TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 4, pp. 1185-1202, August 2014
DOI: 10.11650/tjm.18.2014.3451
This paper is available online at http://journal.taiwanmathsoc.org.tw

MULTIPLE SOLUTIONS FOR PERIODIC SCHRÖDINGER EQUATIONS WITH SPECTRUM POINT ZERO

Dongdong Qin, Fangfang Liao and Yi Chen

Abstract. This paper is concerned with the following Schrödinger equation:

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as} & |x| \to \infty, \end{cases}$$

where the potential V and f are periodic with respect to x and 0 is a boundary point of the spectrum $\sigma(-\triangle + V)$. By a generalized variant fountain theorem and an approximation technique, for old f, we are able to obtain the existence of infinitely many large energy solutions.

1. Introduction

In this paper, we consider the semilinear Schrödinger equation:

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as} & |x| \to \infty, \end{cases} \tag{1.1}$$

where $V(x) : \mathbb{R}^N \to \mathbb{R}$ is a potential and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonlinear coupling which is superlinear as $|u| \to \infty$. As we know, the nonlinear Schrödinger equation with periodic potential and nonlinearities has been widely investigated in the literature over the past several decades for both its importance in applications and mathematical interest, see, e.g., [1-3], [7-11] and [21-24]. It is well know (see, e.g., [13]) that the spectrum of the self-adjoint operator $A = -\triangle + V$ is purely absolutely continuous and bounded below. There are many results on the existence and multiplicity of solutions for problem (1.1) depending on the location of 0 in $\sigma(A)$.

For the case of $0 < \sigma(A)$, Coti-zelati and Rabinowitz proved in [25] the existence of infinitely many solutions with $f \in C^2$ and the so called Ambrosetti-Rabinowitz superquadratic condition. In [9], under a general superlinear assumption and monotone

Received September 6, 2013, accepted January 6, 2014.
Communicated by Neil Trudinger.
2010 *Mathematics Subject Classification*: 35Q55, 58E05.
Key words and phrases: Schrödinger equation, Spectrum point zero, Infinitely many solutions.

condition on f, Li, Wang and Zeng obtained the existence of ground state solutions by concentration compactness argument. We also refer reader to [10] and [14-19] where the condition (AR) was replaced by more general superlinear assumptions.

A lot work has been done under assumption that 0 lies in a spectral gap of $\sigma(A)$. In [8], relying on a degree theory and a linking-type argument developed there, Kryszeuski and Szulkin obtained a nontrival solution under condition (AR) and infinitely many geometrically distinct nontrival solutions with additional locally Lipschitzian assumption on f (see (A8) in [8]). The stronger results to date appear to be those of Szulkin and Weth [18], following the approach of Pankov [11], they proved the existence of ground state solutions under hypotheses weaker than those previously assumed. Yang [23] also obtained the same results using a different method (based on the approach of [16]) which is much simpler. In recent paper [2], using a generalized variant fountain theorem established there, Batkan and Colin proved the existence of infinitely many large energy solutions for (1.1).

As far as we know, there are only several papers deal with the case that 0 is a boundary point of the spectrum $\sigma(A)$. In [3], Bartsch and Ding obtained a nontrival solution with condition (AR). Later, this result was improved by Willem and Zou in [22] by using an improved generalized weak link theorem. In [24], Yang, Chen, Ding proved a nontrival solution under the following assumptions:

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_i, $i = 1, 2, \cdots, N$;

(V2) $0 \in \sigma(A)$ and there exists $\beta > 0$ such that $(0, \beta] \cap \sigma(A) = \emptyset$;

$(S1)$ $f \in C(\mathbb{R}^{N+1}, \mathbb{R})$ is 1-periodic in x_i, $i = 1, 2, \cdots, N$ and there exist constants $c > 0, \quad 2 < \mu \leq p < 2^*$ such that

$$|f(x,u)| \leq c(1 + |u|^{p-1}), \quad \forall\ (x,u) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.2}$$

where $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := +\infty$ if $N = 1$ or 2;

$(S2)$ $f(x,u) = o(u)$ as $|u| \to 0$ uniformly in x;

$(S3)$ There exists constant $c_0 > 0$ such that

$$F(x,u) \geq c_0 |u|^\mu, \quad \forall\ (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where $F(x,u)$ is the primitive function of f;

$(S4)$ $u \to \frac{f(x,u)}{|u|}$ is strictly increasing on $\mathbb{R} \setminus \{0\}$.

To the best of our knowledge, there are no results concerning the existence of infinitely many large energy solutions for (1.1) with spectrum point zero, which is exactly what we will do in this paper with assumptions (V1), (V2) and (S2). Instead of (S1), (S3) and (S4), we give the following assumptions:

(S1$'$) $f \in C(\mathbb{R}^{N+1}, \mathbb{R})$ is 1-periodic in x_i, $i = 1, 2, \cdots, N$ and there exist constants $c > 0$,

$2 < \mu \leq p < 2^*$ and $a(x) \in L^{\frac{2^*}{2^*-p}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$|f(x,u)| \leq c(1 + a(x)|u|^{p-1}), \quad \forall\ (x,u) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.3}$$

where $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := +\infty$ if $N = 1$ or 2;

(S3′) $f(x,u)u \geq 0,\ \forall\ (x,u) \in \mathbb{R}^N \times \mathbb{R}$, and there exists constant $c_0 > 0$ such that

$$F(x,u) \geq c_0 a(x)|u|^\mu, \quad \forall\ (x,u) \in \mathbb{R}^N \times \mathbb{R};$$

(S4′) There exists a function $W(x) \in L^1(\mathbb{R}^N)$ such that

$$F(x,u+v) - F(x,u) + \left[\frac{(t-1)^2 u}{2} - tv\right] f(x,u) \geq W(x),\ \ \forall\ x \in \mathbb{R}^N,\ u, v \in \mathbb{R},\ t \in [0,1];$$

(S5) $f(x,-u) = -f(x,u), \quad \forall\ (x,u) \in \mathbb{R}^N \times \mathbb{R}$.

Now, we are ready to state the main results of this paper.

Theorem 1.1 *Suppose that* (V1), (V2), (S1′), (S2), (S3′), (S4′) and (S5) *are satisfied, then problem* (1.1) *possesses infinitely many large energy solutions in* $H^2_{loc}(\mathbb{R}^N) \cap L^t(\mathbb{R}^N)$ *for* $\mu \leq t \leq 2^*$.

Obviously, condition (V2) implies that V can not be constant. Condition (S4′) first introduced in [15] is weaker than the monotonicity condition (S4), which together with (S2) and (S3), by an standard argument (see [18]), implies that

$$F(x,u+v) - F(x,u) + \left[\frac{(t-1)^2 u}{2} - tv\right] f(x,u) > 0,$$
$$\forall\ x \in \mathbb{R}^N,\ u \in \mathbb{R},\ v \in \mathbb{R} \setminus \{0\},\ t \in [0, +\infty).$$

One point we need to mention is that on multiple solutions for (1.1) with spectrum point zero, Bartsch and Ding obtained the existence of infinitely many geometrically distinct solutions with condition (AR) and the following assumption, see (g_4) in [3], i.e.,

(g_4) There are constant a_3, $\varepsilon > 0$ such that for all x, u, v

$$|f(x,u+v) - f(x,u)| \leq a_3(|u|^{p-2} + |v|^{p-2} + |u|^{p-1})|v|, \qquad if\ |v| \leq \varepsilon,$$

which implies that f is locally Lipschitzian with respect to u and can also be found in a similar fashion in [8] as condition (A8). Consequently, $f(x,u) = f(x,0) + \int_0^u f'_u(x,\xi)d\xi$ for each $x \in \mathbb{R}^N$. It is therefore easy to see that (g_4) is equivalent to f being locally Lipschitzian in u and satisfying $|f'_u(x,u)| \leq a_3(|u|^{p-2} + |u|^{p-1})$ for some a_3 and all $x \in \mathbb{R}^N,\ u \in \mathbb{R}$ for which the derivative $f'_u(x,u)$ exists.

At the end of this introduction, we state another result dealing with the case that 0 is a left end point of $\sigma(A)$, i.e. we replace (V2) by

(V2′) $0 \in \sigma(A)$ and there exists $\beta > 0$ such that $[-\beta, 0) \cap \sigma(A) = \emptyset$.

Theorem 1.2 *Suppose that* (V1), (V2′) *hold and* $-f$ satisfies (S1′), (S2), (S3′), (S4′) and (S5), *then problem* (1.1) *possesses infinitely many large energy solutions in* $H^2_{loc}(\mathbb{R}^N) \cap L^t(\mathbb{R}^N)$ *for* $\mu \leq t \leq 2^*$.

The proof of theorem 1.2 is analogous to theorem 1.1 working with $-\Phi$ instead of Φ, where functional Φ is defined in Section 2.

Inspired by the above recent works and using an argument of concentration compactness type and an approximation technique (see e.g., [2], [3], [24]), we are able to obtain the existence of infinitely many large energy solutions for problem (1.1). This paper is organized as follows. In Section 2, we introduce the variational framework and main variational tool. In Section 3, the existence of critical points for functional Φ restricted on suitable subspace of $H^1(\mathbb{R}^N)$ is proved. In the last Section, The proof of theorem 1.1 is given.

2. Variational Setting

Throughout this paper, we denote by $|\cdot|_s$ with the usual $L^s(\mathbb{R}^N)$ norm for $s \in [1, \infty) \cup \{\infty\}$. For any $s \in [2, 2^*]$, by Soblev embedding theorem, there exists an embedding constant $\gamma_s \in (0, \infty)$ such that

$$|u|_s \leq \gamma_s \|u\|_{H^1}, \quad \forall\ u \in H^1(\mathbb{R}^N). \tag{2.1}$$

Proof of the main result are based on variational methods applied to the following functional:

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx - \int_{\mathbb{R}^N} F(x, u)dx = \frac{1}{2}(Au, u)_{L^2} - \Psi(u) \tag{2.2}$$

where and in the sequel $\Psi(u) := \int_{\mathbb{R}^N} F(x, u)dx$, $(\cdot, \cdot)_{L^2}$ denote the inner product of $L^2(\mathbb{R}^N)$. The hypotheses on $f(x, u)$ imply that $\Phi \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx - \int_{\mathbb{R}^N} f(x, u)v dx, \quad \forall\ u, v \in H^1(\mathbb{R}^N), \tag{2.3}$$

and a standard argument shows that the critical points of Φ are weak solutions of (1.1).

Under assumption (V1), $A = -\triangle + V$ is a self-adjoint operator, acting on $H := L^2(\mathbb{R}^N)$ with domain $D(A) = H^2(\mathbb{R}^N, \mathbb{R})$. Setting $H^- := P_0 H$ and $H^+ := (id - P_0)H$, where $(P_\lambda)_{\lambda\in\mathbb{R}} : H \to H$ denote the spectral family of A, then we have the orthogonal decomposition $H = H^- + H^+$. Let $E = D(|A|^{\frac{1}{2}})$ be equipped with the inner product

$$(u, v) = (|A|^{\frac{1}{2}}u, |A|^{\frac{1}{2}}v)_{L^2} \tag{2.4}$$

and norm $\|u\| = ||A|^{\frac{1}{2}}u|_2$. Then $H^1(\mathbb{R}^N) \subset E$ and we have the decomposition

$$E = E^- \oplus E^+, \quad u = Pu + Qu$$

with $Pu \in E^-$ and $Qu \in E^+$, where $E^\pm = E \cap H^\pm$ orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$, and the orthogonal projections are denoted by

$$P : E \to E^-, \quad Q : E \to E^+. \tag{2.5}$$

By (2.2) and (2.4), we have

$$\Phi(u) = \frac{1}{2}\|Qu\| - \frac{1}{2}\|Pu\| - \Psi(u), \quad \forall\ u \in H^1(\mathbb{R}^N). \tag{2.6}$$

Since the spectrum of A restricted on H^+ is contained in $(\beta, +\infty)$, the norm $\|\cdot\|$ is equivalent to the $H^1(\mathbb{R}^N)$ norm on E^+. But it is not true on $H^1(\mathbb{R}^N) \cap H^- = H^-$ because of $0 \in \sigma(A)$ as a right end point of $\sigma(A)$, thus the norm $\|\cdot\|$ is weaker than $H^1(\mathbb{R}^N)$ norm and H^- is not complete with respect to $\|\cdot\|$. Moreover, we can not look for solutions of (1.1) in the completion E of $H^1(\mathbb{R}^N)$ under norm $\|\cdot\|$, because $\Psi(u)$ is not well defined due to our assumption on $f(x,u)$.

To solve this problem, we set

$$E_n{}^- := E^- \cap P_{-\frac{1}{n}}H \subset E^-, \quad E_n := E_n{}^- \oplus E^+ \subset E, \quad \forall\ n \in \mathbb{N}^*. \tag{2.7}$$

Since the spectrum of A restricted on E_n is bounded away from 0, the norm $\|\cdot\|$ is equivalent to the $H^1(\mathbb{R}^N)$ norm on E_n, i.e., there exist positive constants c_1, c_2 such that

$$c_1\|u\|_E \le \|u\|_{H^1} \le c_2\|u\|, \quad \forall\ u \in E_n. \tag{2.8}$$

Denote orthogonal projection Q_n as follows

$$Q_n = P_{-\frac{1}{n}} + (id - P_0) : E \to E_n. \tag{2.9}$$

Then for any $u\ \in H^1(\mathbb{R}^N)$,

$$Q_n u \to u \text{ as } n \to \infty, \text{ with respect to } \|\cdot\| \text{ and } |\cdot|_s, \quad 2 \le s < 2^*. \tag{2.10}$$

Define another norm on E

$$\|u\|_\mu := (\|u\|^2 + |u|_\mu^2)^{\frac{1}{2}}, \quad \forall\ u \in\ E. \tag{2.11}$$

Let E_μ^- be the completion of H^- with respect to $\|\cdot\|_\mu$, then $E_\mu := E_\mu^- \oplus E^+$ is the completion of $H^1(\mathbb{R}^N)$ with respect to $\|\cdot\|_\mu$, moreover, E_μ is a reflective Banach space such that $H^1(\mathbb{R}^N) \subset E_\mu \subset E$ and all norms $\|\cdot\|$, $\|\cdot\|_{H^1}$, $\|\cdot\|_\mu$ are equivalent on E^+ (see e.g., [3]), i.e.,

$$\|u\|_\mu \sim \|u\| \sim \|u\|_{H^1}, \quad \forall\ u \in E^+. \tag{2.12}$$

The following abstract critical point theorem plays an important role in proving our main results.

Let X be a Hilbert space with norm $\|\cdot\|$ and has an orthogonal decomposition

$$X = Y \oplus Z, \quad Z = Y^\perp = \overline{\oplus_{j=0}^\infty \mathbb{R}e_j},$$

where $\|e_j\| = 1,\ Y \subset X$ is a closed and separable subspace. Let $P : X \to Y$ and $Q : X \to Z$ be the orthogonal projections and $\{b_n\}_n \subset Y$ be an orthogonal base of Y. Define another norm on X by setting

$$\|u\|_\tau = \max\left\{\sum_{i=0}^\infty \frac{|(Pu, b_i)|}{2^{i+1}}, \|Qu\|\right\}, \quad \forall\ u = Pu + Qu \in X. \tag{2.13}$$

The topology generated by $\|u\|_\tau$ is called τ-topology (see [8], [21]). Observe that $\|Qu\| \le \|u\|_\tau \le \|u\|$ for all $u \in X$, moreover, if u_n is a bounded sequence in X, then

$$\|u_n - u\|_\tau \to 0 \iff Pu_n \rightharpoonup u,\ Qu_n \to u.$$

For $0 < r_k < \rho_k$, $k \in \mathbb{N}^*$ and $\lambda \in [1, 2]$, define the following notations:

$$Y_k := Y \oplus (\oplus_{j=0}^k \mathbb{R}e_j), \quad Z_k := \overline{\oplus_{j=k}^\infty \mathbb{R}e_j}, \quad B_k := \{Y_k : \|u\| \le \rho_k\}$$

$\Gamma_k(\lambda)$:=\{ $\gamma|\gamma\ :\ B_k \to X$ is odd, τ-continuous and $\gamma|_{tialB_k} = id$; $\Phi_\lambda(\gamma(u)) \le \Phi_\lambda(u),\ \forall\ u \in B_k$; For every $u \in int(B_k)$, there is a τ-neighborhood N_u in Y_k such that $(id-\gamma)(N_u \cap int(B_k))$ is contained in a finite-dimensional subspace of X\}.

For any $\lambda\ \in [1, 2]$, functional $\Phi_\lambda(\gamma(u)) : X \to \mathbb{R}$ is defined as follows

$$\Phi_\lambda(\gamma(u)) := L(u) - \lambda J(u), \quad \forall\ u \in\ E, \tag{2.14}$$

with the following assumptions:

(A_1) $J(u) \ge 0$ for every $u \in X$, $L(u) \to \infty$ or $J(u) \to \infty$ as $\|u\| \to \infty$;

(A_2) Φ_λ is a τ-upper semicontinuous and $\nabla\Phi_\lambda$ is weakly sequentially continuous, $\forall\ \lambda \in [1, 2]$.

(A_3) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$ and $\Phi_\lambda(-u) = \Phi_\lambda(u),\ \ \forall\ (\lambda, u) \in [1, 2] \times X$;

The generalized variant fountain theorem is:

Lemma 2.1. [2]. *Under assumptions* $(A_1)-(A_3)$, *if there are* $0<r_k<\rho_k$ *such that* $b_k(\lambda)>a_k(\lambda)$ *for all* $\lambda\in[1,2]$, *then* $c_k(\lambda)>b_k(\lambda)$ *for all* $\lambda\in[1,2]$. *Moreover, for a.e.* $\lambda\in[1,2]$, *there exists a sequence* $\{u_k^n(\lambda)\}_n\subset X$ *such that*

$$\sup_n\|u_k^n(\lambda)\|<\infty,\quad \Phi_\lambda(u_k^n(\lambda))\to c_k(\lambda),\quad \Phi'_\lambda(u_k^n(\lambda))\to 0,$$

as $n\to\infty$, *where*

$$a_k(\lambda):=\sup_{u\in Y_k,\|u\|=\rho_k}\Phi_\lambda(u),\ b_k(\lambda)=\inf_{u\in Z_k,\|u\|=r_k}\Phi_\lambda(u),\ c_k(\lambda)=\inf_{\gamma\in\Gamma_k(\lambda)}\sup_{u\in B_k}\Phi_\lambda(\gamma(u)).$$

3. Critical Points for Φ_n

In this section, we assume that (V1), (V2), (S1′), (S2), (S3′), (S4′) and (S5) are all satisfied. In order to apply Lemma 2.1, we consider the family of modified functionals $\Phi_\lambda:E\to\mathbb{R}$

$$\begin{aligned}\Phi_\lambda(u)&=\frac{1}{2}\|Qu\|^2-\frac{\lambda}{2}\|Pu\|^2-\lambda\int_{\mathbb{R}^N}F(x,u)dx\\&=\frac{1}{2}\|Qu\|^2-\frac{\lambda}{2}\|Pu\|^2-\lambda\Psi(u),\quad\forall\ \lambda\in[1,2].\end{aligned}\tag{3.1}$$

Set

$$\Phi_{n,\lambda}=\Phi_\lambda|_{E_n},\ \Psi_n=\Psi|_{E_n},\ \forall\ n\in\mathbb{N}^*.$$

Then Ψ_n is well defined in E, moreover $\Phi_{n,\lambda},\ \Psi_n\in C^1(E,\mathbb{R})$ with

$$\langle\Psi'_n(u),v\rangle=\int_{\mathbb{R}^N}f(x,u)vdx,\ \ \forall\ u,\ v\in E,\tag{3.2}$$

and

$$\langle\Phi'_{n,\lambda}(u),v\rangle=(Qu,Qv)-\lambda(Pu,Pv)-\lambda\int_{\mathbb{R}^N}f(x,u)vdx.\tag{3.3}$$

Obviously, $\Phi_{n,1}(u)=\Phi_n(u),\ \forall\ u\in E$.

For $0<r_k<\rho_k$, $k\in\mathbb{N}^*$ and $\lambda\in[1,2]$, we define

$$Y_k:=E^-\oplus(\oplus_{j=0}^k\mathbb{R}e_j),\ \ Z_k:=\overline{\oplus_{j=k}^\infty\mathbb{R}e_j},\ \ B_k:=\{Y_k:\|u\|\le\rho_k\},\ \ E=Y_k\oplus Z_{k+1},$$

$$\begin{aligned}a_{n,k}(\lambda)&:=\sup_{u\in Y_k,\|u\|=\rho_k}\Phi_{n,\lambda}(u),\quad b_{n,k}(\lambda)=\inf_{u\in Z_k,\|u\|=r_k}\Phi_{n,\lambda}(u),\\c_{n,k}(\lambda)&=\inf_{\gamma\in\Gamma_k(\lambda)}\sup_{u\in B_k}\Phi_{n,\lambda}(\gamma(u)).\end{aligned}$$

Lemma 3.1. *$\Phi_{n,\lambda}$ is of τ-upper semicontinuous and $\nabla\Phi_{n,\lambda}$ is weakly sequentially continuous. Moreover, $\Phi_{n,\lambda}$ maps bounded sets to bounded sets.*

The proof of the preceding Lemma is standard (see for example [8, 21]).

Lemma 3.2. *For any fixed $k, n \in \mathbb{N}^*$ and almost every $\lambda \in [1,2]$, there exists a sequence $\{u_k^m(\lambda)\}_m \subset E$ such that*

$$\sup_m \|u_k^m(\lambda)\| < \infty, \quad \Phi_{n,\lambda}(u_k^m(\lambda)) \to c_{n,k}(\lambda), \quad \Phi'_{n,\lambda}(u_k^m(\lambda)) \to 0, \tag{3.4}$$

as $m \to \infty$.

Proof. By (S3′), (S5), (3.1) and Lemma 3.1, condition $(A_1)-(A_3)$ are all satisfied, in order to use Lemma 2.1, we only need to prove $b_{n,k}(\lambda) > a_{n,k}(\lambda)$, $\forall\ \lambda \in [1,2]$.

Step 1. We claim that for every $\lambda \in [1,2]$, there exists $\rho_k > 0$ (independent of n) big enough such that $\Phi_{n,\lambda}(u) \le 0$ for all $u \in Y_k$, with $\|u\| = \rho_k$. If $k = 0$, (S3′) yields that $F(x,t) \ge 0$ for any $(x,t) \in \mathbb{R}^{N+1}$, so we have $\Phi_{n,\lambda}(u) \le 0$ for $u \in E^-$. If $k \in \mathbb{N}^*$, Arguing by contradiction, suppose that there exist sequences $\{\lambda_m\} \subset [1,2]$, $\{u_m\} \subset Y_k$, with $u_m = Qu_m + Pu_m$, $Pu_m \in E^-,\ Qu_m \in \oplus_{j=0}^k \mathbb{R}e_j$ such that

$$\Phi_{n,\lambda_m}(u_m) \ge 0, \quad \|u_m\| \to \infty, \qquad m \to \infty.$$

Let $w_m = u_m/\|u_m\| = Pw_m + Qw_m$, then

$$1 = \|w_m\|^2 = \|Pw_m\|^2 + \|Qw_m\|^2 \tag{3.5}$$

and

$$0 \le \frac{\Phi_{n,\lambda_m}(u_m)}{\|u_m\|^2} = \frac{1}{2}\|Qw_m\|^2 - \frac{\lambda_m}{2}\|Pw_m\|^2 - \lambda_m \int_{\mathbb{R}^N} \frac{F(x,u_m)}{|u_m|^2}|w_m|^2 dx. \tag{3.6}$$

Then we deduce from (S3′), (3.5), (3.6) and the fact $\lambda_m \in [1,2]$ that

$$\|Pw_m\|^2 \le \lambda_m \|Pw_m\|^2 \le \|Qw_m\|^2 = 1 - \|Pw_m\|^2, \tag{3.7}$$

thereforce

$$0 \le \|Pw_m\| \le \frac{1}{\sqrt{2}}, \quad \frac{1}{\sqrt{2}} \le \|Qw_m\| \le 1. \tag{3.8}$$

Note that w_m is bounded, $Qw_m \in \oplus_{j=0}^k \mathbb{R}e_j$ and all norms are equivalent in finite-dimentional vector space, then we may assume

$$w_m \rightharpoonup w = Pw + Qw, \quad Pw_m \rightharpoonup Pw, \quad Qw_m \to Qw.$$

By (3.8), then $Qw \neq 0$ and $|u_m| = \|u_m\||w_m| \to \infty$ as $m \to \infty$. By (S3′) and Fatou's Lemma, we have

$$\int_{\mathbb{R}^N} \frac{F(x, u_m)}{|u_m|^2} |w_m|^2 dx \to \infty, \tag{3.9}$$

as $m \to \infty$. By (3.6), this is a contradiction.

Step 2. For any fixed $k, n \in \mathbb{N}^*$, we certify that $b_{n,k}(\lambda) > 0$, $\forall\ \lambda \in [1, 2]$. For any $\varepsilon > 0$, by (S1′) and (S2), there exists $C_\varepsilon > 0$ such that

$$|f(x, u)| \leq \varepsilon|u| + C_\varepsilon a(x)|u|^{p-1}, \quad \forall\ (x, u) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.10}$$

and $R_\varepsilon > 0$ such that

$$|a(x)|_{L^{\frac{2^*}{2^*-p}}(\mathbb{R}^N \setminus \Omega_\varepsilon)} = \left\{ \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} |a(x)|^{\frac{2^*}{2^*-p}} dx \right\}^{\frac{2^*-p}{2^*}} \leq \varepsilon, \tag{3.11}$$

where

$$\Omega_\varepsilon = \{x \in \mathbb{R}^N \mid 0 \leq |x| \leq R_\varepsilon\}. \tag{3.12}$$

By (2.1), (2.12), (3.1), (3.10), (3.11) and the fact $\lambda \in [1, 2]$, for any $u \in E^+$, we get

$$\begin{aligned}
\Phi_{n,\lambda}(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda\varepsilon}{2} \int_{\mathbb{R}^N} |u|^2 dx - \frac{\lambda C_\varepsilon}{p} \int_{\mathbb{R}^N} a(x)|u|^p dx \\
&\geq \frac{1}{2}\|u\|^2 - \varepsilon|u|_2^2 - \frac{2C_\varepsilon}{p} \left(\int_{\Omega_\varepsilon} a(x)|u|^p dx + \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} a(x)|u|^p dx \right) \\
&\geq \frac{1}{2}\|u\|^2 - \varepsilon c_2^2 \gamma_2^2 \|u\|^2 - \frac{2C_\varepsilon |a(x)|_\infty}{p} |u|^p_{L^p(\Omega_\varepsilon)} - \frac{2C_\varepsilon}{p} |u|^p_{2^*} |a(x)|_{L^{\frac{2^*}{2^*-p}}(\mathbb{R}^N \setminus \Omega_\varepsilon)} \\
&\geq (\frac{1}{2} - \varepsilon c_2^2 \gamma_2^2)\|u\|^2 - \frac{2C_\varepsilon |a(x)|_\infty}{p} \beta_k^p \|u\|^p - \frac{2\varepsilon C_\varepsilon}{p} c_2^p \gamma_{2^*}^p \|u\|^p \\
&\geq \frac{1}{2}(\frac{1}{2} - \frac{c}{p}\beta_k^p \|u\|^{p-2})\|u\|^2
\end{aligned}$$

where $\varepsilon := \min\left\{ \frac{1}{4c_2^2\gamma_2^2}, \frac{\beta_k^p |a(x)|_\infty}{c_2^p \gamma_{2^*}^p} \right\}$, $\beta_k := \sup_{v \in Z_k, \|v\|=1} |v|_{L^p_{loc}} > 0,\ k \in \mathbb{N}^*$, and c is a positive constant. Similar to the proof of Lemma 3.8 in [21], by (2.12), we have

$$\lim_{k \to \infty} \beta_k = 0. \tag{3.13}$$

For $u \in Z_k$ with $\|u\| = r_k$, if we choose $r_k := (c\beta_k^p)^{\frac{1}{2-p}}$ (independent of n), we have

$$b_{n,k}(\lambda) \geq \Phi_{n,\lambda}(u) \geq \widetilde{b_k} := \frac{1}{2}(\frac{1}{2} - \frac{1}{p})(c\beta_k^p)^{\frac{2}{2-p}} > 0$$

and by (3.13), $\widetilde{b_k} \to \infty$, as $k \to \infty$. Step 1 and step 2 imply that $b_{n,k}(\lambda) > 0 \geq a_{n,k}(\lambda)$ for all $\lambda \in [1, 2]$. By Lemma 2.1, we get the conclusion. ∎

Observe that, $id \in \Gamma_k(\lambda)$, by (3.1), (S3′) and the definition of $c_{n,k}(\lambda)$, we know

$$c_{n,k}(\lambda) \leq \sup_{u \in Y_k, \|u\| = \rho_k} \Phi_n(u) \leq \widetilde{c_k} := \frac{1}{2}\rho_k^2,$$

thus $c_{n,k}(\lambda) \in [\widetilde{b_k}, \widetilde{c_k}],\ \forall\ k, n \in \mathbb{N}^*$.

The following result can be found in [3] as Lemma 2.1 and corollary 2.3, see also Lemma 3.1 in [22].

Lemma 3.3. ([3]). *E_μ embeds continuously into $H^2_{loc}(\mathbb{R}^N)$, hence compactly into $L^t_{loc}(\mathbb{R}^N)$ for $2 \leq t < 2^*$. Furthermore, E^-_μ embeds continuously into $L^t(\mathbb{R}^N)$ for $\mu \leq t \leq 2^*$. $Au \in L^2(\mathbb{R}^N)$ for $u \in E^-_\mu$. On the other hand, if $u \in E_\mu$ solves* (1.1), *then $u(x) \to 0$ as $|x| \to \infty$.*

Lemma 3.4. *For any fixed $k, n \in \mathbb{N}^*$ and almost every $\lambda \in [1, 2]$, there exists $u_k(\lambda) \in E \backslash \{0\}$ such that*

$$\Phi_{n,\lambda}(u_k(\lambda)) = c_{n,k}(\lambda) \in [\widetilde{b_k}, \widetilde{c_k}], \quad \Phi'_{n,\lambda}(u_k(\lambda)) = 0. \tag{3.14}$$

Proof. Let $\{v^m_k(\lambda)\}_m \subset E_n$ be the sequence obtained in Lemma 3.2. Here for notational simplicity, throughout this paragraph, for any fixed $k \in \mathbb{N}^*$, we set $v_m = v^m_k(\lambda)$. by (3.4), we have

$$\sup_m \|v_m\| < \infty, \quad \Phi_{n,\lambda}(v_m) \to c_{n,k}(\lambda), \quad \Phi'_{n,\lambda}(v_m) \to 0. \tag{3.15}$$

If

$$\delta := \limsup_{m\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |Qv_m|^2 dx = 0$$

then by (2.12) and Lion's concentration compactness principle (see [21], Lemma 1.21), we have that $Qv_m \to 0$ in $L^s(\mathbb{R}^N)$, for $2 < s < 2^*$. By (2.1), (2.8), (3.10), (3.15) and Hölder's inequality, we know

$$\begin{aligned}\int_{R^N} |f(x, v_m)Qv_m|dx &\leq \varepsilon \int_{R^N} |v_m||Qv_m|dx + C_\varepsilon \int_{R^N} a(x)|v_m|^{p-1}|Qv_m|dx \\ &\leq \varepsilon|v_m|_2|Qv_m|_2 + C_\varepsilon|a(x)|_\infty|v_m|_p^{p-1}|Qv_m|_p \\ &\leq \varepsilon c_2^2\gamma_2^2\|v_m\|^2 + C_\varepsilon c_2^{p-1}\gamma_p^{p-1}|a(x)|_\infty\|v_m\|^{p-1}|Qv_m|_p \to 0,\end{aligned}$$

since ε is chosen arbitrarily. By (3.1), (3.3), (S3′) and (3.15), we get

$$\Phi_{n,\lambda}(v_m) \leq \frac{1}{2}\|Qv_m\|^2 = \frac{1}{2}\langle\Phi'_{n,\lambda}(v_m), Qv_m\rangle + \frac{1}{2}\lambda\int_{R^N} f(x, v_m)Qv_m dx \to 0.$$

This contradicts with the fact that $\Phi_{n,\lambda}(v_k^m) \geq \widetilde{b_k} > 0$, thus $\delta > 0$.

Going to a subsequence if necessary, we may assume the existence of $k_m \in \mathbb{Z}^N$, such that

$$\int_{B_{1+\sqrt{N}}(k_m)} |Qv_m|^2 dx > \frac{\delta}{2}, \quad \forall\ m \in \mathbb{N}^*.$$

Define $u_m(x) = v_m(x + k_m)$, then

$$\int_{B_{1+\sqrt{N}}(0)} |Qu_m|^2 dx > \frac{\delta}{2}, \quad \forall\ m \in \mathbb{N}^*. \tag{3.16}$$

(V1) and (S1$'$) imply that $\Phi_{n,\lambda}$ and $\Phi'_{n,\lambda}$ are invariant by above translation, hence by (3.15),

$$\Phi_{n,\lambda}(u_m) \to c_{n,k}(\lambda) \in [\widetilde{b_k}, \widetilde{c_k}], \quad \Phi'_{n,\lambda}(u_m) \to 0, \tag{3.17}$$

and $\|u_m\| = \|v_m\|$ is bounded. Going to a subsequence if necessary, we may assume $u_m \rightharpoonup u$ in E_μ as $m \to \infty$. By Lemma 3.3, we have

$$u_m \to u, \quad in\ L^s_{loc}(\mathbb{R}^N), \quad 2 \leq s < 2^* \tag{3.18}$$

and $u_m \to u$ a.e. on $\mathbb{R}^N$. It then follows from (3.16) and (3.17) that $Qu \neq 0$ in $E_\mu \subset E$ and

$$\Phi'_{n,\lambda}(u) = 0. \tag{3.19}$$

By (2.1), (2.8), (3.10)–(3.12) and (3.18), for u restricted on E_n, we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} F(x, u_m - u) dx \Big| \\
&\leq \int_{\mathbb{R}^N} |F(x, u_m - u)| dx \\
&\leq \varepsilon \int_{\mathbb{R}^N} |u_m - u|^2 dx + C_\varepsilon \int_{\Omega_\varepsilon} a(x)|u_m - u|^p dx + C_\varepsilon \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} a(x)|u_m - u|^p dx \\
&\leq \varepsilon c_2^2 \gamma_2^2 \|u_m - u\|^2 + C_\varepsilon |a(x)|_\infty |u_m - u|^p_{L^p(\Omega_\varepsilon)} + C_\varepsilon |u_m - u|^p_{2^*} |a(x)|_{L^{\frac{2^*}{2^*-p}}(\mathbb{R}^N \setminus \Omega_\varepsilon)} \\
&\leq \varepsilon c_2^2 \gamma_2^2 \|u_m - u\|^2 + o(1) + \varepsilon C_\varepsilon c_2^p \gamma_{2^*}^p \|u_m - u\|^p \to 0,
\end{aligned}$$

since ε is chosen arbitrarily. Similarly, we have

$$\int_{\mathbb{R}^N} f(x, u_m - u)(u_m - u) dx \to 0.$$

Since the function $s \mapsto F(x, s)$ satisfies the conditions of Brézis-Lieb Lemma (see [21], Lemma 1.32), it then follows that

$$\int_{\mathbb{R}^N} F(x, u_m) dx \to \int_{\mathbb{R}^N} F(x, u) dx. \tag{3.20}$$

Similarly we have

$$\int_{\mathbb{R}^N} f(x,u_m)u_m dx \to \int_{\mathbb{R}^N} f(x,u)udx. \tag{3.21}$$

Observe that

$$\Phi_{n,\lambda}(u_m) = \frac{1}{2}\langle \Phi'_{n,\lambda}(u_m), u_m\rangle + \frac{\lambda}{2}\int_{\mathbb{R}^N} f(x,u_m)u_m dx - \lambda\int_{\mathbb{R}^N} F(x,u_m)dx. \tag{3.22}$$

Then, by (3.17), (3.20) and (3.21), taking the limit $m\to\infty$ in (3.22), we obtain

$$c_{n,k}(\lambda) = \frac{\lambda}{2}\int_{\mathbb{R}^N} f(x,u)udx - \lambda\int_{\mathbb{R}^N} F(x,u)dx$$

This implies, by (3.19), that $\Phi_{n,\lambda}(u) = c_{n,k}(\lambda) \in [\widetilde{b_k}, \widetilde{c_k}]$, thus (3.14) holds. ■

By Lemma 3.4, we directly obtain the following lemma:

Lemma 3.5. *For any fixed $k,n\in\mathbb{N}^*$, there are sequences $\{u_k(\lambda_m)\}_m \subset E\setminus\{0\}$ and $\{\lambda_m\}_m \subset [1,2]$ with $\lambda_m \to 1$ such that*

$$\Phi'_{n,\lambda_m}(u_k(\lambda_m)) = 0, \qquad \Phi_{n,\lambda_m}(u_k(\lambda_m)) = c_{n,k}(\lambda_m) \in [\widetilde{b_k}, \widetilde{c_k}]. \tag{3.23}$$

Lemma 3.6. *For any $u\in H^1(\mathbb{R}^N)$, there is a constant C such that*

$$\begin{aligned}\Phi(u) \ge\ & \Phi(tQu) + \frac{t^2\|Pu\|^2}{2} + \frac{1-t^2}{2}\langle \Phi'(u), u\rangle \\ & + t^2\langle \Phi'(u), Pu\rangle + C, \qquad\qquad \forall\ t\in[0,1].\end{aligned} \tag{3.24}$$

Proof. Take $v = (t-1)u - tPu$ in (S5), then we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[F(x,tQu) - F(x,u) + \left(\frac{1-t^2}{2}u + t^2Pu\right)f(x,u)\right]dx \\ \ge & \int_{\mathbb{R}^N} W(x)dx, \quad \forall\ t\in[0,1].\end{aligned} \tag{3.25}$$

Then

$$\begin{aligned}&\Phi(u) - \Phi(tQu) - \langle \Phi'(u), w\rangle \\ = & \frac{1-t^2}{2}\|Qu\|^2 - \frac{1}{2}\|Pu\|^2 - (Qu,Qw) + (Pu,Pw) \\ & + \int_{\mathbb{R}^N}[F(x,tQu) - F(x,u) + f(x,u)w]dx, \quad \forall\ w\in H^1(\mathbb{R}^N).\end{aligned} \tag{3.26}$$

Take $w = \frac{1-t^2}{2}u + t^2 Pu$ in (3.26) and by (3.25), we have

$$\begin{aligned}&\Phi(u) - \Phi(tQu) - \langle \Phi'(u), \frac{1-t^2}{2}u + t^2 Pu\rangle \\ =\ & \frac{t^2}{2}\|Pu\|^2 + \int_{\mathbb{R}^N}\left[F(x,tQu) - F(x,u) + \left(\frac{1-t^2}{2}u + t^2 Pu\right) f(x,u)\right]dx,\end{aligned}$$

which implies (3.24) with $C = \int_{\mathbb{R}^N} W(x)dx$. ■

Lemma 3.7. *The sequence* $\{u_k(\lambda_m)\}_m$ *obtained in Lemma* 3.4 *is bounded.*

Proof. Here for notational simplicity, throughout this paragraph, for any fixed $k, n \in \mathbb{N}^*$, we set $u_m = u_k(\lambda_m)$, then

$$\Phi'_{n,\lambda_m}(u_m) = 0, \qquad \Phi_{n,\lambda_m}(u_m) = c_{n,k}(\lambda_m) \in [\widetilde{b_k}, \widetilde{c_k}]. \tag{3.27}$$

Arguing by contradiction that $\|u_m\| \to \infty$ as $m \to \infty$. Define $w_m = u_m/\|u_m\|$, then $\|w_m\| = 1$. If

$$\delta := \limsup_{m\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |Qw_m|^2 dx = 0,$$

by (2.12) and Lion's concentration compactness principle (see [21], Lemma 1.21), we have $Qw_m \to 0$ in $L^s(\mathbb{R}^N)$, for $2 < s < 2^*$. By (S1′), (2.1), (2.8) and (3.10), for any $r > 0$,

$$\begin{aligned}\int_{\mathbb{R}^N} F(x, rQw_m)dx &\le \varepsilon \int_{R^N} |rQw_m|^2 dx + C_\varepsilon \int_{R^N} a(x)|rQw_m|^p dx \\ &\le \varepsilon r^2 |Qw_m|_2^2 + r^p C_\varepsilon |a(x)|_\infty |Qw_m|_p^p \\ &\le \varepsilon c_2^2 \gamma_2^2 r^2 \|Qw_m\|^2 + o(1) \to 0,\end{aligned} \tag{3.28}$$

since ε is chosen arbitrarily. By (3.1) and (3.27), we get

$$0 \le \frac{\Phi_{n,\lambda_m}(u_m)}{\|u_m\|^2} = \frac{1}{2}\|Qw_m\|^2 - \frac{\lambda_m}{2}\|Pw_m\|^2 - \lambda_m \int_{\mathbb{R}^N} \frac{F(x,u_m)}{|u_m|^2}|w_m|^2 dx.$$

Then, by (S3′) and the fact $\{\lambda_m\} \subset [1,2]$, one has $\|Qw_m\| \ge \|Pw_m\|$. Observe that

$$1 = \|w_m\|^2 = \|Qw_m\|^2 + \|Pw_m\|^2$$

then $\|Qw_m\|^2 \ge \frac{1}{2}$. By (3.27), (3.28) and take $t = r/\|u_m\|$ in Lemma 3.6, then for m big enough, we have $t \in [0,1]$ and

$$\begin{aligned}\widetilde{c_k} - C \ \ge\ \Phi_{n,\lambda_m}(u_m) - C &\ge \Phi_{n,\lambda_m}(rQw_m) + \frac{r^2\|Pw_m\|^2}{2} \\ &= \frac{r^2\|Qw_m\|^2}{2} - \lambda_m \int_{\mathbb{R}^N} F(x, rQw_m)dx \ge \frac{r^2}{4} + o(1).\end{aligned}$$

This leads to a contradiction if we take r big enough, thus $\delta > 0$.

Going to a subsequence if necessary, we may assume the existence of $k_m \in \mathbb{Z}^N$, such that

$$\int_{B_{1+\sqrt{N}}(k_m)} |Qw_m|^2 dx > \frac{\delta}{2}, \quad \forall\ m \in \mathbb{N}^*.$$

Define $v_m(x) = w_m(x + k_m)$, then

$$\int_{B_{1+\sqrt{N}}(0)} |Qv_m|^2 dx > \frac{\delta}{2}, \quad \forall\ m \in \mathbb{N}^*. \tag{3.29}$$

Since $V(x)$ is periodic, $\|v_m\| = \|w_m\| = 1$. Passing to a subsequence if necessary, we may assume that $v_m \rightharpoonup v$ in E. By Lemma 3.3, we have

$$v_m \to v, \quad in\ L^2_{loc}(\mathbb{R}^N) \tag{3.30}$$

and $v_m \to v$ a.e. on $\mathbb{R}^N$. By (3.29), $Qv \neq 0$.

Now we define $\tilde{u}_m(x) = u_m(x + k_m)$, then $\tilde{u}_m/\|u_m\| = v_m \to v$ a.e. on $\mathbb{R}^N$, $v \neq 0$ in E_n.

For $x \in \{y \in \mathbb{R}^N : v(y) \neq 0\}$, we have $\lim_{m\to\infty} \tilde{u}_m(x) = \infty$. Hence, it follows from (3.1), (3.27), (S1′), (S3′), Fatou's lemma and the fact $\{\lambda_m\} \subset [1, 2]$ that

$$\begin{aligned}
0 &= \lim_{m\to\infty} \frac{c_{n,k}(\lambda_m) + o(1)}{\|u_m\|^2} = \lim_{m\to\infty} \frac{\Phi_{n,\lambda_m}(u_m)}{\|u_m\|^2} \\
&= \lim_{m\to\infty} \left[\frac{1}{2}\|Qw_m\|^2 - \frac{\lambda_m}{2}\|Pw_n\|^2 - \lambda_m \int_{\mathbb{R}^N} \frac{F(x, \tilde{u}_m)}{\tilde{u}_m^2} v_m^2 dx\right] \\
&\leq \frac{1}{2} - \liminf_{m\to\infty} \int_{\mathbb{R}^N} \frac{F(x, \tilde{u}_m)}{\tilde{u}_m^2} v_m^2 dx \\
&\leq \frac{1}{2} - \int_{\mathbb{R}^N} \liminf_{m\to\infty} \frac{F(x, \tilde{u}_m)}{\tilde{u}_m^2} v_m^2 dx = -\infty.
\end{aligned}$$

This contradiction shows that $\{u_m\}$ is bounded, i.e. $\{u_k(\lambda_m)\}_m$ is bounded. ■

Corollary 3.8. *If $\{u_k(\lambda_m)\}_m$ is the sequence obtained in Lemma* 3.5*, then for any fixed $k, n \in \mathbb{N}^*$, there exists $c_{n,k} \in [\widetilde{b_k}, \widetilde{c_k}]$ such that $\{u_k(\lambda_m)\}_m$ is a* $(\mathrm{PS})_{c_{n,k}}$ *sequence for Φ_n, i.e.,*

$$\lim_{m\to\infty} \Phi_n'(u_k(\lambda_m)) = 0, \ \lim_{m\to\infty} \Phi_n(u_k(\lambda_m)) = c_{n,k} \in \ [\widetilde{b_k}, \widetilde{c_k}]. \tag{3.31}$$

Proof. By Lemma 3.7, $\{u_k(\lambda_m)\}_m$ is bounded, then we obtain the conclusion from (3.23) and the following relations

$$\begin{aligned}
&\lim_{m\to\infty} \Phi_n(u_k(\lambda_m)) \\
= &\lim_{m\to\infty} \left\{\Phi_{n,\lambda_m}(u_k(\lambda_m)) + (\lambda_m - 1)\left[\frac{1}{2}\|Pu_k(\lambda_m)\|^2 + \int_{\mathbb{R}^N} F(x, u_k(\lambda_m)) dx\right]\right\}
\end{aligned}$$

and

$$\lim_{m\to\infty}\langle\Phi_n'(u_k(\lambda_m)),v\rangle$$
$$=\lim_{m\to\infty}\left\{\langle\Phi_{n,\lambda_m}'(u_k(\lambda_m)),v\rangle+(\lambda_m-1)\left[-b(Pu_k(\lambda_m),v)+\int_{\mathbb{R}^N}f(x,u_k(\lambda_m))vdx\right]\right\}$$

uniformly in $\|v\|\leq 1$. ∎

Lemma 3.9. *For any fixed $k\in\mathbb{N}^*$, there is $v_k^n\in E_n\setminus\{0\}$ such that*

$$\Phi_n'(v_k^n)=0,\quad \Phi_n(v_k^n)=c_{n,k}\in\ [\widetilde{b_k},\widetilde{c_k}]. \tag{3.32}$$

Proof. By Lemma 3.7 and Corollary 3.8, we know $\{u_k(\lambda_m)\}_m$ is a bounded (PS)$_{c_{n,k}}$ sequence for Φ_n. By a similar argument as in Lemma 3.4, we can prove that there exists $v_k^n\in E_n\subset E\setminus\{0\}$, such that $\Phi_n'(v_k^n)=0$ and $\Phi_n(v_k^n)=c_{n,k}\in\ [\widetilde{b_k},\widetilde{c_k}]$. ∎

4. Proof of Main Results

Lemma 4.1. *For any fixed $k\in\mathbb{N}^*$, if v_k^n is the critical point obtained in Lemma* 3.9 for Φ_n, then $\{v_k^n\}_n$ is bounded in E_μ.

Proof. By (3.32), there exists a sequence still denote by $\{v_k^n\}_n$ such that

$$\Phi_n'(v_k^n)=0,\quad \lim_{n\to\infty}\Phi_n(v_k^n)=c_k \tag{4.1}$$

where $c_k\in\ [\widetilde{b_k},\widetilde{c_k}]$. Hence $\{v_k^n\}_n$ is a (PS)$_{c_k}^*$ sequence for Φ. By a similar fashion as in the the proof of Lemma 3.7, we can prove that $\{v_k^n\}_n$ is bounded in E. Thus by (2.11), we only need to prove $\{|v_k^n|_\mu\}_n$ is bounded. Arguing by contradiction, suppose that $|v_k^n|_\mu\to\infty$ as $n\to\infty$. Let $w_k^n=v_k^n/|v_k^n|_\mu$, by the boundedness of $\{\|v_k^n\|\}_n$, then we have $|w_k^n|_\mu=1$, $\|w_k^n\|=\|v_k^n\|/|v_k^n|_\mu\to 0$ as $n\to\infty$. Thus $\{\|w_k^n\|_\mu\}_n$ is bounded, passing to a subsequence in necessary, we may assume $w_k^n\rightharpoonup w_k$ in E_μ. By Lemma 3.3, $w_k^n\to w_k$ in $L^t_{loc}(\mathbb{R}^N)$, $2\leq t<2^*$, $w_k^n\to w_k$ a.e. on $\mathbb{R}^N$ as $n\to\infty$. By a similar argument as in the proof of Lemma 3.7, we can prove that $\{Qw_k^n\}_n$ is non-vanishing and passing to a $\mathbb{Z}^N$-translation if necessary, we may assume that $Qw_k^n\neq 0$. Then $|v_k^n|=|w_k^n||v_k^n|_\mu\to\infty$, as $n\to\infty$. By (S3$'$), (4.1) and Fatou's Lemma, we have

$$0\leq\frac{\Phi_n(v_k^n)}{|v_k^n|_\mu^2}=\frac{1}{2}\|Qw_k^n\|^2-\frac{1}{2}\|Pw_k^n\|^2-\int_{\mathbb{R}^N}\frac{F(x,v_k^n)}{|v_k^n|^2}|w_k^n|^2dx\to-\infty,$$

as $n\to\infty$. This contradiction implies that $\{v_k^n\}_n$ is bounded in E_μ. ∎

Proof of Theorem 1.1. By Lemma 4.1, $\{v_k^n\}_n \subset E_n$ is bounded in E_μ. Going to a subsequence if necessary, we may assume $v_k^n \rightharpoonup v_k$ in E_μ, as $n \to \infty$. By Lemma 3.3, we have $v_k^n \to v_k$ in $L^s_{loc}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $v_k^n \to v_k$ a.e. on $\mathbb{R}^N$, as $n \to \infty$. By a similar fashion as in the the proof of Lemma 3.7, we can prove that $\{Qv_k^n\}_n$ is non-vanishing and passing to a $\mathbb{Z}^N$-translation if necessary, we may assume that $Qv_k \neq 0$. For each $\phi \in C_0^\infty(\mathbb{R}^N)$, by (3.10), Lemma 3.3, and Hölder's inequality, we have

$$\begin{aligned}&|\int_{\mathbb{R}^N} f(x, v_k^n)(id - Q_n)\phi dx| \\ &\le \varepsilon \int_{\mathbb{R}^N} |v_k^n||(id-Q_n)\phi|dx + C_\varepsilon |a(x)|_\infty \int_{\mathbb{R}^N} |v_k^n|^{p-1}|(id-Q_n)\phi|dx \to 0,\end{aligned} \tag{4.2}$$

as $n \to \infty$. Since

$$\begin{aligned}&(Av_k^n, \phi)_{L^2} = (Av_k^n, Q_n\phi)_{L^2} \\ &= (\Phi_n'(v_k^n), Q_n\phi)_{L^2} + \int_{\mathbb{R}^N} f(x, v_k^n)\phi dx - \int_{\mathbb{R}^N} f(x, v_k^n)(id - Q_n)\phi dx.\end{aligned} \tag{4.3}$$

By (4.1) and (4.2), taking limit $n \to \infty$ in (4.3), we have

$$(Av_k, \phi)_{L^2} = \int_{\mathbb{R}^N} f(x, v_k)\phi dx,$$

this implies v_k is a weak solution of problem (1.1). By the same fashion as in the the proof of Lemma 3.4, we can prove that $\Phi(v_k) = c_k \in [\widetilde{b_k}, \widetilde{c_k}]$. By Lemma 3.3, since $\widetilde{b_k} \to \infty$ as $k \to \infty$, we know that problem (1.1) possesses infinitely many large energy solutions in $H^2_{loc}(\mathbb{R}^N) \cap L^t(\mathbb{R}^N)$ for $\mu \le t \le 2^*$. ■

REFERENCES

1. S. Alama and Y. Y. Li, On multibump bound states for certain semilinear elliptic equations, *Indiana Univ. Math. J.*, **41** (1992), 983-1026.
2. C. J. Batkam and F. Colin, On multiple solutions of a semilinear Schrӧinger equation with periodic potential, *Nonlinear Analysis*, **84** (2013), 2677-2694.
3. T. Bartsch and Y. H. Ding, On a nonlinear Schrӧinger equation with periodic potential, *Math. Ann.*, **313** (1999), 15-37.
4. T. Bartsch, A. Pankov and Z. Q. Wang, Nonlinear Schrӧinger equations with steep potential well, *Commun. Contemp. Math.*, **3(4)** (2011), 549-569.
5. Y. H. Ding and C. Lee, Multiple solutions of Schrӧdinger equations with indefinite linear part and super or asymptotically linear terms, *J. Differential Equations*, **222** (2006), 137-163.

6. Y. H. Ding, *Variational Methods for Strongly Indefinite Problems*, Interdiscip. Math. Sci., Vol. 7, World Scientific Publishing Co., Hackensack, NJ, 2007.

7. L. Jeanjean, On the existence of bounded Palais-Smale sequence and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$, *Peoc. Roy. Soc. Edinburgh Sect. A*, **129** (1999), 787-809.

8. W. Kryszewski and A. Szulkin, Generalized linking theorem with an application to semilinear Schrӧinger equations, *Adv. Differential Equation*, **3** (1998), 441-472.

9. Y. Q. Li, Z. Q. Wang and J. Zeng, Ground states of nonlinear Schrӧinger equations with potentials, *Ann. Inst. H. Poincare Anal. Non Lineaire*, **23(6)** (2006), 829-837.

10. Z. L. Liu and Z. Q. Wang, On the Ambrosetti-Rabinowitz superlinear condition, *Adv. Nolinear Studies*, **4** (2004), 561-572.

11. A. Pankov, Periodic nonlinear Schrӧinger equation with application to photonic crystals, *Milan J. Math.*, **73** (2005), 259-287.

12. D. D. Qin, X. H. Tang and J. Zhang, Multiple solutions for semilinear elliptic equations with sign-changing potential and nonlinearity, *Electron. J. Diff. Equ.*, **2013** (2013), No. 207, pp. 1-9.

13. M. Reed and B. Simon, *Methods of Mordern Mathematical Physics*, Vol. IV, Analysis of Operators, Academic Press, New York, 1978.

14. P. H. Rabinowitz, On a class of nonlinear Schrӧinger equations, *Z. Angew. Math. Phys.*, **43** (1992), 270-291.

15. M. Schechter, Superlinear Schrӧinger operators, *J. Func. Anal.*, **262** (2012), 2677-2694.

16. M. Schechter and W. Zou, Weak linking theorems and Schrӧinger equations with critical Soblev exponent, *ESAIM Contral Optim. Calc. Var.*, **9** (2003), 601-619 (electronic).

17. M. Struwe, *Variational Methods*, Applications to Nonlinear Partial Differential Equations and Hamiltonion Systems, Springer-Verlag, Berlin, 2000.

18. A. Szulkin and T. Weth, Ground state solutions for some indefinite variational problems, *J. Funct. Anal.*, **257(12)** (2009), 3802-3822.

19. X. H. Tang, Infinitely many solutins for semilinear Schrӧdinger equation with sign-changing potential and nonlinearity, *J. Math. Anal. Appl.*, **401** (2013), 407-415.

20. X. H. Tang, Ground states solutions for superlinear asymptotically periodic Schrӧdinger equations, submitting *Commun. Contemp. Math*.

21. M. Willem, *Minimax Theorems*, Birkhӓuser, Boston, 1996.

22. M. Willem and W. Zou, On a Schrӧdinger equation with periodic potential and spectrum point zero, *Indiana. Univ. Math. J.*, **52** (2003), 109-132.

23. M. Yang, Ground state solutions for a periodic Schrӧinger equation with superlinear nonlinearities, *Noli. Anal.*, **72** (2010), 2620-2627.

24. M. Yang, W. Chen and Y. Ding, Solutions for periodic Schrӧinger equation with spectrum zero and general superlinear nonlinearities, *J. Math. Anal. Appl.*, **364(2)** (2010), 404-413.

25. Coti-Zelati and P. Rabinowitz, Homoclinic type solutions for a smilinear elliptic PDE on $\mathbb{R}^N$, *Comm. Pure Appl. Math.*, **46** (1992), 1217-1269.

Dongdong Qin, Fangfang Liao and Yi Chen
School of Mathematics and Statistics
Central South University
Changsha 410083, Hunan
P. R. China
E-mail: qindd132@163.com
liaofangfang1981@126.com
mathcyt@163.com