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E-PROPER SADDLE POINTS AND E-PROPER DUALITY IN VECTOR OPTIMIZATION WITH SET-VALUED MAPS

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Abstract. In this paper, based on a kind of unified proper efficiency named as E-Benson proper efficiency, we present E-proper saddle points theorems and E-proper duality results including as weak duality and strong duality theorems of vector optimization problems with set-valued maps. Our main results unify and extend the cases of proper saddle points and proper duality as well as ε-proper saddle points and ε-proper duality.

1. INTRODUCTION

It is well known that the concepts of approximate solution have been playing an important role when there are no exact solutions in optimization problems. Kutateladze initially introduced the concept of approximate solution named as ε-efficient solution in [1]. In recent years, some scholars presented several kinds of concepts of approximate efficiency as well as approximate proper efficiency and studied some characterizations and applications in vector optimization problems, see e.g. [2-7] and the references therein.

Recently, Chicco et al. proposed a new concept of approximate efficiency named as E-efficiency based on improvement sets in [8]. E-efficiency unifies the concepts of optimal points, approximate optimal points in scalar optimization, Pareto equilibria and approximate equilibra. Furthermore, Zhao and Yang presented a unified stability result with perturbations in vector optimization in [9]. Gutiérrez et al. generalized the concepts of improvement set and E-efficiency to a real locally convex Hausdorff topological vector space in [10]. Zhao and Yang introduced a kind of unified proper

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efficiency named as E-Benson proper efficiency and established some scalarization and Lagrange multiplier theorems under the E-subconvexlikeness in [11].

Moreover, some saddle points and duality results have been established based on some exact and approximate solutions of vector optimization problems with set-valued maps, see e.g. [12-16] and the references therein. Especially, Li established proper saddle points criteria and proper duality theorems of Benson proper efficiency for vector optimization problems with set-valued maps in [12]. Rong and Ma proposed the concept of ε-proper efficiency by the idea of Benson proper efficiency and established some ε-saddle-point theorems and some ε-proper duality theorems in [16].

Motivated by the works of [11, 12, 16], in this paper, we present some E-proper saddle points theorems and E-proper duality results including as weak duality and strong duality theorems of vector optimization problems with set-valued maps. The main results unify and extend some exact and approximate cases.

2. Preliminaries

In this paper, let X be a linear space, Y and Z be real locally convex Hausdorff topological vector spaces. For a set $A \subseteq Y$, int$A$ and cl$A$ denote interior and closure of A, respectively. A cone $K \subseteq Y$ is called pointed if $K \cap (-K) = \{0\}$. Let K and P be positive closed convex pointed cones in Y and Z with nonempty interiors, respectively. For any $x, y \in Y$,

$$x \leqq_K y \Leftrightarrow y - x \in K$$

and $\langle x, y \rangle$ indicates the inner product of x and y. The generated cone of $A \subseteq Y$ is

$$\text{cone}(A) = \{\lambda a |\ \lambda \geq 0, a \in A\}.$$

Y^* denotes the topological dual space of Y. The positive dual cone of A is defined as

$$A^+ = \{\mu \in Y^* | \langle y, \mu \rangle \geq 0, \forall y \in A\},$$

the quasi-interior of A^+ is defined as

$$A^{+i} = \{\mu \in Y^* | \langle y, \mu \rangle > 0, \forall y \in A \backslash \{0\}\}.$$

Definition 2.1. ([10]). A set $E \subseteq Y$ is said to be an improvement set if $0 \notin E$ and $E + K = E$. We denote the family of the improvement sets in Y by $\mathfrak{T}_Y$.

Definition 2.2. ([11]). Let $E \in \mathfrak{T}_Y$ and $A \subseteq Y$. $a \in A$ is an E-efficient point of A if $(a - E - K\backslash\{0\}) \cap A = \emptyset$ and we denote this by $a \in O^{E+K\backslash\{0\}}(A)$.

Definition 2.3. ([11]). Let $E \in \mathfrak{T}_Y$ and $A \subseteq Y$. $a \in A$ is an E-Benson proper efficient point of A if clcone$(A+E-a) \cap (-K) = \{0\}$. We denote this by $a \in O^E_{BS}(A)$, i.e.,

$$O_{BS}^E(A) = \{a \in A | \text{clcone}(A + E - a) \cap (-K) = \{0\}\}.$$

Let

$$\overline{O_{BS}^E}(A) = \{a \in A | \text{clcone}(A - E - a) \cap K = \{0\}\}.$$

Consider the following vector optimization problem with set-valued maps:

$$\begin{aligned} &\text{(VP)} \ \min F(x) \\ &\text{s.t. } x \in D = \{x \in S | G(x) \cap (-P) \neq \emptyset\}, \end{aligned}$$

where $S \subseteq X$, $F : S \rightrightarrows Y$ and $G : S \rightrightarrows Z$ are set-valued maps with nonempty value. Assume that the feasible set D of (VP) be nonempty. Let $L(Z, Y)$ be the space of continuous linear operator from Z to Y and

$$L^+ = L^+(Z, Y) = \{T \in L(Z, Y) | T(P) \subseteq K\}.$$

Denote by (F, G) the set-valued map from S to $Y \times Z$ defined by

$$(F, G)(x) = F(x) \times G(x).$$

If $\mu \in Y^*, T \in L(Z, Y)$, we also define $\mu F : S \rightrightarrows \mathbb{R}$ and $F + TG : S \rightrightarrows Y$ by

$$(\mu F)(x) = \langle F(x), \mu \rangle \text{ and } (F + TG)(x) = F(x) + T(G(x)),$$

respectively. Moreover, (VP) satisfies the generalized Slater constraint qualification if there exists $\hat{x} \in S$ such that $G(\hat{x}) \cap (-\text{int}P) \neq \emptyset$.

Definition 2.4. ([11]). (i) $x_0 \in D$ is called an E-efficient solution of (VP) if there exists $y_0 \in F(x_0)$ such that $(y_0 - E - K \backslash \{0\}) \cap F(D) = \emptyset$; (ii) $x_0 \in D$ is called an E-Benson proper efficient solution of (VP) if $F(x_0) \cap O_{BS}^E(F(D)) \neq \emptyset$; (iii) (x_0, y_0) is called an E-Benson proper efficient point of (VP) if $x_0 \in D$ and $y_0 \in F(x_0) \cap O_{BS}^E(F(D))$.

Definition 2.5. ([11]). Let $F : S \rightrightarrows Y$ and $E \in \mathfrak{T}_Y$. F is said to be E-subconvexlike on S if $F(S) + \text{int}E$ is a convex set.

Consider the following scalar optimization problem of (VP):

$$\text{(VP)}_\mu \ \min_{x \in D} \langle F(x), \mu \rangle, \ \mu \in Y^* \backslash \{0_{Y^*}\}.$$

Definition 2.6. ([15]). $x_0 \in D$ is called an E-optimal solution of (VP)_μ if there exists $y_0 \in F(x_0)$ such that $\langle y - y_0, \mu \rangle \geq \sigma_{-E}(\mu), \forall x \in D, \forall y \in F(x)$, where

$$\sigma_{-E}(\mu) = \sup_{y \in (-E)} \{\mu(y)\}, \forall \mu \in Y^*.$$

3. E-Proper Saddle Points

In this section, we first give the concept of E-proper saddle points for a set-valued Lagrangian map and then we establish E-proper saddle points theorems.

Definition 3.1. The Lagrangian map for (VP) is the set-valued map $L : S \times L^+(Z,Y) \to 2^Y$ defined by $L(x,T) = F(x) + T(G(x)), (x,T) \in S \times L^+(Z,Y)$.

Definition 3.2. An ordered pair $(\bar{x},\bar{T}) \in S \times L^+(Z,Y)$ is called an E-proper saddle point of the set-valued Lagrangian map $L(x,T)$ if

$$L(\bar{x},\bar{T}) \cap O^E_{BS}(L(S,\bar{T})) \cap \overline{O^E_{BS}}(L(\bar{x},L^+)) \neq \emptyset.$$

Theorem 3.1. *Let $E \in \mathfrak{T}_Y$ be a convex set, $E \subseteq K$. An ordered pair $(\bar{x},\bar{T}) \in S \times L^+(Z,Y)$ is an E-proper saddle point of the set-valued Lagrangian map $L(x,T)$ if and only if there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that the following (i)-(iv):*

(i) $\bar{y} + \bar{T}(\bar{z}) \in O^E_{BS}(L(S,\bar{T}))$;

(ii) $G(\bar{x}) \subseteq -P$;

(iii) $-\bar{T}(\bar{z}) \in K \backslash \mathrm{int}E$;

(iv) $\mathrm{clcone}(F(\bar{x}) - \bar{y} - \bar{T}(\bar{z}) - E) \cap K = \{0\}$.

Proof. Suppose that $(\bar{x},\bar{T})$ is an E-proper saddle point of $L(x,T)$. From definition 3.2, there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that

$$\bar{y} + \bar{T}(\bar{z}) \in O^E_{BS}(L(S,\bar{T})) \text{ and } \bar{y} + \bar{T}(\bar{z}) \in \overline{O^E_{BS}}(L(\bar{x},L^+)).$$

Hence, (i) holds and

$$\text{clcone}\left(L(\bar{x},L^+) - E - (\bar{y} + \bar{T}(\bar{z}))\right) \cap K = \{0\}. \tag{1}$$

From (1), we get

$$\left(F(\bar{x}) + T(G(\bar{x})) - E - (\bar{y} + \bar{T}(\bar{z}))\right) \cap K \subseteq \{0\}, \forall T \in L^+(Z,Y).$$

This implies

$$y + T(z) - (\bar{y} + \bar{T}(\bar{z})) \notin K\backslash\{0\} + E,\ \forall y \in F(\bar{x}), \forall z \in G(\bar{x}), \forall T \in L^+(Z,Y). \tag{2}$$

Since, for any $T \in L^+(Z,Y)$,

$$\begin{aligned} T(\bar{z}) - \bar{T}(\bar{z}) &= (\bar{y} + T(\bar{z})) - \left(\bar{y} + \bar{T}(\bar{z})\right) \in F(\bar{x}) + T(G(\bar{x})) - \left(\bar{y} + \bar{T}(\bar{z})\right) \\ &= L(\bar{x},T) - \left(\bar{y} + \bar{T}(\bar{z})\right), \end{aligned}$$

then we can deduce that

$$\left(\bigcup_{T\in L^+} T(\bar z)\right) - \bar T(\bar z) - E \subseteq L(\bar x, L^+) - E - \left(\bar y + \bar T(\bar z)\right).$$

Hence from (1) and $0 \in K$, we can obtain that

$$\text{clcone}\left(\left(\bigcup_{T\in L^+} T(\bar z)\right) - \bar T(\bar z) - E\right) \cap K = \{0\}.$$

Let $f : L^+(Z,Y) \to Y$ be defined by $f(T) = -T(\bar z),\ T \in L^+(Z,Y)$. Then, the above expression can be written as

$$\text{clcone}\left(f(L^+) + E - f(\bar T)\right) \cap (-K) = \{0\}.$$

This implies that $\bar T \in L^+(Z,Y)$ is an E-Benson proper efficient point of the following vector optimization problem

$$\begin{aligned} &\min\ f(T)\\ &\text{s.t. } T \in L^+(Z,Y). \end{aligned}$$

Since f is a linear map and E is a convex set, then $f - \bar y$ is E-subconvexlike on $L^+(Z,Y)$. Hence, by Theorem 7.2 in [11], there exists $\mu \in K^{+i}$ such that $\bar T$ is an E-optimal solution of problem (VP)$_\mu$, i.e.,

$$\langle f(T) - f(\bar T + e), \mu\rangle \geq 0,\ \forall T \in L^+(Z,Y),\ \forall e \in E.$$

Then

$$\langle -\bar T(\bar z), \mu\rangle \leq \langle -T(\bar z), \mu\rangle + \langle e, \mu\rangle,\ \forall T \in L^+(Z,Y),\ \forall e \in E. \tag{3}$$

We assert that $\bar z \in -P$. Otherwise, since P is a closed convex set, then there exists $\sigma \in Z^*\backslash\{0_{Z^*}\}$ such that

$$\langle -\bar z, \sigma\rangle < \inf_{y\in P}\langle y, \sigma\rangle \leq \langle \delta p, \sigma\rangle,\ \forall \delta > 0,\ \forall p \in P.$$

Taking $p = 0 \in P$, we obtain $\langle \bar z, \sigma\rangle > 0$. Moreover,

$$\frac{\langle -\bar z, \sigma\rangle}{\delta} < \langle p, \sigma\rangle,\ \forall \delta > 0,\ \forall p \in P.$$

Letting $\delta \to +\infty$, we obtain $\langle p, \sigma\rangle \geq 0, \forall p \in P$, and so $\sigma \in P^+\backslash\{0_{Z^*}\}$. Let $\hat e \in E$, $\hat k \in \text{int}K$ be fixed, then $\hat e + \hat k \in E + \text{int}K = \text{int}E$. Define $\hat T : Z \to Y$ as

$$\hat T(z) = \frac{\langle z, \sigma\rangle}{\langle \bar z, \sigma\rangle}(\hat e + \hat k) + \bar T(z). \tag{4}$$

It is evident that $\hat{T} \in L(Z, Y)$ and

$$\hat{T}(p) = \frac{\langle p, \sigma \rangle}{\langle \bar{z}, \sigma \rangle}(\hat{e} + \hat{k}) + \bar{T}(p) \in K + \text{int}K + K \subseteq K, \ \forall p \in P.$$

Hence, $\hat{T} \in L^+(Z, Y)$. Taking $z = \bar{z}$ in (4), we have $\hat{T}(\bar{z}) - \bar{T}(\bar{z}) = \hat{e} + \hat{k}$. This implies

$$\langle \hat{T}(\bar{z}), \mu \rangle - \langle \bar{T}(\bar{z}), \mu \rangle = \langle \hat{e}, \mu \rangle + \langle \hat{k}, \mu \rangle. \tag{5}$$

From $\mu \in K^{+i}$ and $\hat{k} \in$ int K, it follows that $\langle \hat{k}, \mu \rangle > 0$. Moreover, from (5),

$$\langle \hat{T}(\bar{z}), \mu \rangle - \langle \bar{T}(\bar{z}), \mu \rangle > \langle \hat{e}, \mu \rangle.$$

This contradiets to (3) and so $-\bar{z} \in P$. Now, we show that $G(\bar{x}) \subseteq -P$. On the contrary, then there exists $z_0 \in G(\bar{x})$ such that $-z_0 \notin P$. We can verify that there exists $\lambda_0 \in P^+$ such that $\langle z_0, \lambda_0 \rangle > 0$. Taking a fixed $e_0 \in \text{int}E$, we define a map $T_0 : Z \to Y$ as

$$T_0(z) = \frac{\langle z, \lambda_0 \rangle}{\langle z_0, \lambda \rangle_0} e_0, \ z \in Z.$$

Obviously, $T_0 \in L^+(Z, Y)$. Noticing that $-\bar{T}(\bar{z}) \in K$, we can obtain that

$$T_0(z_0) - \bar{T}(\bar{z}) = e_0 - \bar{T}(\bar{z}) \in \text{int}E + K = \text{int}E. \tag{6}$$

On the other hand, taking $T = T_0$, $y = \bar{y}$ and $z = z_0$ in (2), we obtain that

$$T_0(z_0) - \bar{T}(\bar{z}) \notin K \backslash \{0\} + E \subseteq K + E = E.$$

This contradicts to (6) and (ii) holds.

Since $-\bar{z} \in P$ and $\bar{T} \in L^+(Z, Y)$ implies that $-\bar{T}(\bar{z}) \in K$, taking $y = \bar{y} \in F(\bar{x})$ and $T = 0 \in L^+(Z, Y)$ in (2), we can obtain that

$$-\bar{T}(\bar{z}) \notin K \backslash \{0\} + E. \tag{7}$$

Noticing int$E$ = int$K + E \subseteq K \backslash \{0\} + E$. Hence, it follows from (7) that $-\bar{T}(\bar{z}) \notin$ int$E$. Thus, $-\bar{T}(\bar{z}) \in K \backslash$int$E$ and so (iii) holds.

Furthermore, by (1), we have

$$\text{clcone}\left(F(\bar{x}) + T(G(\bar{x})) - E - (\bar{y} + \bar{T}(\bar{z}))\right) \cap K = \{0\}, \ \forall T \in L^+(Z, Y).$$

Taking $T = 0 \in L^+(Z, Y)$, then

$$\text{clcone}\left(F(\bar{x}) - \bar{y} - \bar{T}(\bar{z}) - E\right) \cap K = \{0\}.$$

Thus, (iv) holds.

Conversely, from $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$, it follows that

$$\bar{y} + \bar{T}(\bar{z}) \in F(\bar{x}) + \bar{T}(G(\bar{x})) = L(\bar{x}, \bar{T}). \tag{8}$$

By (ii), we have $-T(G(\bar{x})) \subseteq T(P) \subseteq K,\ \forall T \in L^+(Z,Y)$. Noticing that $E \in \mathfrak{T}_Y$, then

$$E - T(G(\bar{x})) \subseteq E + K = E,\ \forall T \in L^+(Z,Y). \tag{9}$$

From (9) and (iv), it follows that

$$\text{clcone}(F(\bar{x}) - \bar{y} - \bar{T}(\bar{z}) - (E - T(G(\bar{x})))) \cap K = \{0\}, \forall T \in L^+,$$

i.e.,

$$\text{clcone}(L(\bar{x}, L^+) - \bar{y} - \bar{T}(\bar{z}) - E) \cap K = \{0\}.$$

Then

$$\bar{y} + \bar{T}(\bar{z}) \in \overline{O^E_{BS}}(L(\bar{x}, L^+)). \tag{10}$$

From (i), (8) and (10), we can obtain that

$$\bar{y} + \bar{T}(\bar{z}) \in L(\bar{x}, \bar{T}) \cap O^E_{BS}(L(S, \bar{T})) \cap \overline{O^E_{BS}}(L(\bar{x}, L^+)).$$

Therefore, $(\bar{x}, \bar{T})$ is an E-proper saddle point of $L(x, T)$. ■

Example 3.1. Let $X = Y = Z = \mathbb{R}^2$, $K = P = \mathbb{R}^2_+$, $E = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 \geq 0, y_2 \geq 0, y_1 + y_2 \geq 3\}$, $S = [-1, 1] \times \{0\}$. The set-valued maps $F : S \rightrightarrows Y$ and $G : S \rightrightarrows Z$ are defined as

$$F(x_1, x_2) = \{(y_1, y_2) \in \mathbb{R}^2 | (y_1, y_2) \in [-1 + x_1, 1 + x_1] \times \{1\}\}, \forall (x_1, x_2) \in S,$$

$$G(x_1, x_2) = \{(z_1, z_2) \in \mathbb{R}^2 | (z_1, z_2) = \lambda(1 + x_1, 1 + x_1), \forall \lambda \in [0, 1]\}, \forall (x_1, x_2) \in S.$$

Let $\bar{x} = (-1, 0)$ and $\bar{T}(z_1, z_2) = (0.5z_1, 0.5z_2) \in L^+(Z, Y)$, we can verify that $(\bar{x}, \bar{T})$ is the E-proper saddle points of $L(x, T)$. In fact, we only need to verify that

$$a = (0, 1) \in L(\bar{x}, \bar{T}) \cap O^E_{BS}(L(S, \bar{T})) \cap \overline{O^E_{BS}}(L(\bar{x}, L^+)) \neq \emptyset. \tag{11}$$

Since

$$F(\bar{x}) = \left\{(y_1, y_2) \in \mathbb{R}^2 | (y_1, y_2) \in [-2, 0] \times \{1\}\right\}, G(\bar{x}) = \{(0, 0)\},$$

$$L(\bar{x}, \bar{T}) = F(\bar{x}) + \bar{T}(G(\bar{x})) = \left\{(y_1, y_2) \in \mathbb{R}^2 | (y_1, y_2) \in [-2, 0] \times \{1\}\right\},$$

$$L(S, \bar{T}) = F(S) + \bar{T}(G(S)) = \{(y_1, y_2) | y_1 - y_2 + 3 \geq 0, y_1 - y_2 - 1 \leq 0, 1 \leq y_2 \leq 2\},$$

$$L(\bar{x}, L^+) = F(\bar{x}) + L^+(G(\bar{x})) = F(\bar{x}),$$

then

$$\text{clcone}(L(S,\bar{T})+E-a)\cap(-K)=\{(y_1,y_2)\in\mathbb{R}^2|3y_1+2y_2\geq 0, y_2\geq 0\}\cap(-\mathbb{R}^2_+) = \{0\},$$

$$\text{clcone}(L(\bar{x}, L^+) - E - a) \cap K = \{(y_1, y_2) \in \mathbb{R}^2|y_1 \leq 0, y_2 \leq 0\} \cap \mathbb{R}^2_+ = \{0\},$$

Hence, (11) holds which implies $(\bar{x}, \bar{T})$ is the E-proper saddle points of $L(x, T)$.

Moreover, let $\bar{y} = (-1, 1) \in F(\bar{x}), \bar{z} = (0, 0) \in G(\bar{x})$. Clearly, $\bar{y}+\bar{T}(\bar{z}) = (-1, 1)$, (ii) and (iii) hold. Thus, we only need to verify that (i) and (iv) hold. This can be seen from

$$\begin{aligned} &\text{clcone}(L(S,\bar{T}) + E - (\bar{y} + \bar{T}(\bar{z}))) \cap (-K) \\ =\ &\text{clcone}(L(S,\bar{T}) + E - (-1, 1)) \cap (-\mathbb{R}^2_+) \\ =\ &\{(y_1, y_2) \in \mathbb{R}^2|3y_1 + y_2 \geq 0, y_2 \geq 0\} \cap (-\mathbb{R}^2_+) = \{0\} \end{aligned}$$

and

$$\begin{aligned} &\text{clcone}(F(\bar{x}) - (\bar{y} + \bar{T}(\bar{z})) - E) \cap K \\ =\ &\text{clcone}(F(\bar{x}) - (-1, 1) - E) \cap \mathbb{R}^2_+ \\ =\ &\{(y_1, y_2) \in \mathbb{R}^2|3y_1 + y_2 \leq 0, y_2 \leq 0\} \cap \mathbb{R}^2_+ = \{0\}. \end{aligned}$$

Theorem 3.2. *Let $E \in \mathfrak{T}_Y$. If an ordered pair $(\bar{x}, \bar{T}) \in S \times L^+(Z, Y)$ is an E-proper saddle point of the set-valued Lagrangian map $L(x, T)$ and $0 \in G(\bar{x})$, then there exists $\bar{z} \in G(\bar{x})$ such that $\bar{x}$ is a E'-Benson proper efficient point of (VP), where $E' = E - \bar{T}(\bar{z}) \subseteq K$.*

Proof. Since $(\bar{x}, \bar{T}) \in S \times L^+(Z, Y)$ is an E-proper saddle point of $L(x, T)$ and by Definition 3.2, then there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that $\bar{y} + \bar{T}(\bar{z}) \in O^E_{BS}(L(S, \bar{T}))$, i.e.,

$$\text{clcone}\left(F(S) + \bar{T}(G(S)) + E - (\bar{y} + \bar{T}(\bar{z}))\right) \cap (-K) = \{0\}.$$

Taking $E' = E - \bar{T}(\bar{z})$. Obviously, $E' \in \mathfrak{T}_Y$ and

$$E' = E - \bar{T}(\bar{z}) \subseteq K + K = K.$$

Noticing that $0 \in G(\bar{x}) \subseteq G(S)$, we have

$$E' = E - \bar{T}(\bar{z}) \subseteq E + \bar{T}(G(S)) - \bar{T}(\bar{z}).$$

Hence, from $D \subseteq S$, we can obtain that

$$\begin{aligned} &\text{clcone}\left(F(D) + E' - \bar{y})\right) \subseteq \text{clcone}\left(F(S) + E' - \bar{y})\right) \\ &\subseteq \text{clcone}\left(F(S) + \bar{T}(G(S)) + E - (\bar{y} + \bar{T}(\bar{z}))\right). \end{aligned}$$

Consequently, we have

$$\text{clcone}\big(F(D) + E' - \bar{y})\big) \cap (-K) = \{0\}.$$

That is, $\bar{x}$ is a E'-Benson proper efficient point of (VP). ■

Remark 3.1. Theorem 3.1 unifies and extends Proposition 6.1 in [12] and Proposition 5.1 in [16]. Theorem 3.2 unifies and extends Theorem 6.1 in [12] and Theorem 5.1 in [16].

4. E-Proper Duality

A set-valued map $\Phi : L^+(Z, Y) \to 2^Y$ is defined as

$$\Phi(T) = O^E_{BS}(L(S, T),\ T \in L^+(Z, Y),$$

which is said to be an E-proper dual map of (VP). The vector maximization problem with set-valued map Φ,

$$\text{(VD) max} \bigcup_{T \in L^+} \Phi(T)$$

is said to be a dual problem of (VP).

Definition 4.1. Let $E \in \mathfrak{T}_Y$. A point $y \in Y$ is called a feasible point of (VD) if $y \in \bigcup_{T \in L^+} \Phi(T)$. A feasible point $\bar{y}$ is called an E-efficient point of (VD) if

$$(\bar{y} + E + K\backslash\{0\}) \cap \left(\bigcup_{T \in L^+} \Phi(T) \right) = \emptyset.$$

Theorem 4.1. (Weak Duality). *Let $E \in \mathfrak{T}_Y$, If $\bar{x}$ be any feasible solution of (VP), and $\bar{y}$ be any feasible point of (VD). Then $(\bar{y} - F(\bar{x})) \cap (E + K\backslash\{0\}) = \emptyset$.*

Proof. Since $\bar{y}$ is a feasible point of (VD), then $\bar{y} \in \bigcup_{T \in L^+} \Phi(T)$. This implies that there exists $\bar{T} \in L^+(Z, Y)$ such that

$$\bar{y} \in \Phi(\bar{T}) = O^E_{BS}(L(S, \bar{T})) \subseteq O^{E+K\backslash\{0\}}(L(S, \bar{T})),$$

i.e.,

$$(\bar{y} - E - K\backslash\{0\}) \cap (F(S) + \bar{T}(G(S))) = \emptyset.$$

Then,

$$\bar{y} - y - \bar{T}(z) \notin E + K\backslash\{0\}, \forall y \in F(\bar{x}), \forall z \in G(\bar{x}). \tag{12}$$

Moreover, from $G(\bar{x}) \cap (-P) \neq \emptyset$, it follows that there exists $\bar{z} \in G(\bar{x})$ such that $-\bar{z} \in P$. Then

$$-\bar{T}(\bar{z}) \in \bar{T}(P) \subseteq K.$$

Taking $z = \bar{z}$ in (12), then

$$\bar{y} - y - \bar{T}(\bar{z}) \notin E + K\backslash\{0\},\ \forall y \in F(\bar{x}).$$

From $-\bar{T}(\bar{z}) \in K$ and $E \in \mathfrak{T}_Y$, it follows that

$$\bar{y} - y \notin E + K\backslash\{0\},\ \forall y \in F(\bar{x}),$$

which indicates the conclusion. ■

Theorem 4.2. *Let $E \in \mathfrak{T}_Y$. If $\bar{x}$ be any feasible solution of* (*VP*) *and*

$$\bar{y} \in \left(\bigcup_{T \in L^+} \Phi(T)\right) \cap F(\bar{x}).$$

Then, $\bar{x}$ is an E-efficient point of (*VP*) *and $\bar{y}$ is an E-efficient point of* (*VD*).

Proof. Since $\bar{y} \in \bigcup\limits_{T \in L^+} \Phi(T)$, then there exists $\bar{T} \in L^+(Z, Y)$ such that

$$\bar{y} \in \Phi(\bar{T}) = O_{BS}^{E}(L(S, \bar{T}) \subseteq O^{E+K\backslash\{0\}}(L(S, \bar{T}),$$

i.e.,

$$(\bar{y} - E - K\backslash\{0\}) \cap (F(S) + \bar{T}(G(S))) = \emptyset.$$

Then,

$$\bar{y} - y - \bar{T}(z) \notin E + K\backslash\{0\}, \forall x \in S, \forall y \in F(x), \forall z \in G(x). \tag{13}$$

When $x \in D \subseteq S$, there exists $\tilde{z} \in G(x)$ such that $\tilde{z} \in -P$. Hence, $-\bar{T}(\tilde{z}) \in K$. Taking $z = \tilde{z}$ in (13), we have

$$\bar{y} - y \notin E + K\backslash\{0\}, \forall x \in D, \forall y \in F(x),$$

i.e.,

$$(\bar{y} - E - K\backslash\{0\}) \cap F(D) = \emptyset.$$

Since $\bar{y} \in F(\bar{x})$, then $\bar{x}$ is an E-efficient point of (VP).

Moreover, let $\bar{x}$ be a feasible point of (VP). By $\bar{y} \in F(\bar{x})$ and Theorem 4.1, we obtain that

$$(y - \bar{y}) \cap (E + K\backslash\{0\})) = \emptyset,\ \forall y \in \bigcup_{T \in L^+} \Phi(T).$$

That is,

$$(\bar{y} + E + K\backslash\{0\})) \cap \left(\bigcup_{T \in L^+} \Phi(T) \right) = \emptyset.$$

Thus, $\bar{y}$ is an$E$-efficient point of (VD) by using Definition 4.1. ∎

Theorem 4.3. (Strong Duality). *Let $E \in \mathfrak{T}_Y$, $E \subseteq K$, F be E-subconvexlike on D, (F, G) be $(E \times P)$-subconvexlike on S and (VP) satisfy the generalized Slater constraint qualification. If $(\bar{x}, \bar{y})$ is an E-Benson proper efficient point of (VP) and $0 \in G(\bar{x})$, then $\bar{y}$ is an E-efficient point of (VD).*

Proof. According to Theorem 8.1 in [11], then there exists $T \in L^+$ such that $(\bar{x}, \bar{y})$ is an E-Benson proper efficient point of the following problem (UVP):

$$\begin{aligned} &\text{(UVP)} \ \min L(x, T) \\ &\text{s.t.} \ (x, T) \in S \times L^+(Z, Y). \end{aligned}$$

Hence,

$$\bar{y} \in F(\bar{x}) \subseteq F(\bar{x}) + T(G(\bar{x})) \in L(S, T), \forall T \in L^+(Z, Y)$$

and

$$\text{clcone}(L(S, T) + E - \bar{y}) \cap (-K) = \{0\}, \forall T \in L^+.$$

Then,

$$\bar{y} \in O^E_{BS}(L(S, T)) = \Phi(T), \forall T \in L^+(Z, Y).$$

Thus,

$$\bar{y} \in F(\bar{x}) \cap \left(\bigcup_{T \in L^+} \Phi(T) \right).$$

Then, $\bar{y}$ is an E-efficient point of (VD) by means of Theorem 4.2. ∎

Remark 4.1. Theorem 4.1 unifies Theorem 7.1 in [12] and Theorem 6.1 in [16]. Theorem 4.3 unifies and extends Theorem 7.3 in [12] and Theorem 6.2 in [16].

5. Conclusions Remarks

In this paper, we first present the concept of E-proper saddle points via improvement sets. Furthermore, based on the E-Benson proper efficiency proposed by Zhao and Yang in [11], we establish E-proper saddle points theorems. In the end, we also establish E-proper duality results including as weak duality and strong duality theorems of vector optimization problems with set-valued maps. These results unify and extend some known results about exact and approximate cases.

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References

1. S. S. Kutateladze, Convex ε-programming, *Soviet Mathematics Doklady*, **20** (1979), 391-393.
2. P. Loridan, ε-solutions in vector minimization problems, *Journal of Optimization Theory and Applications*, **43** (1984), 265-276.
3. I. Valyi, Approximate saddle-point theorems in vector optimization, *Journal of Optimization Theory and Applications*, **55** (1987), 435-448.
4. J. C. Liu, ε-Properly efficient solutions to nondifferentiable multiobjective programming problems, *Applied Mathematics Letters*, **12** (1999), 109-113.
5. J. Dutta and V. Vetrivel, On approximate minima in vector optimization, *Numerical Functional Analysis and Optimization*, **22** (2001), 845-859.
6. C. Gutiérrez, B. Jiménez and V. Novo, A unified approach and optimality conditions for approximate solutions of vector optimization problems, *SIAM Journal on Optimization*, **17** (2006), 688-710.
7. Y. Gao and X. M. Yang, Optimality conditions for approximate solutions of vector optimization problems, *Journal of Industrial and Management Optimization*, **7** (2011), 483-496.
8. M. Chicco, F. Mignanego, L. Pusillo and S. Tijs, Vector optimization problems via improvement sets, *Journal of Optimization Theory and Applications*, **150** (2011), 516-529.
9. K. Q. Zhao and X. M. Yang, A unified stability result with perturbations in vector optimization, *Optimization Letters*, doi: 10.1007/s11590-012-0533-1, 2012.
10. C. Gutiérrez, B. Jiménez and V. Novo, Improvement sets and vector optimization, *European Journal of Operational Research*, **223** (2012), 304-311.
11. K. Q. Zhao and X. M. Yang, E-Benson proper efficiency in vector optimization, *Optimization*, doi: 10.1080/02331934.2013.798321, 2013.
12. Z. F. Li, Benson proper efficiency in the vector optimization of set-valued maps, *Journal of Optimization Theory and Applications*, **98** (1998), 623-649.
13. G. Y. Chen and W. D. Rong, Characterizations of the Benson proper efficiency for nonconvex vector optimization, *Journal of Optimization Theory and Applications*, **98** (1998), 365-384.

14. W. D. Rong and Y. N. Wu, ε-Weak minimal solutions of vector optimization problems with set-valued maps, *Journal of Optimization Theory and Applications*, **106** (2000), 569-579.

15. K. Q. Zhao, X. M. Yang and J. W. Peng, Weak E-optimal solution in vector optimization, *Taiwanese Journal of Mathematics*, **17** (2013), 1287-1302.

16. W. D. Rong and Y. Ma, ε-Properly efficient solutions of vector optimization problems with set-valued maps, *Operations Research Transactions*, **4** (2000), 21-32.

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