} \in S(p)$ and so $d(\hat{x}, S(p)) = 0$, which contradicts (4.8). This proof is completed.

If I is a singleton, from Theorems 4.4-4.10, one can derive the similar results for (PSSQEP).

Corollary 4.11. *(PSSQEP) is L-P well-posed if and only if for each* $p \in \bigwedge$,

$$\Omega_p(\delta, \epsilon) \neq \emptyset, \quad \forall \delta, \epsilon > 0, \quad \text{and} \quad \text{diam}[\Omega_p(\delta, \epsilon)] \to 0, \quad \text{as} \quad (\delta, \epsilon) \to (0, 0).$$

Corollary 4.12. *Assume that all conditions of Theorem 3.3 are satisfied. Then (PSSQEP) is generalized L-P well-posed if and only if for each* $p \in \bigwedge$,

$$\Omega_p(\delta, \epsilon) \neq \emptyset, \quad \forall \delta, \epsilon > 0 \quad \text{and} \quad \lim_{(\delta,\epsilon)\to(0,0)} \mathcal{M}(\Omega_p(\delta, \epsilon)) = 0.$$

Corollary 4.13. *(PSSQEP) is generalized L-P well-posed if and only if for each* $p \in \bigwedge$, *$S(p)$ is nonempty compact, and* $\mathcal{H}(\Omega_p(\delta, \epsilon), S(p)) \to 0$ *as* $(\delta, \epsilon) \to (0, 0)$ *[or,* $q(\delta, \epsilon) \to 0$ *as* $(\delta, \epsilon) \to (0, 0)$*].*

The following example illustrate that the compactness of the solution set $S(p)$ in Theorems 4.6 and Corollary 4.13 is indispensable.

Example 4.14. Let the spaces $\bigwedge = [-1, 1]$, $X = Y = Z = R = (-\infty, +\infty)$, $C(x) = R^+ = [0, +\infty)$ for all $x \in X$, $H = K = R^+ = [0, +\infty)$ and let the mappings $T(x, p) = [1 + p, x + 2], \Gamma(x, p) = [0, x + 1 + p], F(x, y, z) = [0, 2y + z - x]$ and $\Psi(x, z) = [0, x - \frac{z}{2}]$ for all $x, y, z \in X$ and $p \in \bigwedge$. Clearly, the solution set $S(p) = H$ and so it is not compact. Moreover, $\Omega_p(\delta, \epsilon) = H$, since $S(p) \subseteq \Omega_p(\delta, \epsilon)$ for all $\delta, \epsilon > 0$ and $\delta \in (0, 1 - |p|)$. Therefore, $\mathcal{H}(\Omega_p(\delta, \epsilon), S(p)) = 0$ for all $\delta, \epsilon > 0$ and $\delta \in (0, 1 - |p|)$. Let $\{p^n\} \subseteq \bigwedge$ with $p^n \to p^* \in \bigwedge$. It is easy to check that the sequence $\{n\}$ is a L-P approximating solution sequence corresponding to $\{p^n\}$ of (PSSQEP). However, the sequence $\{n\}$ has no convergent subsequence. Therefore (PSSQEP) is not generalized well-posed.

Corollary 4.15. *Assume that all conditions of Theorem 3.3 are satisfied and,* $\bigwedge$ *is finite dimensional. If for each* $p \in \bigwedge$, *there exist some* $\tilde{\delta} > 0$ *and* $\tilde{\epsilon} > 0$ *such that* $\Omega_p(\tilde{\delta}, \tilde{\epsilon})$ *is nonempty bounded. Then (PSSQEP) is L-P well-posed if and only if for each* $p \in \bigwedge$, $S(p)$ *is a singleton.*

Example 4.16. Let the spaces $\bigwedge = (-1,1)$, $X = Y = Z = R = (-\infty, +\infty)$, $C(x) = R^+ = [0, +\infty)$ and $e(x) = 1$ for all $x \in X$, $H = K = [-1, 0]$ and let the mappings $T(x,p) = \{-1\}, \Gamma(x,p) = [x, 0], F(x,y,z) = -(x-y-z)$ and $\Psi(x,z) = -z$ for all $x, y, z \in X$ and $p \in \bigwedge$. It is easy to verify that the solution set $S(p) = \{-1\}$ for $p \in \bigwedge$, there exist $\tilde{\delta} = \tilde{\epsilon} = \frac{1-|p|}{2} \in (0, 1-|p|)$ such that $\Omega_p(\frac{1-|p|}{2}, \frac{1-|p|}{2}) = [-1, -\frac{|p|+1}{2}]$ is nonempty bounded and the assumptions of Corollary 4.15 are satisfied. So, from Corollary 4.15, (PSSQEP) is L-P well-posed.

Corollary 4.17. *Assume that all conditions of Corollary 4.15 are satisfied. If for each $p \in \bigwedge$, there exist some $\tilde{\delta} > 0$ and $\tilde{\epsilon} > 0$ such that $\Omega_p(\tilde{\delta}, \tilde{\epsilon})$ is nonempty bounded. Then (PSSQEP) is generalized L-P well-posed if and only if for each $p \in \bigwedge, S(p) \neq \emptyset$.*

We remark that the boundness of $\Omega_p(\tilde{\delta}, \tilde{\epsilon})$ is necessary for some $\tilde{\delta}, \tilde{\epsilon} > 0$.

Example 4.18. Let the spaces $\bigwedge = (-1,1)$, $X = Y = Z = R = (-\infty, +\infty)$, $C(x) = R^+ = [0, +\infty)$ for all $x \in X$, $H = K = R^+ = [0, +\infty)$ and let the mappings $T(x,p) = \Gamma(x,p) = [0, x], F(x,y,z) = -(x+5y-z)$ and $\Psi(x,z) = x - z$ for all $x, y, z \in X$ and $p \in \bigwedge$. It is easy to check that the assumptions of Corollary 4.17 hold. But $\Omega_p(\delta, \epsilon)$ is unbounded for all $\delta, \epsilon > 0$ and $\delta \in (0, 1-|p|)$, since the solution set $S(p) = H$ is unbounded. So, from Corollary 4.17, (PSSQEP) is not generalized L-P well-posed.

The following theorem present the sufficient conditions for the Kuratowski measure of noncompactness $\mathcal{M}(\Omega_p(\delta, \epsilon))$ which approaches to zero as $(\delta, \epsilon) \to (0, 0)$.

Theorem 4.19. *Assume that all conditions of Theorem 4.8 are satisfied. Then, for each $p \in \bigwedge, \Omega_p(\delta, \epsilon) \neq \emptyset, \forall \delta, \epsilon > 0$ and $\mathcal{M}(\Omega_p(\delta, \epsilon)) \to 0$ as $(\delta, \epsilon) \to (0, 0)$.*

Proof. It immediately follows from Theorems 4.5 and 4.8. This proof is completed.

Remark 4.20. (i) If the index set I is a singleton, $p \in \bigwedge, H, K$ are nonempty closed and convex subsets of a finite dimensional Euclidean space $X, f : X \to X$, $\Gamma(x,p) = H, C(x) = (-\infty, 0], F(x,y,z) \equiv 0$ and $\Psi(x,y) = \langle f(x), x-y \rangle$ for all $(x,y,z) \in H \times K \times H$, then Theorem 4.4 is reduced to Theorem 3.1 of (Hu and Fang [31],p. 375). Moreover, if X is a Banach space with its dual space $X^*, f : X \to X^*$, then Theorem 4.4 also is reduced to Theorem 2.3 of (Huang, Yang and Zhu [18],p. 164).

(ii) If $\varphi : X \to R \cup \{+\infty\}$ is a proper convex and lower semicontinuous functional, let $\Psi(x,y) = \langle f(x), x-y \rangle + \varphi(x) - \varphi(y)$ for all $x, y \in X$, then Corollaries 4.11–4.13, 4.15 and 4.17 can be applied to the following mixed variational inequality: find $x \in X$ such that

$$\langle f(x), x-y \rangle + \varphi(x) - \varphi(y) \leq 0, \quad \forall y \in X,$$

which has been studied by Fang, Huang and Yao [32].

Acknowledgments

The authors wish to thank for the anonymous referees and associated editor for their very careful and valuable comments which led to an improved presentation of this manuscript.

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Jia-wei Chen
School of Mathematics and Statistics
Southwest University
Chongqing 400715
P. R. China
E-mail: J.W.Chen713@163.com

Yeong-Cheng Liou
Department of Information Management
Cheng Shiu University
Kaohsiung 833, Taiwan
E-mail: simplex_liou@hotmail.com