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TOPOLOGICAL STRUCTURE OF THE SPACE OF COMPOSITION OPERATORS FORM $F(p, q, s)$ SPACE to $\mathcal{B}_\mu$ SPACE

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Abstract. We study the topological structure of the space of all bounded composition operators from $F(p, q, s)$ to $\mathcal{B}_\mu$ on the unit disk $\mathbb{D}$ in the operator norm topology. At the same time, we characterizes the boundedness and compactness of the differences of two composition operators.

1. Introduction

Let $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk of the complex plane $\mathbb{C}$. The collection of all holomorphic self-maps of $\mathbb{D}$ will be denoted by $S(\mathbb{D})$. Let dv denote the Lebesegue measure on $\mathbb{D}$ normalized so that $v(\mathbb{D}) = 1$ and $d\sigma$ the normalized Lebesgue measure on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$.

A positive continuous function μ on $[0, 1)$ is called normal if there exist three constants a, b ($0 < a < b$), and $\delta \in (0, 1)$, such that

(i) $\frac{\mu(r)}{(1-r)^a}$ is decreasing on $[\delta, 1)$ and $\lim\limits_{r\to 1^-} \frac{\mu(r)}{(1-r)^a} = 0$;

(ii) $\frac{\mu(r)}{(1-r)^b}$ is increasing on $[\delta, 1)$ and $\lim\limits_{r\to 1^-} \frac{\mu(r)}{(1-r)^b} = \infty$.

Let $\mu(z) = \mu(|z|)$ be normal on $\mathbb{D}$, the weighed Bloch space $\mathcal{B}_\mu$ consists of all $f \in H(\mathbb{D})$ satisfying

$$|||f||| = \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

Then $|||\cdot|||$ defines a complete semi-norm on $\mathcal{B}_\mu$. And $\mathcal{B}_\mu$ is a Banach space under the norm

$$\|f\|_\mu = |f(0)| + |||f|||.$$

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For $0 < \alpha < \infty$, $\mu(z) = (1 - |z|^2)^\alpha$, $\mathcal{B}_\mu$ is the α-Bloch type space $\mathcal{B}_\alpha$ with the norm $\|f\|_\alpha$.

For $a \in \mathbb{D}$, let $g(z, a) = \log |\varphi_a(z)|^{-1}$ be the Green's function on $\mathbb{D}$ with logarithmic singularity at a, where φ_a is the Möbius transformation of $\mathbb{D}$.

Let $0 < p, s < \infty, -2 < q < \infty$, a function $f \in H(\mathbb{D})$ is said to belong to $F(p, q, s)$ if

$$\|f\|_F^p = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dv(z) < \infty,$$

$F(p, q, s)$ is called the general function space since we can get many function spaces, such as Hardy space, Bergman space, Bloch space, Q_p space, if we take special parameters of p, q, s, and if $q + s \le -1$, then $F(p, q, s)$ is the space of constant functions.

Let $\varphi \in S(\mathbb{D})$, the composition operator C_φ induced by φ is defined as

$$(C_\varphi f)(z) = f(\varphi(z)), \;\; f \in H(\mathbb{D}), \; z \in \mathbb{D}.$$

This operator is well studied for many years, readers interested in this topic can refer to the books [3, 14, 20], which are excellent sources for the development of the theory of composition operators, and the recent papers [8, 15, 16, 19] and the references therein.

For two Banach spaces X and Y of analytic functions on $\mathbb{D}$, let $\mathcal{C}(X \to Y)$ be the set of all bounded composition operators from X to Y with the operator norm topology. For the purpose of this paper, we limit our analysis to the differences of composition operators and topological structure of $\mathcal{C} = \mathcal{C}(F(p, q, s) \to \mathcal{B}_\mu)$. Boundedness and compactness of differences of composition operators on various spaces of analytic functions have been investigated by several authors, see e.g.[1, 7, 9, 10, 13]. The topological structure of the set of composition operators has been studied in [2, 4, 5, 6, 11]. The remainder is assembled as follows: In section 2, we collect the necessary background material and preliminary results. In Section 3, we characterize the boundedness and compactness of the difference $C_\varphi - C_\psi$ acting from $F(p, q, s)$ spaces to $\mathcal{B}_\mu$ space. In the last two sections, we research the topological structure of $\mathcal{C}$.

Throughout the paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. Notations and Some Lemmas

To begin the discussion, let us introduce some notations.

For $a \in \mathbb{D}$, the Möbius transformation of $\mathbb{D}$, is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z},$$

it is easy to see that $\varphi_a(0) = a, \varphi_a(a) = 0$ and $\varphi_a = \varphi_a^{-1}$.

For z, w in $\mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\overline{z}w}\right|,$$

and the Bergman metric is given by

$$\beta\,(z,w) = \inf_{\gamma} \int\limits_{\gamma} \frac{|d\xi|}{1-|\xi|^2} = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)},$$

where γ is any piecewise smooth curve in $\mathbb{D}$ from z to w.

It is well known that

$$1-\rho^2(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2}.$$

For $\varphi \in S(\mathbb{D})$, the Schwarz-Pick lemma shows that $\rho\,(\varphi(z),\varphi(w)) \le \rho(z,w)$, and if equality holds for some $z \ne w$, then φ is an automorphism of the disk.

For $z, w \in \mathbb{D}$, we defined that

$$d_\alpha(z,w) = \sup\{|f(z)-f(w)| : f \in \mathcal{B}_\alpha, \|f\|_\alpha \le 1\}.$$

According to Proposition 16 in [21], we know that d_α is a distance of $\mathbb{D}$. When $\alpha = 1$, the distance d_1 is precisely the Bergman distance $\beta(z,w)$.

Finally, we collect some lemmas which will be needed during the paper.

The following lemma is the crucial criterion for compactness, whose proof is an easy modification of that of Proposition 3.11 in [3].

Lemma 1. *Suppose that* $0 < p, s < \infty, -2 < q < +\infty, q+s > -1$, *and* μ *is normal on* $\mathbb{D}$, $\varphi, \psi \in S(\mathbb{D})$, *then the operator* $C_\varphi - C_\psi : F(p,q,s) \to \mathcal{B}_\mu$ *is compact if and only if* $C_\varphi - C_\psi : F(p,q,s) \to \mathcal{B}_\mu$ *is bounded and for any bounded sequence* $\{f_k\}_{k\in\mathbb{N}}$ *in* $F(p,q,s)$ *which converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $k \to \infty$, *we have* $|||(C_\varphi - C_\psi)\, f_k||| \to 0$, *as* $k \to \infty$.

The following lemma can be found in [18].

Lemma 2. ([18, Lemma 2.5]). *For* $0 < p, s < \infty, -2 < q < +\infty, q+s > -1$, *there exists a constant* $C > 0$ *such that*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|^2)^p}{|1-\overline{w}z|^{2+q+p}} (1-|z|^2)^q g^s(z,a)\,dv(z) \le C$$

for every $w \in \mathbb{D}$.

Lemma 3. ([17, Lemma 2.3]). *Suppose that* $0 < p, s < \infty, -2 < q < \infty$ *and* $q+s > -1$. *If* $f \in F(p,q,s)$, *then* $f \in \mathcal{B}_{(2+q)/p}$, *and* $\|f\|_{\mathcal{B}_{(2+q)/p}} \le C\|f\|_F$.

Lemma 4. ([12, p. 192]). *If $f \in \mathcal{B}_\alpha$, then there exists a positive constant C satisfing*

$$|f(z)| \le C \begin{cases} \|f\|_\alpha & \alpha \in (0,1), \\ \|f\|_\alpha \log \frac{2}{1-|z|^2} & \alpha = 1, \\ \frac{\|f\|_\alpha}{(1-|z|^2)^{\alpha-1}} & \alpha > 1. \end{cases}$$

By Lemma 3 and Lemma 1 in [9], we can obtain following lemma.

Lemma 5. *Suppose that $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, If $f \in F(p,q,s)$, then there is a positive constant C independent of f such that*

$$|(1-|z|^2)^\beta f'(z) - (1-|w|^2)^\beta f'(w)| \le C\|f\|_F \rho(z,w)$$

for all $z, w \in \mathbb{D}$, where $\beta = \frac{q+2}{p}$.

Lemma 6. ([21, Theorem 18]). *Suppose $\alpha > 0$ and f is analytic on $\mathbb{D}$. Then f is in $\mathcal{B}_\alpha$ if and only if there exists a constant $C > 0$ such that*

$$|f(z) - f(w)| \le C d_\alpha(z,w), \quad z, w \in \mathbb{D}.$$

The next lemma can be found in [6].

Lemma 7. ([6, Lemma 4.1]). *Let $z, w \in \mathbb{D}$, and $\rho(z,w) = \lambda < 1$, for $t \in [0,1]$, put $z_t = (1-t)z + tw$. Then the map $t \mapsto \rho(z_t, w)$ is continuous and decreasing on $[0,1]$.*

Lemma 8. *For each $r \in (0,1)$, there exists a positive constant C_r such that*

$$C_r \le \frac{1-|w|^2}{1-|z|^2} \le C_r$$

for all z and w in $\mathbb{D}$ with $\rho(z,w) \le r$.

Proof. When $\rho(z,w) \le r$, we have $\beta(z,w) \le \frac{1}{2}\log\frac{1+r}{1-r}$, so according to Lemma 2.20 in [22], this lemma is complete. ∎

Lemma 9. *For each $r \in (0,1)$ and $b > 0$, there exists a positive constant C_r such that*

$$\left|1 - \left(\frac{1-|w|^2}{1-|z|^2}\right)^b\right| \le C_r \rho(z,w)$$

for all z and w in $\mathbb{D}$ with $\rho(z,w) \le r$.

Proof. According to Lemma 2.27 in [22] and Lemma 8, we obtain

$$\begin{aligned}
&\left|1-\left(\frac{1-|w|^2}{1-|z|^2}\right)^b\right| \\
&\leq \left|1-\left(\frac{1-|w|^2}{1-w\overline{z}}\right)^b\right| + \frac{(1-|w|^2)^b}{(1-|z|^2)^b}\left|1-\left(\frac{1-|z|^2}{1-w\overline{z}}\right)^b\right| \\
&\leq C_r\rho(z,w).
\end{aligned}$$

■

3. The Boundedness and Compactness of $C_\varphi - C_\psi : F(p,q,s) \to \mathcal{B}_\mu$

In this section, we characterize the boundedness and compactness of $C_\varphi - C_\psi$ from $F(p,q,s)$ to $\mathcal{B}_\mu$, and suppose that $\beta = \frac{q+2}{p}$. According to the main results in [16], we get that:

(i) $C_\varphi : F(p,q,s) \to \mathcal{B}_\mu$ is bounded if and only if $\lim\limits_{z\in\mathbb{D}} \frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta} < \infty$.

(ii) $C_\varphi : F(p,q,s) \to \mathcal{B}_\mu$ is compact if and only if C_φ is bounded and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta} = 0.$$

Theorem 1. *Assume that* $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$. μ *is normal on* $\mathbb{D}$ *and* $\varphi, \psi \in S(\mathbb{D})$, *then* $C_\varphi - C_\psi : F\,(p,q,s) \to \mathcal{B}_\mu$ *is bounded if and only if* (1), (2) *and* (3) *hold, where*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\varphi'(z)|}{\left(1-|\varphi\,(z)|^2\right)^\beta}\rho(\varphi(z),\psi(z)) < \infty, \tag{1}$$

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi'(z)|}{\left(1-|\psi\,(z)|^2\right)^\beta}\rho(\varphi(z),\psi(z)) < \infty, \tag{2}$$

$$\sup_{z\in\mathbb{D}} \left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta} - \frac{\mu(z)\psi'(z)}{(1-|\psi(z)|^2)^\beta}\right| < \infty. \tag{3}$$

Proof. Assume that $C_\varphi - C_\psi$ is bounded, then there is a positive constant M such that $\|(C_\varphi - C_\psi)f\|_\mu < M\|f\|_F$ for every $f \in F(p,q,s)$. Fix $w \in \mathbb{D}$, if $\varphi(w) = \psi(w)$, then $\rho(\varphi(w),\psi(w)) = 0$, it is easy to see that

$$\frac{\mu(w)|\varphi'(w)|}{\left(1-|\varphi\,(w)|^2\right)^\beta}\rho(\varphi(w),\psi(w)) = 0.$$

If $\varphi(w) \neq \psi(w)$, and $\varphi(w) \neq 0$, we define two functions

$$f_w(z) = \frac{1-|\varphi(w)|^2}{\beta\overline{\varphi(w)}(1-\overline{\varphi(w)}z)^\beta},$$

$$h_w(z) = \int_0^z \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}u)^{\beta+1}}\varphi_{\varphi(w)}(u)du.$$

By Lemma 2, it is easy to check that f_w and h_w belong to $F(p,q,s)$, and $\|f_w\|_F \leq C$, $\|h_w\|_F \leq C$ for a positive constant C independent of w.

So we have

$$\begin{aligned} MC &> \|(C_\varphi - C_\psi)f_w\|_\mu \\ &\geq \sup_{z\in\mathbb{D}} \mu(z)|f'_w(\varphi(z))\varphi'(z) - f'_w(\psi(z))\psi'(z)| \\ &\geq \mu(w)|f'_w(\varphi(w))\varphi'(w) - f'_w(\psi(w))\psi'(w)| \\ &\geq \mu(w)\left|\frac{\varphi'(w)}{(1-|\varphi(w)|^2)^\beta} - \frac{(1-|\varphi(w)|^2)\psi'(w)}{(1-\overline{\varphi(w)}\psi(w))^{\beta+1}}\right|, \end{aligned} \tag{4}$$

and

$$\begin{aligned} MC &> \|(C_\varphi - C_\psi)h_w\|_\mu \\ &\geq \sup_{z\in\mathbb{D}} \mu(z)|h'_w(\varphi(z))\varphi'(w) - h'_w(\psi(z))\psi'(w)| \\ &\geq \mu(w)|h'_w(\varphi(w))\varphi'(w) - h'_w(\psi(w))\psi'(w)| \\ &\geq \mu(w)|\psi'(w)|\rho(\varphi(w),\psi(w))\left|\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}\psi(w))^{\beta+1}}\right|. \end{aligned} \tag{5}$$

Due to $0 < \rho(\varphi(w),\psi(w)) < 1$, according to (4) and (5), we can get

$$\frac{\mu(w)|\varphi'(w)|\rho(\varphi(w),\psi(w))}{(1-|\varphi(w)|^2)^\beta} < C.$$

If $\varphi(w) = 0$, $\psi(w) \neq 0$, letting $f(z) = z$, $g(z) = \frac{z^2}{2}$, $\|f\|_F \leq C$ and $\|g\|_F \leq C$ for a positive constant C, we get

$$MC > \|(C_\varphi - C_\psi)f\|_\mu \geq \mu(w)|\varphi'(w) - \psi'(w)| \tag{6}$$

and

$$MC > \|(C_\varphi - C_\psi)g\|_\mu \geq \mu(w)|\varphi(w)\varphi'(w) - \psi(w)\psi'(w)| = \mu(w)|\psi(w)\psi'(w)|. \tag{7}$$

By (6), (7) and $\rho(\varphi(w),\psi(w)) < 1$, we obtain $\mu(w)|\varphi'(w)|\rho(\varphi(w),\psi(w)) < C$.

Since w is an arbitrary element, the inequality (1) holds. Analogously we can obtain (2).

Next we prove that (3) is true. For given $w \in \mathbb{D}$, if $\psi(w) \neq 0$, we consider the test function

$$Q_w(z) = \frac{1-|\psi(w)|^2}{\beta\overline{\psi(w)}(1-\overline{\psi(w)}z)^\beta},$$

by Lemma 2, we can obtain that $Q_w \in F(p,q,s)$ with $\|Q_w\|_F \leq C$ for a constant C independent of w.

$$\begin{aligned} MC &> \left\|\left(C_\varphi - C_\psi\right)Q_w\right\|_\mu \\ &\geq \mu(w)\left|\left(\left(C_\varphi - C_\psi\right)Q_w\right)'(w)\right| \\ &= \mu(w)|Q_w'(\varphi(w))\varphi'(w) - Q_w'(\psi(w))\psi'(w)| \\ &= \left|\frac{\mu(w)\left(1-|\psi(w)|^2\right)\varphi'(w)}{\left(1-\overline{\psi(w)}\varphi(w)\right)^{\beta+1}} - \frac{\mu(w)\psi'(w)}{\left(1-|\psi(w)|^2\right)^\beta}\right| \\ &= |I(w)+J(w)|, \end{aligned} \tag{8}$$

where

$$I(w) = \frac{\mu(w)\varphi'(w)}{\left(1-|\varphi(w)|^2\right)^\beta} - \frac{\mu(w)\psi'(w)}{\left(1-|\psi(w)|^2\right)^\beta},$$

and

$$\begin{aligned} J(w) &= \frac{\mu(w)\varphi'(w)(1-|\psi(w)|^2)}{(1-\overline{\psi(w)}\varphi(w))^{\beta+1}} - \frac{\mu(w)\varphi'(w)}{(1-|\varphi(w)|^2)^\beta} \\ &= \frac{\mu(w)\varphi'(w)}{(1-|\varphi(w)|^2)^\beta}[(1-|\varphi(w)|^2)^\beta Q_w'(\varphi(w)) - (1-|\psi(w)|^2)^\beta Q_w'(\psi(w))]. \end{aligned}$$

By Lemma 5, we conclude that

$$\begin{aligned} |J(w)| &\leq C\frac{\mu(w)|\varphi'(w)|}{(1-|\varphi(w)|^2)^\beta}\rho(\varphi(w),\psi(w))\|Q_w\|_F \\ &\leq C\frac{\mu(w)|\varphi'(w)|}{(1-|\varphi(w)|^2)^\beta}\rho(\varphi(w),\psi(w)). \end{aligned}$$

Thus we obtain that $|J(w)| < C$ for all $w \in \mathbb{D}$ by (1). Combining with (8) we obtain $|I(w)| < C$.

If $\psi(w) = 0$, and $\varphi(w) = 0$, letting $f(z) = z$, we can get

$$|\mu(w)\psi'(w) - \mu(w)\varphi'(w)| < M\|f\|_F.$$

If $\psi(w) = 0$, $\varphi(w) \neq 0$, then $\rho(\varphi(w),\psi(w)) \neq 0$, by (1) and (2), we can get

$$\left|\frac{\mu(w)\varphi'(w)}{(1-|\varphi(w)|^2)^\beta}-\mu(w)\psi'(w)\right|<C.$$

Since the arbitrariness of w, the desired result (3) follows.

Conversely, we assume that (1), (2) and (3) hold. For any $f\in F(p,q,s)$, we have

$$\begin{aligned}&\|(C_\varphi-C_\psi)f\|_\mu\\ =\ &|f(\varphi(0))-f(\psi(0))|+\sup_{z\in\mathbb{D}}\mu(z)|f'(\varphi(z))\varphi'(z)-f'(\psi(z))\psi'(z)|\end{aligned}$$

By Lemma 3 and Lemma 4, it is obvious that $|f(\varphi(0))-f(\psi(0))|\le C\|f\|_F$.

$$\begin{aligned}&\sup_{z\in\mathbb{D}}\mu(z)|f'(\varphi(z))\varphi'(z)-f'(\psi(z))\psi'(z)|\\ \le\ &\sup_{z\in\mathbb{D}}\frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta}|(1-|\varphi(z)|^2)^\beta f'(\varphi(z))-(1-|\psi(z)|^2)^\beta f'(\psi(z))|\\ &\quad+\sup_{z\in\mathbb{D}}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\mu(z)\psi'(z)}{(1-|\psi(z)|^2)^\beta}\right|(1-|\psi(z)|^2)^\beta|f'(\psi(z))|\\ \le\ &C\left[\sup_{z\in\mathbb{D}}\frac{\mu(z)|\varphi'(z)|\rho(\varphi(z),\psi(z))}{(1-|\psi(z)|^2)^\beta}+\sup_{z\in\mathbb{D}}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\mu(z)\psi'(z)}{(1-|\psi(z)|^2)^\beta}\right|\right]\|f\|_F\\ \le\ &C\,\|f\|_F\,.\end{aligned}$$

So $C_\varphi-C_\psi:F(p,q,s)\to\mathcal{B}_\mu$ is bounded. The proof is finished. ■

In the progress of proving, we can see that when (1) and (3), or (2) and (3) hold, $C_\varphi-C_\psi$ is bounded. Next, we characterize the compactness of $C_\varphi-C_\psi$. Since when $\sup_{z\in\mathbb{D}}|\varphi(z)|<1$, it is easy to see that C_φ is compact, so we always suppose $\sup_{z\in\mathbb{D}}|\varphi(z)|=\sup_{z\in\mathbb{D}}|\psi(z)|=1$.

Theorem 2. *Assume that* $0<p,s<\infty$, $-2<q<\infty$, $q+s>-1$, μ *is normal on* $\mathbb{D}$ *and* $\varphi,\psi\in S(\mathbb{D})$, C_φ *and* C_ψ *are bounded, then* $C_\varphi-C_\psi:F(p,q,s)\to\mathcal{B}_\mu$ *is compact if and only if* (9),(10) *and* (11) *hold, where*

$$\lim_{|\varphi(z)|\to1}\frac{\mu(z)|\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^\beta}\rho(\varphi(z),\psi(z))=0,\tag{9}$$

$$\lim_{|\psi(z)|\to1}\frac{\mu(z)|\psi'(z)|}{\left(1-|\psi(z)|^2\right)^\beta}\rho(\varphi(z),\psi(z))=0,\tag{10}$$

$$\lim_{\min\{|\varphi(z)|,|\psi(z)|\}\to1}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\mu(z)\psi'(z)}{(1-|\psi(z)|^2)^\beta}\right|=0.\tag{11}$$

Proof. Assume that $C_\varphi - C_\psi$ is compact. For any sequence $\{w_n\}_{n=1}^\infty$ such that $|\varphi(w_n)| \to 1$ as $n \to \infty$. Here, we can suppose that $\varphi(w_n) \neq 0$ for every n, put

$$f_n(z) = \frac{1 - |\varphi(w_n)|^2}{\beta\overline{\varphi(w_n)}(1 - \overline{\varphi(w_n)}z)^\beta}.$$

By Lemma 2, we can prove that $\|f_n\|_F \leq C$, and the function sequence $\{f_n\}_{n=1}^\infty$ converges to 0 uniformly in every compact subset of $\mathbb{D}$, then by Lemma 1, we know $\||(C_\varphi - C_\psi)f_n\|| \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \sup_{z\in\mathbb{D}} \mu(z)|f_n'(\varphi(z))\varphi'(z) - f_n'(\psi(z))\psi'(z)| \\
&\geq \lim_{n\to\infty} \mu(w_n)|f_n'(\varphi(w_n))\varphi'(w_n) - f_n'(\psi(w_n))\psi'(w_n)| \\
&= \lim_{n\to\infty} \mu(w_n)\left|\frac{\varphi'(w_n)}{(1 - |\varphi(w_n)|^2)^\beta} - \frac{\psi'(w_n)(1 - |\varphi(w_n)|^2)}{(1 - \overline{\varphi(w_n)}\psi(w_n))^{\beta+1}}\right|.
\end{aligned} \tag{12}$$

Since $0 \leq \rho(\varphi(w_n), \psi(w_n)) \leq 1$, we obtain

$$\lim_{n\to\infty} \mu(w_n)\rho(\varphi(w_n), \psi(w_n))\left|\frac{\varphi'(w_n)}{(1-|\varphi(w_n)|^2)^\beta} - \frac{\psi'(w_n)(1 - |\varphi(w_n)|^2)}{(1-\overline{\varphi(w_n)}\psi(w_n))^{\beta+1}}\right| = 0. \tag{13}$$

Similarly, for function sequence $\{g_n\}_{n=1}^\infty$,

$$g_n(z) = \int_0^z \frac{1 - |\varphi(w_n)|^2}{(1 - \overline{\varphi(w_n)}u)^{\beta+1}}\varphi_{\varphi(w_n)}(u)du,$$

we have

$$\lim_{n\to\infty} \frac{\mu(w_n)|\psi'(w_n)|(1 - |\varphi(w_n)|^2)}{|1 - \overline{\varphi(w_n)}\psi(w_n)|^{\beta+1}}\rho(\varphi(w_n), \psi(w_n)) = 0. \tag{14}$$

So according to (13) and (14), we get (9).

Analogously we can obtain (10).

Next we prove (11), let $\{w_n\}_{n=1}^\infty$ be a sequence such that $|\varphi(w_n)| \to 1$ and $|\psi(w_n)| \to 1$ as $n \to \infty$. Put

$$Q_n(z) = \frac{1 - |\psi(w_n)|^2}{\beta\overline{\psi(w_n)}(1 - \overline{\psi(w_n)}z)^\beta}.$$

Then $\{Q_n\}_{n=1}^\infty$ is a bounded sequence in $F(p,q,s)$ and converges to 0 uniformly in compact subset of $\mathbb{D}$ as $n \to \infty$.

$$
\begin{aligned}
0 &= \lim_{n\to\infty} |||(C_\varphi - C_\psi)Q_n||| \\
&\geq \lim_{n\to\infty} \left| \frac{\mu(w_n)\varphi'(w_n)(1-|\psi(w_n)|^2)}{(1-\overline{\psi(w_n)}\varphi(w_n))^{\beta+1}} - \frac{\mu(w_n)\psi'(w_n)}{(1-|\psi(w_n)|^2)^\beta} \right| \\
&\geq \lim_{n\to\infty} \Bigg| \left| \frac{\mu(w_n)\varphi'(w_n)}{(1-|\varphi(w_n)|^2)^\beta} - \frac{\mu(w_n)\psi'(w_n)}{(1-|\psi(w_n)|^2)^\beta} \right| \\
&\qquad - \left| \frac{\mu(w)\varphi'(w_n)}{(1-|\varphi(w)|^2)^\beta}[(1-|\varphi(w)|^2)^\beta Q_w(\varphi(w)) - (1-|\psi(w)|^2)^\beta Q_w(\psi(w))] \right| \Bigg| \\
&\geq \lim_{n\to\infty} \left| \left| \frac{\mu(w_n)\varphi'(w_n)}{(1-|\varphi(w_n)|^2)^\beta} - \frac{\mu(w_n)\psi'(w_n)}{(1-|\psi(w_n)|^2)^\beta} \right| - C\frac{\mu(w_n)\varphi'(w_n)\rho(\varphi(w_n),\psi(w_n))}{(1-|\varphi(w_n)|^2)^\beta} \right|,
\end{aligned}
$$

so we get (11) by (9).

Conversely, assume that (9), (10) and (11) hold. Let $\{f_n\}_{n=1}^\infty$ be a sequence in $F(p,q,s)$ such that $\|f\|_F \leq 1$ and f_n convergence to 0 uniformly on every compact subset of $\mathbb{D}$. We need to prove $|||(C_\varphi - C_\psi)f_n||| \to 0$, suppose that it does not hold. Thus for some $\varepsilon > 0$, $|||(C_\varphi - C_\psi)f_n||| > \varepsilon$, then for each n, we can find a sequence $\{z_n\}_{n=1}^\infty \subset \mathbb{D}$ such that

$$
\left|\mu(z_n)\varphi'(z_n)f'(\varphi(z_n)) - \mu(z_n)\psi'(z_n)f'(\psi(z_n))\right| > \varepsilon. \tag{15}
$$

Since C_φ and C_ψ are bounded, and f_n' convergence to 0 uniformly on compact subset of $\mathbb{D}$, we obtain that either $|\varphi(z_n)|$ or $|\psi(z_n)|$ tends to 1. Suppose that $|\varphi(z_n)| \to 1$ and $|\psi(z_n)| \not\to 1$, then there exists a subsequence $\{z_{n_k}\}$ such that $|\psi(z_{n_k})| < 1$, hence $\liminf_{k\to\infty} \rho(\varphi(z_{n_k}), \psi(z_{n_k})) > 0$. So by (9), we have

$$
\frac{\mu(z_{n_k})|\varphi'(z_{n_k})|}{(1-|\varphi(z_{n_k})|^2)^\beta} \to 0.
$$

On the other hand, since $|\psi(z_{n_k})| < 1$, $|f'(\psi(z_{n_k}))| \to 0$,

$$
\begin{aligned}
&\left|\mu(z_{n_k})\varphi'(z_{n_k})f'(\varphi(z_{n_k})) - \mu(z_{n_k})\psi'(z_{n_k})f'(\psi(z_{n_k}))\right| \\
\leq &\left| \frac{\mu(z_{n_k})|\varphi'(z_{n_k})|}{(1-|\varphi(z_{n_k})|^2)^\beta}(1-|\varphi(z_{n_k})|^2)^\beta f'(\varphi(z_{n_k})) \right| + |\mu(z_{n_k})\psi'(z_{n_k})f'(\psi(z_{n_k}))| \\
\to &\ 0,
\end{aligned}
$$

this contradicts (15). So we get $|\psi(z_n)| \to 1$. This implies that $|\varphi(z_n)| \to 1$ and $|\psi(z_n)| \to 1$. Then we have

$$
\begin{aligned}
&\left|\mu(z_n)\varphi'(z_n)f'(\varphi(z_n)) - \mu(z_n)\psi'(z_n)f'(\psi(z_n))\right| \\
\leq &\lim_{n\to\infty} \frac{\mu(z_n)|\varphi'(z_n)|\rho(\varphi(z_n),\psi(z_n))}{(1-|\varphi(z_n)|^2)^\beta} + \lim_{n\to\infty} \left| \frac{\mu(z_n)\varphi'(z_n)}{(1-|\varphi(z_n)|^2)^\beta} - \frac{\mu(z_n)\psi'(z_n)}{(1-|\psi(z_n)|^2)^\beta} \right| \\
= &\ 0.
\end{aligned}
$$

This contradicts (15). Thus we complete the proof. ∎

4. Compact Differences and Path Connected I

In this section we consider the relationship between the compact difference and the path connection when $\beta = 1$, to discuss the behavior, we need some notions.

For $\varphi \in S(\mathbb{D})$, let $\Gamma_r(\varphi) = \{z \in \mathbb{D}, |\varphi(z)| > r\}$ for $r \in (0,1)$. Let $\Gamma(\varphi)$ be the set of sequences $\{z_k\}$ in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$, and $\Gamma^{\#}(\varphi)$ the set of sequences $\{z_k\}$ in $\Gamma(\varphi)$ such that $\frac{\mu(z_k)|\varphi'(z_k)|}{1-|\varphi(z_k)|^2} \not\to 0$.

Using these notions, we can obtain that C_φ is compact if and only if $\Gamma^{\#}(\varphi) = \emptyset$. And under the conditions of Theorem 2, we can get the following theorem.

Theorem 3. *$C_\varphi - C_\psi$ is compact if and only if (a) and (b) hold:*

(a) $\Gamma^{\#}(\varphi) = \Gamma^{\#}(\psi)$, *and they are included in* $\Gamma(\varphi) \cap \Gamma(\psi)$*;*

(b)

$$\lim_{n\to\infty} \frac{\mu(z_n)|\varphi'(z_n)|\rho(\varphi(z_n),\psi(z_n))}{1-|\varphi(z_n)|^2} = \lim_{n\to\infty} \frac{\mu(z_n)|\psi'(z_n)|\rho(\varphi(z_n),\psi(z_n))}{1-|\psi(z_n)|^2}$$
$$= \lim_{n\to\infty} \left| \frac{\mu(z_n)\varphi'(z_n)}{1-|\varphi(z_n)|^2} - \frac{\mu(z_n)\psi'(z_n)}{1-|\psi(z_n)|^2} \right| = 0,$$

when $\{z_n\} \in \Gamma(\varphi) \cap \Gamma(\psi)$.

Proof. Assume $C_\varphi - C_\psi$ is compact, then we can get (b) by Theorem 2. When $\Gamma^{\#}(\varphi) = \emptyset$, we know C_φ is compact, then C_ψ is also compact, we have $\Gamma^{\#}(\psi) = \emptyset$. When $\Gamma^{\#}(\varphi) \neq \emptyset$, for any sequence $\{z_n\} \in \Gamma^{\#}(\varphi)$, we know $\frac{\mu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2} \not\to 0$. Thus according to (12) and (14), we have $\lim\limits_{n\to\infty} \rho(\varphi(z_n),\psi(z_n)) = 0$, so $|\psi(z_n)| \to 1$. Then we can get $\frac{\mu(z_n)|\psi'(z_n)|}{1-|\psi(z_n)|^2} \not\to 0$ by (11), so $\{z_n\} \in \Gamma^{\#}(\psi)$, $\Gamma^{\#}(\varphi) \subset \Gamma^{\#}(\psi)$. Similarly we can get $\Gamma^{\#}(\psi) \subset \Gamma^{\#}(\psi)$, thus we obtain (a).

If (a) and (b) hold, we only need to prove that (9),(10) and (11) are true when $\beta = 1$. It is obvious that (11) holds by (b). For every sequence $\{z_n\} \in \Gamma(\varphi)$, if $\{z_n\} \notin \Gamma^{\#}(\varphi)$, then we get (9) according to $\rho(\varphi(z_n),\psi(z_n)) \le 1$. If $\{z_n\} \in \Gamma^{\#}(\varphi)$, then $|\psi(z_n)| \to 1$ by (a), thus we have (9) by (b). Similarly, we can get (10). ∎

Next, we character the connection of $\mathcal{C}$. Put $\varphi_t(z) = (1-t)\varphi(z) + t\psi(z)$ for $t \in [0,1]$, we give a sufficient condition for the compactness of $C_\varphi - C_{\varphi_t}$.

Theorem 4. *Assume that $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\beta = 1$, μ is a normal weighted function on $\mathbb{D}$ and $\varphi, \psi \in S(\mathbb{D})$, C_φ and C_ψ are bounded from $F(p,q,s)$ to $\mathcal{B}_\mu$, $C_\varphi - C_\psi$ is compact. Then for any $t \in [0,1]$, the following hold:*

(i) $\Gamma^{\#}(\varphi_t) \subset \Gamma(\varphi) \cap \Gamma(\psi)$*;*

(ii) *For any* $\{z_n\} \subset \Gamma(\varphi) \cap \Gamma(\psi)$,

$$\lim_{n\to\infty} \left| \frac{\mu(z_n)\varphi'(z_n)}{1-|\varphi(z_n)|^2} - \frac{\mu(z_n)\varphi_t'(z_n)}{1-|\varphi_t(z_n)|^2} \right| = \lim_{n\to\infty} \frac{\mu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2} \rho(\varphi(z_n), \varphi_t(z_n)) = 0.$$

Moreover, $C_\varphi - C_{\varphi_t}$ *is compact from* $F(p,q,s)$ *to* $\mathcal{B}_\mu$ *for any* $t \in [0,1]$.

Proof. Because $\Gamma^{\#}(\varphi_t) \subset \Gamma(\varphi_t)$, and $\Gamma(\varphi_t) \subset \Gamma(\varphi) \cap \Gamma(\psi)$, we get (i). Since $C_\varphi - C_\psi$ is compact, then

$$\lim_{n\to\infty} \frac{\mu(z_n)|\varphi'(z_n)|\rho(\varphi(z_n), \psi(z_n))}{1-|\varphi(z_n)|^2} = \lim_{n\to\infty} \left| \frac{\mu(z_n)\varphi'(z_n)}{1-|\varphi(z_n)|^2} - \frac{\mu(z_n)\psi'(z_n)}{1-|\psi(z_n)|^2} \right| = 0,$$

when $\{z_n\} \in \Gamma(\varphi) \cap \Gamma(\psi)$. By Lemma 7, we know

$$\frac{\mu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2} \rho(\varphi(z_n), \varphi_t(z_n)) \le \frac{\mu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2} \rho(\varphi(z_n), \psi(z_n)) \to 0$$

as $n \to \infty$. And

$$\begin{aligned}
&\mu(z) \left| \frac{\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\varphi_t'(z)}{1-|\varphi_t(z)|^2} \right| \\
\le &\left| \frac{\mu(z)\varphi'(z)}{1-|\varphi(z)|^2} \left(1 - \frac{(1-t)(1-|\varphi(z)|^2)}{1-|\varphi_t(z)|^2} - \frac{t(1-|\psi(z)|^2)}{1-|\varphi_t(z)|^2} \right) \right| \\
&+\mu(z) \left| \frac{(1-t)\varphi'(z)}{1-|\varphi_t(z)|^2} + \frac{t(1-|\psi(z)|^2)\varphi'(z)}{(1-|\varphi(z)|^2)(1-|\varphi_t(z)|^2)} - \frac{(1-t)\varphi'(z)+t\psi'(z)}{1-|\varphi_t(z)|^2} \right| \\
= &\frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \left| 1 - \frac{(1-t)(1-|\varphi(z)|^2)}{1-|\varphi_t(z)|^2} - \frac{t(1-|\psi(z)|^2)}{1-|\varphi_t(z)|^2} \right| \\
&+\mu(z) \left| \frac{t(1-|\psi(z)|^2)\varphi'(z)}{(1-|\varphi_t(z)|^2)(1-|\varphi(z)|^2)} - \frac{t\psi'(z)}{1-|\varphi_t(z)|^2} \right| \\
\le &\frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \left| 1 - \frac{(1-t)(1-|\varphi(z)|^2)}{1-|\varphi_t(z)|^2} - \frac{t(1-|\psi(z)|^2)}{1-|\varphi_t(z)|^2} \right| \\
&+\mu(z) \frac{t(1-|\psi(z)|^2)}{1-|\varphi_t(z)|^2} \left| \frac{\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\psi'(z)}{1-|\psi(z)|^2} \right|.
\end{aligned}$$

According to the proof of Lemma 4.2 in [6], we get that

$$\left| 1 - (1-t)\frac{1-|\varphi(z)|^2}{1-|\varphi_t(z)|^2} - t\frac{1-|\psi(z)|^2}{1-|\varphi_t(z)|^2} \right| \le \rho(\varphi(z), \psi(z))^2 \le \rho(\varphi(z), \psi(z)),$$

and

$$\frac{t(1-|\psi(z)|^2)}{1-|\varphi_t(z)|^2} \le 1.$$

So we have

$$\lim_{n\to\infty}\left|\frac{\mu(z_n)\varphi'(z_n)}{1-|\varphi(z_n)|^2}-\frac{\mu(z_n)\psi'(z_n)}{1-|\varphi_t(z_n)|^2}\right|=0.$$

Moreover, when C_φ and C_ψ are bounded, by Theorem 1 in [16], it is easy to check that C_{φ_t} is bounded, so we can get $C_\varphi - C_{\varphi_t}$ is compact by Theorem 3. ■

Theorem 5. *Assume that* $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\beta = 1$, μ *is a normal weighted function on* $\mathbb{D}$ *and* $\varphi, \psi \in S(\mathbb{D})$, C_φ *and* C_ψ *are bounded from* $F(p, q, s)$ *to* $\mathcal{B}_\mu$, $C_\varphi - C_\psi$ *is compact, then the following are equivalent:*

(i) $\frac{\mu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2} \to 0$, *when* $\{z_n\} \in \Gamma(\psi)\setminus\Gamma(\varphi)$ *and* $\frac{\mu(z_n)|\psi'(z_n)|}{1-|\psi(z_n)|^2} \to 0$, *when* $\{z_n\} \in \Gamma(\varphi)\setminus\Gamma(\psi)$.

(ii) *The map* $t \longmapsto C_{\varphi_t}$ *is continuous from* $[0, 1]$ *to* $\mathcal{C}$.

Proof. (i)⇒(ii). We will show that $t \in [0, 1] \to C_{\varphi_t}$ is a continuous path in $\mathcal{C}$, thus we need to show

$$\lim_{t\to t'}\|C_{\varphi_{t'}} - C_{\varphi_t}\|_{F(p,q,s)\to\mathcal{B}_\mu} = 0$$

for each $t' \in [0, 1]$. To prove this we show

$$\lim_{t\to t'+}\|C_{\varphi_{t'}} - C_{\varphi_t}\|_{F(p,q,s)\to\mathcal{B}_\mu} = 0$$

for each $t' \in [0, 1)$, the proof for the left-hand limits is similar. Note that $C_{\varphi_t} - C_\psi$ is compact for every $t' \in [0, 1]$ by Theorem 4. Setting $(\varphi_{t'})_r = (1 - r)\varphi_{t'} + r\psi$ for $r \in [0, 1]$, we have

$$\varphi_{t'} - \varphi_t = (\varphi_{t'})_0 - (\varphi_{t'})_{r_1}, \quad 0 \le t' \le t < 1,$$

where $r_1 = \frac{t-t'}{1-t'}$. Thus, to prove this implication, it is sufficient to consider only the continuity at $t' = 0$.

$$\begin{aligned}&\|C_\varphi - C_{\varphi_t}\|_{F(p,q,s)\to\mathcal{B}_\mu}\\ =&\sup_{\|f\|_F\le 1}|f(\varphi(0)) - f(\varphi_t(0))| + \sup_{\|f\|_F\le 1}\sup_{z\in\mathbb{D}}\mu(z)|f'(\varphi(z))\varphi'(z) - f'(\varphi_t(z))\varphi_t'(z)|\end{aligned}$$

By Lemma 3 and Lemma 6, we obtain

$$\sup_{\|f\|_F\le 1}|f(\varphi(0)) - f(\varphi_t(0))| \le Cd_1(\varphi(0), \varphi_t(0)) \to 0$$

as $t \to 0$.

$$\begin{aligned}&\sup_{\|f\|_F\le 1}\sup_{z\in\mathbb{D}}\mu(z)|f'(\varphi(z))\varphi'(z) - f'(\varphi_t(z))\varphi_t'(z)|\\ \le\ & C\sup_{z\in\mathbb{D}}\left[\left|\frac{\mu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\mu(z)\varphi_t'(z)}{1-|\varphi_t(z)|^2}\right| + \frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\rho(\varphi(z), \varphi_t(z))\right].\end{aligned}$$

Then for every $\varepsilon > 0$, by Theorem 4, the compactness of $C_\varphi - C_\psi$ and (i), there is a constant $r_1 \in (0,1)$ such that

$$\left|\frac{\mu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\mu(z)\varphi_t'(z)}{1-|\varphi_t(z)|^2}\right| + \frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\rho(\varphi(z),\varphi_t(z)) < \varepsilon,$$

when $z \in \Gamma_{r_1}(\varphi) \bigcap \Gamma_{r_1}(\psi)$, and

$$\frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\rho(\varphi(z),\psi(z)) < \varepsilon, \quad \frac{\mu(z)|\psi'(z)|}{1-|\psi(z)|^2} < \varepsilon,$$

when $z \in \Gamma_{r_1}(\varphi) \setminus \Gamma_{r_1}(\psi)$.

When $z \in \Gamma_{r_1}(\varphi) \setminus \Gamma_{r_1}(\psi)$, there exists a constant m such that $\rho(\varphi(z),\psi(z)) \geq m > 0$, then $\frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|^2} < \frac{\varepsilon}{m}$. So we have $\left|\frac{\mu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\mu(z)\psi(z)}{1-|\psi(z)|^2}\right| < C\varepsilon$ for some positive constant C.

According to the proving process of Theorem 4, we know that

$$\begin{aligned}
&\sup_{z\in\Gamma_{r_1}(\varphi)\setminus\Gamma_{r_1}(\psi)} \left|\frac{\mu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\mu(z)\varphi_t'(z)}{1-|\varphi_t(z)|^2}\right| \\
&\leq \sup_{z\in\Gamma_{r_1}(\varphi)\setminus\Gamma_{r_1}(\psi)} \left[\left|\frac{\mu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\mu(z)\psi'(z)}{1-|\psi(z)|^2}\right| + \frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\rho(\varphi(z),\psi(z))\right] \\
&\leq C\varepsilon.
\end{aligned}$$

So

$$\sup_{z\in\Gamma_{r_1}(\varphi)} \left[\left|\frac{\mu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\mu(z)\varphi_t'(z)}{1-|\varphi_t(z)|^2}\right| + \frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\rho(\varphi(z),\varphi_t(z))\right] < M_1\varepsilon.$$

On $\mathbb{D}\setminus\Gamma_{r_1}(\varphi)$, $\left|\frac{\mu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\mu(z)\varphi_t'(z)}{1-|\varphi_t(z)|^2}\right|$ and $\frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\rho(\varphi(z),\varphi_t(z))$ converges uniformly to 0 as $t \to 0$. Thus there is some t_1 so close to 0 that for any $t < t_1$,

$$\sup_{z\in\mathbb{D}\setminus\Gamma_{r_1}(\varphi)} \left[\left|\frac{\mu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\mu(z)\varphi_t'(z)}{1-|\varphi_t(z)|^2}\right| + \frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\rho(\varphi(z),\varphi_t(z))\right] < \varepsilon.$$

Hence we get $\|C_\varphi - C_{\varphi_t}\|_{F(p,q,s)\to\mathcal{B}_\mu} \to 0$ as $t \to 0$.

(ii)⇒(i). Suppose that there exists a sequence $\{z_n\} \in \Gamma(\psi)\setminus\Gamma(\varphi)$ such that $\frac{\mu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2} \to a \neq 0$.

Put $f_w(z) = \frac{1-|\varphi(w)|^2}{\overline{\varphi(w)}(1-\overline{\varphi(w)}z)}$, since $\|f_w\|_F < C$ for every $w \in \mathbb{D}$, we have that

$$\begin{aligned}
C\|C_\varphi - C_{\varphi_t}\|_{F(p,q,s)\to\mathcal{B}_\mu} &\geq \|(C_\varphi - C_{\varphi_t})f_{z_n}\|_\mu \\
&\geq \frac{\mu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2} - \frac{\mu(z_n)|\varphi_t'(z_n)|}{1-|\varphi_t(z_n)|^2}(1-\rho(\varphi(z_n),\varphi_t(z_n))^2.
\end{aligned}$$

So taking the limit, we obtain that $\|C_\varphi - C_{\varphi_t}\|_{F(p,q,s)\to\mathcal{B}_\mu} > 0$ for $t \in (0,1]$, this implies that the map $t \mapsto C_{\varphi_t}$ is not continuous at $t = 0$. This contradicts the condition (ii). ■

As an immediate consequence, we obtain the next corollary.

Corollary 1. *Assume that* $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\beta = 1$ *and* μ *is normal on* $\mathbb{D}$ *and* $\varphi, \psi \in S(\mathbb{D})$, C_φ *and* C_ψ *are bounded from* $F(p, q, s)$ *to* $\mathcal{B}_\mu$, $C_\varphi - C_\psi$ *is compact. If* $\Gamma(\varphi) = \Gamma(\psi)$, *then* C_φ *and* C_ψ *in the same connected component.*

Yet another consequence is that the compact composition operators belong to the same component.

Corollary 2. *Assume that* $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\beta = 1$ *and* μ *is normal on* $\mathbb{D}$ *and* $\varphi, \psi \in S(\mathbb{D})$, *then the compact composition operators from* $F(p, q, s)$ *to* $\mathcal{B}_\mu$ *form an connected set in* $\mathcal{C}$.

5. Compact Difference and Path Connectedness II

We continue the research of path connectedness when $\beta = \frac{q+2}{p} \neq 1$ in this section. Now, we study the compactness of $C_\varphi - C_{\varphi_t}$.

Theorem 6. *Assume that* $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, μ *is a normal weighted function on* $\mathbb{D}$ *and* $\varphi, \psi \in S(\mathbb{D})$, C_φ *and* C_ψ *are bounded from* $F(p, q, s)$ *to* $\mathcal{B}_\mu$. *If* $C_\varphi - C_\psi$ *is compact and* $\sup_{z\in\mathbb{D}} \rho(\varphi(z), \psi(z)) < 1$, *then* $C_\varphi - C_{\varphi_t}$ *is compact for* $t \in [0, 1]$.

Proof. It is obvious that $C_\varphi - C_{\varphi_t}$ is compact when $t = 0$ or $t = 1$.
For fixed $t \in (0, 1)$, since C_φ and C_ψ are bounded, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \frac{\mu(z)|\varphi_t'(z)|}{(1 - |\varphi_t(z)|^2)^\beta} \\
&\leq \sup_{z\in\mathbb{D}} \frac{(1-t)\mu(z)|\varphi'(z)|}{(1 - |\varphi_t(z)|^2)^\beta} + \sup_{z\in\mathbb{D}} \frac{t\mu(z)|\psi'(z)|}{(1 - |\varphi_t(z)|^2)^\beta} \\
&= \sup_{z\in\mathbb{D}} \frac{(1-t)\mu(z)|\varphi'(z)|}{(1 - |\varphi(z)|^2)^\beta} \frac{(1 - |\varphi(z)|^2)^\beta}{(1 - |\varphi_t(z)|^2)^\beta} + \sup_{z\in\mathbb{D}} \frac{t\mu(z)|\psi'(z)|}{(1 - |\psi(z)|^2)^\beta} \frac{(1 - |\psi(z)|^2)^\beta}{(1 - |\varphi_t(z)|^2)^\beta} \\
&\leq C \sup_{z\in\mathbb{D}} \frac{(1-t)(1 - |\varphi(z)|^2)^\beta}{(1 - |\varphi_t(z)|^2)^\beta} + C \sup_{z\in\mathbb{D}} \frac{t(1 - |\psi(z)|^2)^\beta}{(1 - |\varphi_t(z)|^2)^\beta} \\
&\leq C[(1-t)^{1-\beta} + t^{1-\beta}],
\end{aligned}$$

then C_{φ_t} is a bounded operator.

By Lemma 7 and the compactness of $C_\varphi - C_\psi$, we obtain

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)|}{(1 - |\varphi(z)|^2)^\beta} \rho(\varphi(z), \varphi_t(z)) = 0.$$

And when $|\varphi_t(z)| \to 1$, we have $|\varphi(z)| \to 1$ and $|\psi(z)| \to 1$, then

$$
\begin{aligned}
&\lim_{|\varphi_t(z)|\to 1} \frac{\mu(z)|\varphi_t'(z)|}{(1-|\varphi_t(z)|^2)^\beta}\rho(\varphi(z),\varphi_t(z))\\
\le\ & \lim_{|\varphi(z)|\to 1} \frac{(1-t)^{1-\beta}\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta}\rho(\varphi(z),\psi(z))\\
&+\lim_{|\psi(z)|\to 1} \frac{t^{1-\beta}\mu(z)|\psi'(z)|}{(1-|\psi(z)|^2)^\beta}\rho(\varphi(z),\psi(z))\\
=\ & 0.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\left|\frac{\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\varphi_t'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right|\\
=&\left|\frac{\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{(1-t)\varphi'(z)}{(1-|\varphi_t(z)|^2)^\beta}-\frac{t\psi'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right|\\
\le&\left|\frac{\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\psi'(z)}{(1-|\psi(z)|^2)^\beta}\right|\cdot\frac{t(1-|\psi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}\\
&+\left|\frac{\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}\right|\left|1-\frac{(1-t)(1-|\varphi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}-\frac{t(1-|\psi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}\right|\\
\le& C\left|\frac{\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\psi'(z)}{(1-|\psi(z)|^2)^\beta}\right|\cdot\frac{(1-|\psi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}+\left|\frac{\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}\right|\\
&\cdot\left|1-\frac{(1-|\varphi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}\right|+t\left|\frac{\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}\right|\left|\frac{(1-|\varphi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}-\frac{(1-|\psi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}\right|.
\end{aligned}
$$

Because $\sup_{z\in\mathbb{D}}\rho(\varphi(z),\psi(z))\le r_0<1$, we get $\rho(\varphi(z),\varphi_t(z))\le\rho(\varphi(z),\psi(z))$ and $\rho(\psi(z),\varphi_t(z))\le\rho(\varphi(z),\psi(z))$ by Lemma 7, then we have by Lemma 8 and Lemma 9

$$
\begin{aligned}
&\lim_{\min\{|\varphi(z)|,|\varphi_t(z)|\}\to 1}\mu(z)\left|\frac{\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\varphi_t'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right|\\
\le\ & C\lim_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\mu(z)\psi'(z)}{(1-|\psi(z)|^2)^\beta}\right|\cdot\frac{(1-|\psi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}\\
&+\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta}\left|1-\frac{(1-|\varphi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}\right|\\
&+\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta}\left|\frac{(1-|\psi(z)|^2)^\beta}{(1-|\varphi(z)|^2)^\beta}-1\right|\cdot\frac{(1-|\varphi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}\\
\le\ & C\lim_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\mu(z)\psi'(z)}{(1-|\psi(z)|^2)^\beta}\right|\\
&+C\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta}\rho(\varphi(z),\psi(z))=0.
\end{aligned}
$$

Thus we know $C_\varphi - C_{\varphi_t}$ is compact by Theorem 2. ■

Theorem 7. *Assume that* $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, μ *is a normal weighted function on* $\mathbb{D}$ *and* $\varphi, \psi \in S(\mathbb{D})$, C_φ *and* C_ψ *are bounded from* $F(p,q,s)$ *to* $\mathcal{B}_\mu$*. If* $C_\varphi - C_\psi$ *is compact and* $\sup\limits_{z\in\mathbb{D}} \rho(\varphi(z), \psi(z)) < 1$, *then* C_φ *and* C_ψ *are in the same path component.*

Proof. To prove this theorem, we only need to consider

$$\lim_{t\to 0} \|C_\varphi - C_{\varphi_t}\|_{F(p,q,s)\to\mathcal{B}_\mu} = 0.$$

By Lemma 5 and Lemma 6, we have

$$\begin{aligned}
&\|C_\varphi - C_{\varphi_t}\|_{F(p,q,s)\to\mathcal{B}_\mu}\\
\le& \sup_{\|f\|_F\le 1} |f(\varphi(0)) - f(\varphi_t(0))| + \sup_{\|f\|_F\le 1}\sup_{z\in\mathbb{D}} \mu(z)|f'(\varphi(z))\varphi'(z) - f'(\varphi_t(z))\varphi_t'(z)|\\
\le& C d_\alpha(\varphi(0), \varphi_t(0)) + \sup_{z\in\mathbb{D}} \frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta}\rho(\varphi(z), \varphi_t(z))\\
&+ \sup_{z\in\mathbb{D}} \left| \frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta} - \frac{\mu(z)\varphi_t'(z)}{(1-|\varphi_t(z)|^2)^\beta} \right|.
\end{aligned}$$

It is obvious that $\lim\limits_{t\to 0} d_\alpha(\varphi(0), \varphi_t(0)) = 0$.

Since we assume

$$\sup_{z\in\mathbb{D}} \rho(\varphi(z), \psi(z)) < 1,$$

we can find a $\lambda < 1$ such that $\sup\limits_{z\in\mathbb{D}} \rho(\varphi(z), \psi(z)) \le \lambda$. And

$$\begin{aligned}
\rho(\varphi(z), \varphi_t(z)) &= \left| \frac{\varphi(z) - \varphi_t(z)}{1 - \overline{\varphi(z)}\varphi_t(z)} \right|\\
&= \frac{t|\varphi(z) - \psi(z)|}{|1 - \overline{\varphi(z)}\varphi(z) + t\overline{\varphi(z)}\varphi(z) - t\overline{\varphi(z)}\psi(z)|}\\
&\le \frac{t|\varphi(z) - \psi(z)|}{|1 - \overline{\varphi(z)}\psi(z)| - |(1-t)\overline{\varphi(z)}(\psi(z) - \varphi(z))|}\\
&\le \frac{t}{\rho^{-1}(\varphi(z), \psi(z)) - (1-t)|\varphi(z)|}\\
&\le \frac{t\lambda}{1-(1-t)\lambda},
\end{aligned}$$

so $\sup\limits_{z\in\mathbb{D}} \rho(\varphi(z), \varphi_t(z)) \to 0$ if $t \to 0$. Finally, by the boundedness of C_φ, we have

$$\lim_{t\to 0}\sup_{z\in\mathbb{D}} \frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta}\rho(\varphi(z), \varphi_t(z)) = 0.$$

Since $\sup_{z\in\mathbb{D}}\rho(\varphi(z),\psi(z)) \le \lambda < 1$, for every sequence $\{z_n\} \subset \mathbb{D}$, we have $|\varphi(z_n)| \to 1$ if and only if $|\psi(z_n)| \to 1$. And $C_\varphi - C_\psi$ is compact, so for every $\varepsilon > 0$, there exists a constant $r \in (0,1)$ such that

$$\sup_{|\varphi(z)|>r}\left[\frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta}\rho(\varphi(z),\psi(z)) + \left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta} - \frac{\mu(z)\psi'(z)}{(1-|\psi(z)|^2)^\beta}\right|\right] < \varepsilon.$$

We know

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta} - \frac{\mu(z)\varphi_t'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right| \\
\le &\sup_{|\varphi(z)|\le r}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta} - \frac{\mu(z)\varphi_t'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right| \\
&+ \sup_{|\varphi(z)|>r}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta} - \frac{\mu(z)\varphi_t'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right|.
\end{aligned}$$

According to the process of Theorem 6, we have

$$\begin{aligned}
&\sup_{|\varphi(z)|>r}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta} - \frac{\mu(z)\varphi_t'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right| \\
\le\ & C\sup_{|\varphi(z)|>r}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta} - \frac{\mu(z)\psi'(z)}{(1-|\psi(z)|^2)^\beta}\right| + C\sup_{|\varphi(z)|>r}\frac{\mu(z)|\varphi'(z)|\rho(\varphi(z),\psi(z))}{(1-|\varphi(z)|^2)^\beta} \\
<\ & C\varepsilon.
\end{aligned}$$

There is a constant $r' \in (0,1)$ such that $|\psi(z)| \le r'$ when $|\varphi(z)| \le r$, then $|\varphi_t(z)| \le \max\{r, r'\}$. C_φ and C_ψ are bounded operators, then we obtain

$$\begin{aligned}
&\sup_{|\varphi(z)|\le r}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta} - \frac{\mu(z)\varphi_t'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right| \\
\le &\sup_{|\varphi(z)|\le r}\frac{t\mu(z)|\varphi'(z)-\psi'(z)|}{(1-|\varphi_t(z)|^2)^\beta} + \sup_{|\varphi(z)|\le r}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta} - \frac{\mu(z)\varphi'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right| \\
\le &\sup_{|\varphi(z)|\le r}\frac{t\mu(z)(|\varphi'(z)|+|\psi'(z)|)}{(1-|\varphi_t(z)|^2)^\beta} + \sup_{|\varphi(z)|\le r}\frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta}\left|1 - \frac{(1-|\varphi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}\right| \\
\le &\sup_{|\varphi(z)|\le r}\frac{t\mu(z)|\varphi'(z)|}{(1-|\varphi_t(z)|^2)^\beta} + \sup_{|\psi(z)|\le r'}\frac{t\mu(z)|\psi'(z)|}{(1-|\varphi_t(z)|^2)^\beta} \\
&+ \sup_{|\varphi(z)|\le r}\frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta}\left|1 - \frac{(1-|\varphi(z)|^2)^\beta}{(1-|\varphi_t(z)|^2)^\beta}\right|,
\end{aligned}$$

Since the boundedness of C_φ and C_ψ, we obtain $\sup_{|\varphi(z)|\le r}\mu(z)|\varphi'(z)| < C$, and $\sup_{|\psi(z)|\le r'}\mu(z)|\psi'(z)| < C$, by Lemma 7 and Lemma 9, we have

$$\sup_{|\varphi(z)|\leq r}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\mu(z)\varphi_t'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right|\to 0$$

as $t\to 0$, thus, we get

$$\lim_{t\to 0}\sup_{z\in\mathbb{D}}\left|\frac{\mu(z)\varphi'(z)}{(1-|\varphi(z)|^2)^\beta}-\frac{\mu(z)\varphi_t'(z)}{(1-|\varphi_t(z)|^2)^\beta}\right|=0.$$

Then $t\mapsto C_{\varphi_t}$ is a continuous curve in $\mathcal{C}$, C_φ and C_ψ is in the same path component. ■

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