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TOPOLOGICAL ENTROPY OF PROPER MAP

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Abstract. By using the Carathéodory-Pesin structure (C-P structure), the topological entropy on the whole space introduced for a proper map, is generalized to the cases of arbitrary subset, i.e., we introduce three notions of topological entropy. Some of the properties of these notions are provided. As some applications, for the proper map of locally compact separable metric space, we prove the following variational principles: (1) The upper capacity topological entropy on any subset and the minimum of the Bowen-Dinaburg entropies always coincide; (2) For any invariant probability measure, the measure-theoretic entropy and the infimum of the topological entropies on all sets which are of full measures always coincide; (3) The relationship between the topological entropies of level sets of the ergodic average of some continuous functions and the measure-theoretic entropies are given. These are the extensions of results of Patrão and Pesin, etc.

1. INTRODUCTION

Let f be a continuous map acting on a compact topological space X, we call (X, f) a compact system. The concept of topological entropy plays a central role in topological dynamics. The notion of topological entropy was introduced by Adler, Konheim and McAndrew [1] as an invariant of topological conjugacy. We call it the AKM entropy. Later, Bowen[4] and Dinaburg [10] presented equivalent approach to the notion of topological entropy in the case when the domain of considered map is a metrizable space. We call it the BD entropy. Since the topological entropy appeared to be a very useful invariant in dynamical systems and ergodic theory, there were several attempts to find its suitable generalizations. By using the approach of the definition of the Hausdorff dimension, Bowen [5] introduced the topological entropy on any subset. We call it the Bowen dimensional entropy. Let X be a compact metric space, Pesin and

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Pitskel [14] introduced the topological entropy (topological pressure) on any subset of X. We call it the PP entropy and it coincides with the Bowen dimensional entropy. Let X be a topological space and $f : X \to X$ be a proper map. Patrão[13] introduced a topological entropy, it is the extension of AKM entropy. We call it the Patrão entropy. The other extensions of topological entropy, one can see, for example, [8, 9, 11, 12, 18].

The classical Carathéodory construction in the general measure theory was originated by Carathéodory in [6]. Pesin [15] introduced a construction which is a generalization of the classical Carathéodory construction. It produces various characteristics of dimension type. For example, Hausdorff dimension, topological entropy, etc. We call it the Carathéodory-Pesin structure(or briefly, C-P structure). It is a very powerful tool to study dimension theory and dynamical systems.

In this paper, by using the C-P structure, the notion of topological entropy on the whole space introduced for a proper map, is generalized to the case of arbitrary subset. We introduce three notions of topological entropy. The lower and upper capacity topological entropies are the extensions of the Patrão entropy and AKM entropy. The topological entropy is the extension of PP entropy and Bowen dimensional entropy. Some of the properties of these notions are provided. As some applications, for the proper map of locally compact separable metric space, we prove the following variational principles: (1) The upper capacity topological entropy on any subset and the minimum of the Bowen-Dinaburg entropies always coincide; (2) For any invariant probability measure, the measure-theoretic entropy and the infimum of the topological entropies on all sets which are of full measures always coincide; (3) The relationship between the topological entropies of level sets of the ergodic average of some continuous functions and the measure-theoretic entropies are given. These are the extensions of results of Patrão and Pesin, etc.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we introduce the notions of the topological entropy, the lower and upper capacity topological entropies and some basic properties of them. In Section 4, we give some further properties. In Section 5, we give some variational principles.

2. Preliminaries

We describe the Carathéodory-Pesin structure(for the full description see refs. [15]). Let X and $\mathcal{S}$ be arbitrary sets and $\mathcal{F} = \{U_s : s \in \mathcal{S}\}$ a collection of subsets in X. We assume that there exist two functions $\eta, \psi : \mathcal{S} \to \mathbb{R}^+$ satisfying the following conditions:

(1) there exists $s_0 \in \mathcal{S}$ such that $U_{s_0} = \phi$; if $U_s = \phi$ then $\eta(s) = 0$ and $\psi(s) = 0$; if $U_s \neq \phi$ then $\eta(s) > 0$ and $\psi(s) > 0$;

(2) for any $\delta > 0$ one can find $\varepsilon > 0$ such that $\eta(s) \leq \delta$ for any $s \in \mathcal{S}$ with $\psi(s) \leq \varepsilon$;

(3) for any $\varepsilon > 0$ there exists a finite or countable subcollection $\mathcal{G} \subset \mathcal{S}$ which covers X (i.e., $\cup_{s\in\mathcal{G}} U_s \supset X$) and $\psi(\mathcal{G}) := \sup\{\psi(s) : s \in \mathcal{G}\} \leq \varepsilon$.

Let $\xi : \mathcal{S} \to \mathbb{R}^+$ be a function. We say that the set $\mathcal{S}$, collection of subsets $\mathcal{F}$, and the functions ξ, η, ψ, satisfying Conditions (1), (2), and (3), introduce the Carathéodory-Pesin structure or C-P structure τ on X and write $\tau = (\mathcal{S}, \mathcal{F}, \xi, \eta, \psi)$.

Given a set $Z \subset X$ and numbers $\alpha \in \mathbb{R}$, $\varepsilon > 0$, we define

$$M(Z, \alpha, \varepsilon) := \inf_{\mathcal{G}} \{\sum_{s\in\mathcal{G}} \xi(s)\eta(s)^\alpha\},$$

where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset \mathcal{S}$ covering Z with $\psi(s) \leq \varepsilon$ for any $s \in \mathcal{G}$. The quantity $M(Z, \alpha, \varepsilon)$ is a monotone function with respect to ε, therefore, the following limit exists:

$$m(Z, \alpha) = \lim_{\varepsilon\to 0} M(Z, \alpha, \varepsilon).$$

It was shown in [15] that there exists a critical value $\alpha_c \in [-\infty, +\infty]$ such that

$$m(Z, \alpha) = 0, \alpha > \alpha_c,$$

$$m(Z, \alpha) = \infty, \alpha < \alpha_c.$$

The number α_c is said to be the Carathéodory-Pesin dimension of the set Z.

We shall now assume that the following condition holds:

(3′) there exists $\epsilon > 0$ such that for any $0 < \varepsilon \leq \epsilon$ there exists a finite or countable subcollection $\mathcal{G} \subset \mathcal{S}$ covering X such that $\psi(s) = \varepsilon$ for any $s \in \mathcal{G}$.

Given a set $Z \subset X$ and numbers $\alpha \in \mathbb{R}$, $\varepsilon > 0$, we define

$$R(Z, \alpha, \varepsilon) := \inf_{\mathcal{G}} \{\sum_{s\in\mathcal{G}} \xi(s)\eta(s)^\alpha\},$$

where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset \mathcal{S}$ covering Z such that $\psi(s) = \varepsilon$ for any $s \in \mathcal{G}$. We set

$$\underline{r}(Z, \alpha) = \underline{\lim}_{\varepsilon\to 0} R(Z, \alpha, \varepsilon), \overline{r}(Z, \alpha) = \overline{\lim}_{\varepsilon\to 0} R(Z, \alpha, \varepsilon).$$

It was shown in [15] that there exist $\underline{\alpha}_C, \overline{\alpha}_C \in \mathbb{R}$ such that

$$\underline{r}(Z, \alpha) = 0, \alpha > \underline{\alpha}_C, \underline{r}(Z, \alpha) = \infty, \alpha < \underline{\alpha}_C;$$

$$\overline{r}(Z, \alpha) = 0, \alpha > \overline{\alpha}_C, \overline{r}(Z, \alpha) = \infty, \alpha < \overline{\alpha}_C.$$

The numbers $\underline{\alpha}_C$ and $\overline{\alpha}_C$ are said to be the lower and upper Carathéodory-Pesin capacities of the set Z respectively.

For any $\varepsilon > 0$ and any set $Z \subset X$, let us put

$$\Lambda(Z,\varepsilon) = \inf_{\mathcal{G}}\{\sum_{s\in\mathcal{G}}\xi(s)\},$$

where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset \mathcal{S}$ covering Z for which $\psi(s) = \varepsilon$ for all $s \in \mathcal{G}$.

Let us assume that the following condition holds:

(4) $\eta(s_1) = \eta(s_2)$ for any $s_1, s_2 \in \mathcal{S}$ for which $\psi(s_1) = \psi(s_2)$.

It was shown in [15] that if the function η satisfies Condition (4) then for any $Z \subset X$,

$$\underline{\alpha}_C = \underline{\lim}_{\varepsilon\to 0}\frac{\log\Lambda(Z,\varepsilon)}{\log(1/\eta(\varepsilon))}, \overline{\alpha}_C = \overline{\lim}_{\varepsilon\to 0}\frac{\log\Lambda(Z,\varepsilon)}{\log(1/\eta(\varepsilon))}.$$

Example 2.1. Let X be a compact metric space with metric d, $f : X \to X$ a continuous map and $\mathcal{U}$ a finite open cover of X. Denote by $S_m(\mathcal{U})$ the set of all strings $\mathbf{U} = \{U_{i_0}, U_{i_1}, \cdots, U_{i_{m-1}} : U_{i_j} \in \mathcal{U}\}$ of length $m = m(\mathbf{U})$. We put $S(\mathcal{U}) = \bigcup\limits_{m\geq 0} S_m(\mathcal{U})$.

To a given string $\mathbf{U} = \{U_{i_0}, U_{i_1}, \cdots, U_{i_{m-1}}\} \in S(\mathcal{U})$ we associate the set

$$X(\mathbf{U}) = \{x \in X : f^j(x) \in U_{i_j}, j = 0, 1, \cdots, m(U) - 1\}.$$

It is easy to see that $X(\mathbf{U}) = \bigcap\limits_{j=0}^{m(U)-1} f^{-j}U_{i_j}$. Define the collection of subsets

$$\mathcal{F} = \mathcal{F}(\mathcal{U}) = \{X(\mathbf{U}) : \mathbf{U} \in S(\mathcal{U})\}$$

and three functions $\xi, \eta, \psi : S(\mathcal{U}) \to \mathbb{R}^+$ as follows $\xi(\mathbf{U}) = 1$, $\eta(\mathbf{U}) = \exp(-m(\mathbf{U}))$, $\psi(\mathbf{U}) = m(\mathbf{U})^{-1}$. It is easy to verify that the set $S, \mathcal{F}$ and the functions ξ, η, and ψ satisfy the Conditions (1), (2), and (3) in above and hence they determine a C-P structure $\tau = \tau(\mathcal{U}) = (S, \mathcal{F}, \xi, \eta, \psi)$ on X. We say that a collection of strings $\mathcal{G}$ covers a set $Z \subset X$ if $\bigcup\limits_{\mathbf{U}\in\mathcal{G}} X(\mathbf{U}) \supset Z$. For any set $Z \subset X$ and $\alpha \in \mathbb{R}$, define

$$M(Z,\alpha,\mathcal{U},N) = \inf_{\mathcal{G}}\{\sum_{\mathbf{U}\in\mathcal{G}}\xi(\mathbf{U})\eta(\mathbf{U})^{\alpha}\} = \inf_{\mathcal{G}}\{\sum_{\mathbf{U}\in\mathcal{G}}\exp(-\alpha m(\mathbf{U}))\}$$

and the infimum is taken over all finite or countable collections of strings $\mathcal{G} \subset S(\mathcal{U})$ such that $m(\mathbf{U}) \geq N$ for all $\mathbf{U} \in \mathcal{G}$ and $\mathcal{G}$ covers Z. Defining

$$m(Z,\alpha,\mathcal{U}) = \lim_{N\to+\infty} M(Z,\alpha,\mathcal{U},N).$$

For every real numbers α introduce

$$\underline{r}(Z,\alpha,\mathcal{U}) = \underline{\lim}_{N\to+\infty} R(Z,\alpha,\mathcal{U},N),$$

$$\overline{r}(Z,\alpha,\mathcal{U}) = \overline{\lim_{N\to+\infty}} R(Z,\alpha,\mathcal{U},N),$$

where $R(Z,\alpha,\mathcal{U},N) = \inf_{\mathcal{G}}\{\sum_{\mathbf{U}\in\mathcal{G}} \exp(-\alpha N)\}$ and the infimum is taken over all collections of strings $\mathcal{G} \subset S(\mathcal{U})$ such that $m(\mathbf{U}) = N$ for all $\mathbf{U} \in \mathcal{G}$ and $\mathcal{G}$ covers Z. By the definition of C-P structure, define

$$h_Z^{PP}(f,\mathcal{U}) := \inf\{\alpha : m(Z,\alpha,\mathcal{U}) = 0\} = \sup\{\alpha : m(Z,\alpha,\mathcal{U}) = +\infty\},$$

$$\underline{Ch}_Z^{PP}(f,\mathcal{U}) := \inf\{\alpha : \underline{r}(Z,\alpha,\mathcal{U}) = 0\} = \sup\{\alpha : \underline{r}(Z,\alpha,\mathcal{U}) = +\infty\},$$

$$\overline{Ch}_Z^{PP}(f,\mathcal{U}) := \inf\{\alpha : \overline{r}(Z,\alpha,\mathcal{U}) = 0\} = \sup\{\alpha : \overline{r}(Z,\alpha,\mathcal{U}) = +\infty\}.$$

Moreover, define

$$h_Z^{PP}(f) := \sup_{|\mathcal{U}|\to 0} h_Z^{PP}(f,\mathcal{U}),$$

$$\underline{Ch}_Z^{PP}(f) := \sup_{|\mathcal{U}|\to 0} \underline{Ch}_Z^{PP}(f,\mathcal{U}),$$

$$\overline{Ch}_Z^{PP}(f) := \sup_{|\mathcal{U}|\to 0} \overline{Ch}_Z^{PP}(f,\mathcal{U}),$$

where $|\mathcal{U}|$ denotes the maximum of the diameters of $A \in \mathcal{U}$ in the sense of d. We call the quantities $h_Z^{PP}(f), \underline{Ch}_Z^{PP}(f)$ and $\overline{Ch}_Z^{PP}(f)$, respectively the PP entropy and lower and upper capacity PP entropy of f on the set Z(see [14,15]). The other papers that used the C-P structure, one can see, for example, [2, 12].

Let X be a topological space and $f : X \to X$ be a proper map[13], i.e., f is a continuous map such that the pre-image by f of any compact set is compact. An open set is called an admissible open set if the closure or the complement of it is compact. An admissible cover of X[13] is an open and finite cover $\mathcal{U}$ of X such that, for each $A \in \mathcal{U}$, A is an admissible open set. Given an admissible cover $\mathcal{U}$ of X, for any $n \in \mathbb{N}$, we have that the set given by

$$\mathcal{U}^n := \{A_0 \cap f^{-1}A_1 \cap \cdots \cap f^{-n}A_n : A_i \in \mathcal{U}\}$$

is also an admissible cover of X, since f is a proper map. Given an admissible cover $\mathcal{U}$ of X, we denote by $N(\mathcal{U}^n)$ the smallest cardinality of all subcovers of $\mathcal{U}^n$. The Patrão entropy of the map f is defined as

$$h^P(f) := \sup_{\mathcal{U}} h(T,\mathcal{U}) = \sup_{\mathcal{U}}\{\lim_{n\to\infty}\frac{1}{n}\log N(\mathcal{U}^n)\},$$

where the supremum is taken over all admissible covers of X. We note that, when X is compact, the Patrão entropy coincides with the AKM entropy.

If X is a locally compact separable metric space, we can associate with X its one-point compactification, which we denote by $\widetilde{X}$. We have that $\widetilde{X}$ is defined as the disjoint union of X with $\{\infty\}$, where ∞ is some point not in X called the point at infinity. The topology in $\widetilde{X}$ consist of the former open sets in X and the sets $A \cup \{\infty\}$, where the complement of A in X is compact. Let $f : X \to X$ be a proper map, defining $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ by

$$\widetilde{f}(\widetilde{x}) = \begin{cases} f(\widetilde{x}), & \widetilde{x} \neq \infty \\ \infty, & \widetilde{x} = \infty, \end{cases}$$

we have that $\widetilde{f}$ is also a proper map, called the extension of f to $\widetilde{X}$. We note that the separability of X is equivalent to the metrizability of $\widetilde{X}$.

Let (X, d) be a metric space and denote by $B(x, \delta)$ the open ball centered at x with radius $\delta > 0$. The metric d is called admissible[13] if the following conditions are satisfied:

(1) If $\mathcal{U}_\delta = \{B(x_1, \delta), \cdots, B(x_k, \delta)\}$ is a cover of X, for every $\delta \in (a, b)$, where $0 < a < b$, then there exists $\delta_\varepsilon \in (a, b)$ such that $\mathcal{U}_{\delta_\varepsilon}$ is admissible.

(2) Every admissible cover of X has a Lebesgue number.

It is easy to see that, if (X, d) is compact, then d is automatically admissible.

Let (X, d) be a metric space and $f : X \to X$ a continuous map, we say that f with the specification property if for any $\varepsilon > 0$ there exists an integer $m = m(\varepsilon)$ such that for arbitrary finite intervals $I_j = [a_j, b_j] \subset \mathbb{N}, j = 1, \cdots, k$, such that

$$dist(I_i, I_j) \geq m(\varepsilon), i \neq j,$$

and any $x_1, \cdots, x_k$ in X there exists a point $x \in X$ such that

$$d(f^{p+a_j}(x), f^p(x_j)) < \varepsilon$$

for all $p = 0, \cdots, b_j - a_j$ and every $j = 1, \cdots, k$.

The following general concept of multifractal spectra is introduced in [3].

Let X be a set and $g : X \to [-\infty, +\infty]$ a function. The level sets of g

$$K_\alpha^g = \{x \in X : g(x) = \alpha\}, -\infty \leq \alpha \leq +\infty$$

are disjoint and produce a multifractal decomposition of X, that is

$$X = \bigcup_{-\infty \leq \alpha \leq +\infty} K_\alpha^g.$$

Let now G be a set function, i.e., a real function that is defined on subsets of X. Assume that $G(Z_1) \leq G(Z_2)$ if $Z_1 \subseteq Z_2$. We define the function $\mathcal{F} : [-\infty, +\infty] \to \mathbb{R}$ by $\mathcal{F}(\alpha) = G(K_\alpha^g)$. We call $\mathcal{F}$ the multifractal spectrum specified by the pair of functions (g, G), or the $(g, G)-$multifractal spectrum.

It often happens that the function g is defined only on a subset $Y \subset X$. In this case the decomposition (1) should be replaced by

$$X = \bigcup_{-\infty \leq \alpha \leq +\infty} K_\alpha^g \bigcup (X \setminus Y).$$

We still call this decomposition of X a multifractal decomposition.

The following results appeared in [16] and [7] respectively.

Theorem A. ([16]). *Let X be a compact metric space, $f : X \to X$ be a continuous map with the specification property and $\varphi \in C(X, \mathbb{R})$. Let $g(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=o}^{n-1} \varphi(f^i(x))$. For some $\alpha \in \mathbb{R}$, suppose $K_\alpha^g \neq \emptyset$, then*

$$h_{K_\alpha^g}^{PP}(f) = \sup\{h_\mu(f) : \mu \in M(X, f), \int \varphi d\mu = \alpha\},$$

where $M(X, f)$ denotes the set of all $f-$invariant probability measures on X, $h_\mu(f)$ denotes the measure-theoretic entropy of f with respect to $\mu \in M(X, f)$.

Theorem B. ([7]). *Let X be a compact metric space, $f : X \to X$ be a continuous map with the specification property and $\varphi \in C(X, \mathbb{R})$ satisfying $\inf_{\mu \in M(X,f)} \int \varphi d\mu < \sup_{\mu \in M(X,f)} \int \varphi d\mu$. Let*

$$Q_\varphi := \{x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=o}^{n-1} \varphi(f^i(x)) \quad \text{dose} \quad \text{not} \quad \text{exist.}\}$$

Then

$$h_{Q_\varphi}^{PP}(f) = h^{AKM}(f),$$

where $h^{AKM}(f)$ denotes the AKM entropy of f.

3. Topological Entropy and Lower and Upper Capacity Topological Entropy Introduced in This Paper and Its Some Basic Properties

In this section, by using the C-P structure, the topological entropy and lower and upper capacity topological entropies are introduced for a proper map.

Let X be a topological space, $f : X \to X$ a proper map and $\mathcal{U}$ an admissible cover of X. Denote by $S_m(\mathcal{U})$ the set of all strings $\mathbf{U} = \{U_{i_0}, U_{i_1}, \cdots, U_{i_{m-1}} : U_{i_j} \in \mathcal{U}\}$ of length $m = m(\mathbf{U})$. We put $S(\mathcal{U}) = \bigcup_{m\geq 0} S_m(\mathcal{U})$.

To a given string $\mathbf{U} = \{U_{i_0}, U_{i_1}, \cdots, U_{i_{m-1}}\} \in S(\mathcal{U})$ we associate the set

$$X(\mathbf{U}) = \{x \in X : f^j(x) \in U_{i_j}, j = 0, 1, \cdots, m(U) - 1\}.$$

It is easy to see that $X(\mathbf{U}) = \bigcap_{j=0}^{m(U)-1} f^{-j}U_{i_j}$, then $X(\mathbf{U})$ is an admissible open set. Define the collection of subsets

$$\mathcal{F} = \mathcal{F}(\mathcal{U}) = \{X(\mathbf{U}) : \mathbf{U} \in S(\mathcal{U})\}$$

and three functions $\xi, \eta, \psi : S(\mathcal{U}) \to \mathbb{R}^+$ as follows $\xi(\mathbf{U}) = 1$, $\eta(\mathbf{U}) = \exp(-m(\mathbf{U}))$, $\psi(\mathbf{U}) = m(\mathbf{U})^{-1}$. It is easy to verify that the set $S, \mathcal{F}$ and the functions ξ, η, and ψ satisfy the Conditions (1), (2), and (3) in Section 2 and hence they determine a C-P structure $\tau = \tau(\mathcal{U}) = (S, \mathcal{F}, \xi, \eta, \psi)$ on X. We say that a collection of strings $\mathcal{G}$ covers a set $Z \subset X$ if $\bigcup_{\mathbf{U}\in\mathcal{G}} X(\mathbf{U}) \supset Z$. For any set $Z \subset X$ and $\alpha \in \mathbb{R}$, define

$$M(Z, \alpha, \mathcal{U}, N) = \inf_{\mathcal{G}} \{\sum_{\mathbf{U}\in\mathcal{G}} \xi(\mathbf{U})\eta(\mathbf{U})^{\alpha}\} = \inf_{\mathcal{G}} \{\sum_{\mathbf{U}\in\mathcal{G}} \exp(-\alpha m(\mathbf{U}))\}$$

and the infimum is taken over all finite or countable collections of strings $\mathcal{G} \subset S(\mathcal{U})$ such that $m(\mathbf{U}) \geq N$ for all $\mathbf{U} \in \mathcal{G}$ and $\mathcal{G}$ covers Z. Defining

$$m(Z, \alpha, \mathcal{U}) = \lim_{N\to+\infty} M(Z, \alpha, \mathcal{U}, N).$$

For every real numbers α introduce

$$\underline{r}(Z, \alpha, \mathcal{U}) = \varliminf_{N\to+\infty} R(Z, \alpha, \mathcal{U}, N),$$
$$\overline{r}(Z, \alpha, \mathcal{U}) = \varlimsup_{N\to+\infty} R(Z, \alpha, \mathcal{U}, N),$$

where $R(Z, \alpha, \mathcal{U}, N) = \inf_{\mathcal{G}} \{\sum_{\mathbf{U}\in\mathcal{G}} \exp(-\alpha N)\}$ and the infimum is taken over all collections of strings $\mathcal{G} \subset S(\mathcal{U})$ such that $m(\mathbf{U}) = N$ for all $\mathbf{U} \in \mathcal{G}$ and $\mathcal{G}$ covers Z. By the definition of C-P structure, define

$$h_Z(f, \mathcal{U}) := \inf\{\alpha : m(Z, \alpha, \mathcal{U}) = 0\} = \sup\{\alpha : m(Z, \alpha, \mathcal{U}) = +\infty\},$$
$$\underline{Ch}_Z(f, \mathcal{U}) := \inf\{\alpha : \underline{r}(Z, \alpha, \mathcal{U}) = 0\} = \sup\{\alpha : \underline{r}(Z, \alpha, \mathcal{U}) = +\infty\},$$
$$\overline{Ch}_Z(f, \mathcal{U}) := \inf\{\alpha : \overline{r}(Z, \alpha, \mathcal{U}) = 0\} = \sup\{\alpha : \overline{r}(Z, \alpha, \mathcal{U}) = +\infty\}.$$

Moreover, define

$$h_Z(f) := \sup_{\mathcal{U}} h_Z(f, \mathcal{U}),$$
$$\underline{Ch}_Z(f) := \sup_{\mathcal{U}} \underline{Ch}_Z(f, \mathcal{U}),$$
$$\overline{Ch}_Z(f) := \sup_{\mathcal{U}} \overline{Ch}_Z(f, \mathcal{U}),$$

where the supremum is taken over all admissible covers of X. We call the quantities $h_Z(f), \underline{Ch}_Z(f)$ and $\overline{Ch}_Z(f)$, respectively the topological entropy and lower and upper capacity topological entropy of f on the set Z.

Remark 3.1.

(1) If X be a compact topological space, then it is easy to see that the topological entropy and the Bowen dimensional entropy[5] coincide. If X be a compact metric space, then the topological entropy coincides with the PP entropy(or see the following Theorem 4.2).

(2) It is easy to see that

$$h_Z(f) \le \underline{Ch}_Z(f) \le \overline{Ch}_Z(f), \forall Z \subset X.$$

By the basic properties of the Carathéodory-Pesin dimension[15] and definitions, we get the following three basic properties.

Proposition 3.1. *Let X be a topological space and $f : X \to X$ be a proper map, then*

(1) $h_\emptyset(f) = 0.$

(2) $h_{Z_1}(f) \le h_{Z_2}(f)$ *if* $Z_1 \subset Z_2 \subset X.$

(3) $h_Z(f) = \sup\limits_{i\ge1} h_{Z_i}(f)$, *where* $Z = \bigcup\limits_{i\ge1} Z_i, Z_i \subset X, i = 1, 2, \cdots.$

(4) *If f is a homeomorphism then* $h_Z(f) = h_{f(Z)}(f)$.

Proposition 3.2. *Let X be a topological space and $f : X \to X$ be a proper map, then*

(1) $\underline{Ch}_\emptyset(f) = \overline{Ch}_\emptyset(f) = 0.$

(2) $\underline{Ch}_{Z_1}(f) \le \underline{Ch}_{Z_2}(f)$, $\overline{Ch}_{Z_1}(f) \le \overline{Ch}_{Z_2}(f)$ *if* $Z_1 \subset Z_2 \subset X.$

(3) $\underline{Ch}_Z(f) \ge \sup\limits_{i\ge1} \underline{Ch}_{Z_i}(f)$ *and* $\overline{Ch}_Z(f) \ge \sup\limits_{i\ge1} \overline{Ch}_{Z_I}(f)$, *where* $Z = \bigcup\limits_{i\ge1} Z_i, Z_i \subset X, i = 1, 2, \cdots.$

Proposition 3.3. *Let X be a topological space, $f : X \to X$ be a proper map and $\mathcal{U}$ be an admissible cover of X, then*

$$\underline{Ch}_Z(f,\mathcal{U}) = \varliminf_{N\to+\infty} \frac{1}{N} \log \Lambda(Z,\mathcal{U},N), \overline{Ch}_Z(f,\mathcal{U}) = \varlimsup_{N\to+\infty} \frac{1}{N} \log \Lambda(Z,\mathcal{U},N),$$

where $\Lambda(Z,\mathcal{U},N)$ denotes the minimum cardinalities of all finite or countable collections of strings $\mathcal{G} \subset S(\mathcal{U})$ such that $m(\mathbf{U}) = N$ for all $\mathbf{U} \in \mathcal{G}$ and $\mathcal{G}$ covers Z.

We can also prove the following properties.

Proposition 3.4. *Let X_i be a topological space and $f_i : X_i \to X_i$ be a proper map $(i = 1, 2)$. If f_2 is a factor of f_1, i.e., there exists a continuous surjection $\pi : X_1 \to X_2$ such that $\pi \circ f_1 = f_2 \circ \pi$, then*

$$h_Z(f_1) \geq h_{\pi Z}(f_2),$$

$$\underline{Ch}_Z(f_1) \geq \underline{Ch}_{\pi Z}(f_2),$$

$$\overline{Ch}_Z(f_1) \geq \overline{Ch}_{\pi Z}(f_2), \quad \forall Z \subset X.$$

In particular, if f_1 and f_2 topological conjugate, i.e., the map π is a homeomorphism then

$$h_Z(f_1) = h_{\pi Z}(f_2),$$

$$\underline{Ch}_Z(f_1) = \underline{Ch}_{\pi Z}(f_2),$$

$$\overline{Ch}_Z(f_1) = \overline{Ch}_{\pi Z}(f_2), \quad \forall Z \subset X.$$

Proof. Let $\mathcal{U}_2$ be an admissible cover of X_2. Then

$$M(\pi Z, \alpha, \mathcal{U}_2, N) = \inf_{\mathcal{G}} \sum_{\mathbf{U} \in \mathcal{G}} \exp(-\alpha m(\mathbf{U})),$$

where the infimum is taken over all finite or countable collections of strings $\mathcal{G} \subset S(\mathcal{U}_2)$ such that $m(\mathbf{U}) \geq N$ for all $\mathbf{U} \in \mathcal{G}$ and $\mathcal{G}$ covers πZ. Put

$$\mathcal{U}_1 = \{\pi^{-1} U_i : U_i \in \mathcal{U}_2\}.$$

Then $\mathcal{U}_1$ be an admissible cover of X_1. For each string $\mathbf{U} = \{U_{i_0}, U_{i_1}, \cdots, U_{i_{m-1}}\} \in \mathcal{S}(\mathcal{U}_2)$, let $\pi^{-1}\mathbf{U} = \{\pi^{-1}U_{i_0}, \pi^{-1}U_{i_1}, \cdots, \pi^{-1}U_{i_{m-1}}\}$, then $\pi^{-1}\mathbf{U} \in \mathcal{S}(\mathcal{U}_1)$, Conversely, for each string $\mathbf{V} \in \mathcal{S}(\mathcal{U}_1)$, there is a unique string $\mathbf{U} \in \mathcal{S}(\mathcal{U}_2)$ such that $\mathbf{V} = \pi^{-1}\mathbf{U}$. Furthermore, $m(\mathbf{U}) = m(\pi^{-1}\mathbf{U}) = m(\mathbf{V})$ and

$$X(\mathbf{V}) = X(\pi^{-1}\mathbf{U}) = \pi^{-1} X(\mathbf{U}).$$

So $M(\pi Z, \alpha, \mathcal{U}_2, N) = M(Z, \alpha, \mathcal{U}_1, N)$. Letting $N \to \infty$, we have

$$m(\pi Z, \alpha, \mathcal{U}_2) = m(Z, \alpha, \mathcal{U}_1).$$

Moreover,

$$h_{\pi Z}(f_2, \mathcal{U}_2) = h_Z(f_1, \mathcal{U}_1) \leq h_Z(f_1).$$

Therefore,

$$h_{\pi Z}(f_2) \leq h_Z(f_1).$$

If π is a homeomorphism, then $\pi^{-1}f_2 = f_1\pi^{-1}$ so, by the above, we have that $h_Z(f_1) \leq h_{\pi Z}(f_2)$. Hence,

$$h_Z(f_1) = h_{\pi Z}(f_2).$$

The others can be proved in a similar fashion. ∎

Proposition 3.5. *Let X be a topological space and $f : X \to X$ a proper map, then* $h_{f(Z)}(f) = h_Z(f)$, $\underline{Ch}_{f(Z)}(f) = \underline{Ch}_Z(f)$, $\overline{Ch}_{f(Z)}(f) = \overline{Ch}_Z(f), \forall Z \subset X$.

Proof. Let $\mathcal{U}$ be an admissible cover of X and $\mathcal{G} \subset \bigcup_{m\geq N} S_m(\mathcal{U})$ cover Z, i.e., $Z \subset \bigcup_{\mathbf{U}\in\mathcal{G}} X(\mathbf{U})$. We write $\mathbf{U} = \{U_{i_0}, \cdots, U_{i_{m(\mathbf{U})-1}}\}$ for any $\mathbf{U} \in \mathcal{G}$. Then

$$\begin{aligned} f(Z) &\subset f(\bigcup_{\mathbf{U}\in\mathcal{G}} X(\mathbf{U})) = \bigcup_{\mathbf{U}\in\mathcal{G}} f(X(\mathbf{U})) \\ &\subset \bigcup_{\mathbf{U}\in\mathcal{G}} (f(U_{i_0}) \cap U_{i_1} \cap \cdots \cap f^{-m(\mathbf{U})+2}(U_{i_{m(\mathbf{U})-1}})) \\ &\subset \bigcup_{\mathbf{U}\in\mathcal{G}} (U_{i_1} \cap \cdots \cap f^{-m(\mathbf{U})+2}(U_{i_{m(\mathbf{U})-1}})). \end{aligned}$$

That is, $\{U_{i_1} \cap \cdots \cap f^{-m(\mathbf{U})+2}U_{i_{m(\mathbf{U})-1}}\}_{\mathbf{U}\in\mathcal{G}}$ covers the set $f(Z)$. Therefore,

$$M(f(Z), \alpha, \mathcal{U}, N-1) \leq e^{\alpha} M(Z, \alpha, \mathcal{U}, N).$$

Letting $N \to \infty$, then

$$e^{\alpha} m(Z, \alpha, \mathcal{U}) \geq m(f(Z), \alpha, \mathcal{U}),$$

which implies

$$h_Z(f, \mathcal{U}) \geq h_{f(Z)}(f, \mathcal{U}).$$

Then

$$h_Z(f) \geq h_{f(Z)}(f).$$

On the other hand, For any $\mathcal{G} \subset \bigcup_{m\geq N} S_m(\mathcal{U})$ covering $f(Z)$, i.e.,

$$f(Z) \subset \bigcup_{\mathbf{U}\in\mathcal{G}} X(\mathbf{U}) = \bigcup_{\mathbf{U}\in\mathcal{G}} (U_{i_0} \cap \cdots \cap f^{-m(\mathbf{U})+1}(U_{i_{m(\mathbf{U})-1}})).$$

Then

$$Z \subset f^{-1}(f(Z)) \subset \bigcup_{\mathbf{U}\in\mathcal{G}} (f^{-1}(U_{i_0}) \cap f^{-2}(U_{i_1}) \cap \cdots \cap f^{-m(\mathbf{U})}(U_{i_{m(\mathbf{U})-1}})).$$

Choose $\{U_1, \cdots, U_k\} \subset \mathcal{U}$ such that it covers Z, i.e., $Z \subset \cup_{i=1}^k U_i$. Then

$$\begin{aligned} Z &\subset (\bigcup_{j=1}^{k} U_i) \bigcap (\bigcup_{\mathbf{U}\in\mathcal{G}} (f^{-1}(U_{i_0}) \cap f^{-2}(U_{i_1}) \cap \cdots \cap f^{-m(\mathbf{U})} U_{i_{m(\mathbf{U})-1}})) \\ &= \bigcup_{\mathbf{U}\in\mathcal{G}} \bigcup_{j=1}^{k} (U_j \cap f^{-1}(U_{i_0}) \cap f^{-2}(U_{i_1}) \cap \cdots \cap f^{-m(\mathbf{U})}(U_{i_{m(\mathbf{U})-1}})). \end{aligned}$$

That is to say, $\left\{U_j \cap f^{-1}(U_{i_0}) \cap f^{-2}(U_{i_1}) \cap \cdots \cap f^{-m(\mathbf{U})}(U_{i_{m(\mathbf{U})-1}})\right\}_{1\le j\le k, \mathbf{U}\in\mathcal{G}}$ covers the set Z. Therefore,

$$M(Z, \alpha, \mathcal{U}, N+1) \le kM(f(Z), \alpha, \mathcal{U}, N).$$

Letting $N \to \infty$, we have

$$m(Z, \alpha, \mathcal{U}) \le k \cdot m(f(Z), \alpha, \mathcal{U}).$$

Thus

$$h_Z(f, \mathcal{U}) \le h_{f(Z)}(f, \mathcal{U}).$$

Moreover,

$$h_Z(f) \le h_{f(Z)}(f).$$

The others can be proved in a similar fashion. ∎

Remark 3.2. Proposition 3.4 and 3.5 extend Bowen's results [5].

Example 3.1. Let X be a topological space and $I : X \to X$ the identity map, then $h_Z(I) = \underline{Ch}_Z(I) = \overline{Ch}_Z(I) = 0, \forall Z \subset X$.

In the next example, we consider a non-trivial map and its entropies.

Example 3.2 Let $X = \mathbb{R}, f(x) = x^2$, then

$$h_Z(f) = \underline{Ch}_Z(f) = \overline{Ch}_Z(f) = 0, \forall Z \subset [0, +\infty).$$

Proof. We only need to show the case of $Z = [0, +\infty)$. It is easy to see that f is a proper map. For any admissible cover $\mathcal{U} = \{U_0, U_1, \cdots, U_{m-1}\}$ of $\mathbb{R}$, there exists a $U_p \in \mathcal{U}$ such that $\mathbb{R}\backslash U_p$ is compact, i.e., $\mathbb{R}\backslash U_p$ is a bounded closed set. Letting $K = \sup\{x : x \in \mathbb{R}\backslash U_p\}$. Since $\mathcal{U}$ is a cover of $\mathbb{R}$, assume $0 \in U_0, 1 \in U_1$, then there exist closed intervals $[a_0, b_0]$ and $[a_1, b_1]$, where $b_0 < a_1$, such that $0 \in [a_0, b_0] \subset U_0, 1 \in [a_1, b_1] \subset U_1$. By the monotonicity of f, there exists $n_0 \in \mathbb{N}$ such that $f^{n_0}(a_1) < b_0 < 1, f^{n_0}(b_1) > K$. Let $N > n_0$, we discuss it in the following four cases.

(1) For any $x \in [b_1, +\infty), f^{n_0}(x) \geq f^{n_0}(b_1) > K$, i.e., $x \in U_{i_0}, f(x) \in U_{i_1}, \cdots, f^{n_0-1}(x) \in U_{i_{n_0-1}}, f^{n_0}(x) \in U_p, \cdots, f^{N-1}(x) \in U_p$, where $i_0, i_1, \cdots, i_{n_0-1} \in \{0, 1, \cdots, m-1\}$. Then there exist at most m^{n_0} strings in $S_N(\mathcal{U})$ cover $[b_1, +\infty)$.

(2) For any $x \in [1, b_1)$, if $K \geqslant b_1$, we discuss it in the following three cases.

(2.1) $f^{N-1}(x) > K$. In this case, there exists $j \in \{1, 2, \cdots, N-1\}$ such that $f^{j-1}(x) < b_1, f^j(x) \geq b_1$, i.e., $x \in U_1, f(x) \in U_1, \cdots, f^{j-1}(x) \in U_1, f^j(x) \in U_{i_j}, \cdots, f^{j+n_0-1}(x) \in U_{i_{j+n_0-1}}, f^{j+n_0}(x) \in U_p, \cdots, f^{N-1}(x) \in U_p$, where $i_j, i_{j+1}, \cdots, i_{j+n_0-1} \in \{0, 1, \cdots, m-1\}$. Then there exist at most Nm^{n_0} strings in $S_N(\mathcal{U})$ cover these points.

(2.2) $b_1 \leq f^{N-1}(x) \leq K$. If $f^{N-2}(x) < b_1, f^{N-1}(x) \geq b_1$, i.e., $x \in U_1, f(x) \in U_1, \cdots, f^{N-2}(x) \in U_1, f^{N-1}(x) \in U_{i_{N-1}}$, where $i_{N-1} \in \{0, 1, \cdots, m-1\}$, then there exist at most m strings in $S_N(\mathcal{U})$ cover these points. If $f^{N-3}(x) < b_1, f^{N-2}(x) \geq b_1$, i.e., $x \in U_1, f(x) \in U_1, \cdots, f^{N-3}(x) \in U_1, f^{N-2}(x) \in U_{i_{N-2}}, f^{N-1}(x) \in U_{i_{N-1}}$, where $i_{N-2}, i_{N-1} \in \{0, 1, \cdots, m-1\}$, then there exist at most m^2 strings in $S_N(\mathcal{U})$ cover these points. $\cdots\cdots$. If $f^{N-(n_0+1)}(x) < b_1, f^{N-n_0}(x) \geq b_1$, i.e., $x \in U_1, f(x) \in U_1, \cdots, f^{N-(n_0+1)}(x) \in U_1, f^{N-n_0}(x) \in U_{i_{N-n_0}}, \cdots, f^{N-1}(x) \in U_{i_{N-1}}$, where $i_{N-n_0}, \cdots, i_{N-1} \in \{0, 1, \cdots, m-1\}$, then there exist at most m^{n_0} strings in $S_N(\mathcal{U})$ cover these points. If $f^{N-(n_0+2)}(x) < b_1, f^{N-(n_0+1)}(x) \geq b_1$, then $f^{N-1}(x) \geq f^{n_0}(b_1) > K$ and return to the case (2.1).

(2.3) $f^{N-1}(x) < b_1$. In this case $x \in U_1, f(x) \in U_1, \cdots, f^{N-1}(x) \in U_1$, then there exists one string in $S_N(\mathcal{U})$ cover these points. So we get that there exist at most $1 + m + \cdots + m^{n_0} + Nm^{n_0}$ strings in $S_N(\mathcal{U})$ cover $[1, b_1)$.

If $K < b_1$, similar to the above, we can consider the cases $f^{N-1}(x) > K$ and $f^{N-1}(x) \leq K < b_1$ respectively, there exist at most $1+m^{n_0}$ strings in $S_{N-1}(\mathcal{U})$ cover $[1, b_1)$.

(3) For any $x \in (a_1, 1)$, we discuss it in the following three cases

(3.1) $f^{N-1}(x) \leq b_0$. In this case there exists $j \in \{1, 2, \cdots, N-1\}$ such that $f^{j-1}(x) \geq a_1, f^j(x) < a_1$, then $x \in U_1, f(x) \in U_1, \cdots, f^{j-1}(x) \in U_1, f^j(x) \in U_{i_j}, \cdots, f^{j+n_0-1}(x) \in U_{i_{j+n_0}-1}, f^{j+n_0}(x) \in U_0, \cdots, f^{N-1}(x) \in U_0$, where $j, j+1, \cdots, j+n_0-1 \in \{0, 1, \cdots, m-1\}$. Then there exist at most Nm^{n_0} strings in $S_N(\mathcal{U})$ cover these points.

(3.2) $b_0 < f^{N-1}(x) < a_1$. If $f^{N-2}(x) \geq a_1, f^{N-1}(x) < a_1$, then $x \in U_1, f(x) \in U_1, \cdots, f^{N-2}(x) \in U_1, f^{N-1}(x) \in U_{i_{N-1}}$, where $i_{N-1} \in \{0, 1, \cdots, m-1\}$, there exist at most m strings in $S_N(\mathcal{U})$ cover these points. If $f^{N-3}(x) \geq a_1, f^{N-2}(x) < a_1$, then $x \in U_1, f(x) \in U_1, \cdots, f^{N-3}(x) \in U_1, f^{N-2}(x) \in U_{i_{N-2}}, f^{N-1}(x) \in U_{i_{N-1}}$, where $i_{N-2}, i_{N-1} \in \{0, 1, \cdots,$

$m-1\}$. Then there exist at most m^2 strings in $S_N(\mathcal{U})$ cover these points. $\cdots\cdots$. If $f^{N-(n_0+1)}(x) \geq a_1, f^{N-n_0}(x) < a_1$, then $x \in U_1, f(x) \in U_1, \cdots, f^{N-(n_0+1)}(x) \in U_1, f^{N-n_0}(x) \in U_{i_{N-n_0}}, \cdots, f^{N-1}(x) \in U_{i_{N-1}}$, where $i_{N-n_0}, \cdots, i_{N-1} \in \{0, 1, \cdots, m-1\}$. Then there exist at most m^{n_0} strings in $S_N(\mathcal{U})$ cover these points. If $f^{N-(n_0+2)}(x) \geq a_1, f^{N-(n_0+1)}(x) < a_1$, then $f^{N-1}(x) < f^{n_0}(a_1) < b_0$ and return to the case (3.1).

(3.3) $f^{N-1}(x) \geq a_1$. In this case $x \in U_1, f(x) \in U_1, \cdots, f^{N-1}(x) \in U_1$, there exists one string cover these points. So we get that there exist at most $1 + m + \cdots + m^{n_0} + Nm^{n_0}$ strings in $S_N(\mathcal{U})$ cover $[a_1, 1)$.

(4) For any $x \in [0, a_1]$, since there exists $n_0 \in \mathbb{N}$ such that $f^{n_0}(x) < b_0$, then $x \in U_{i_0}, f(x) \in U_{i_1}, \cdots, f^{n_0-1}(x) \in U_{i_{n_0-1}}, f^{n_0}(x) \in U_0, \cdots, f^{N-1}(X) \in U_0$. Hence there exist at most m^{n_0} strings in $S_{N-1}(\mathcal{U})$ cover $[0, a_1]$.

By the above four cases, there exist at most $L := 2m^{n_0} + 2(1 + m + \cdots + m^{n_0} + Nm^{n_0}) + (1 + m^{n_0})$ strings in $S_N(\mathcal{U})$ cover $Z = [0, +\infty)$. So we get that

$$R(Z, \alpha, \mathcal{U}, N) \leq L \exp(-\alpha N).$$

For any $\alpha > 0$, we have $\overline{r}(Z, \alpha, \mathcal{U}) = 0$, then $\overline{Ch}_Z(f, \mathcal{U}) \leq 0$. Moreover, $\overline{Ch}_Z(f) \leq 0$. By $0 \leq h_Z(f) \leq \underline{Ch}_Z(f) \leq \overline{Ch}_Z(f) \leq 0$, we have $h_Z(f) = \underline{Ch}_Z(f) = \overline{Ch}_Z(f) = 0$. ■

4. Some Further Properties of the Topological Entropy and Lower and Upper Capacity Topological Entropy

In this section, we give some further properties of these entropies introduced in this paper. These results show some relationships among these entropies and some classical entropies and generalize some classical results.

Theorem 4.1.

(1) *Let X be a topological space and $f : X \to X$ be a proper map. For any $Z \subset X$ and satisfy $f^{-1}(Z) = Z$, we have $\underline{Ch}_Z(f) = \overline{Ch}_Z(f)$. In particular, $h^P(f) = \underline{Ch}_X(f) = \overline{Ch}_X(f)$, Where $h^P(f)$ denotes the Patrão entropy of f.*

(2) *Under the conditions of (1), if Z is compact set, then $h_Z(f) = \underline{Ch}_Z(f) = \overline{Ch}_Z(f)$, in particular, if X is a compact space, then $h^P(f) = h^{AKM}(f) = h_X(f) = \underline{Ch}_X(f) = \overline{Ch}_X(f)$, where $h^{AKM}(f)$ denotes the AKM entropy of f.*

We can use the analogous method as that of [15] to prove this theorem, so we omit the proof.

Remark 4.1. It is easy to see that the lower and upper capacity topological entropies are generalizations of the Patrão entropy and AKM entropy. The topological entropy are generalizations of the Bowen dimensional entropy and AKM entropy.

Theorem 4.2. *Let X be a locally compact separable metric space and $f : X \to X$ be a proper map. Let d be the metric given by the restriction to X of some metric $\widetilde{d}$ on $\widetilde{X}$, the one-point compactification of X. Then for any $Z \subset X$, the following limit exist:*

$$\lim_{|\mathcal{U}|\to 0} h_Z(f,\mathcal{U}) = h_Z(f),$$

$$\lim_{|\mathcal{U}|\to 0} \overline{Ch}_Z(f,\mathcal{U}) = \overline{Ch}_Z(f),$$

$$\lim_{|\mathcal{U}|\to 0} \underline{Ch}_Z(f,\mathcal{U}) = \underline{Ch}_Z(f),$$

where $\mathcal{U}$ is any admissible cover of X.

Lemma 4.1. ([13]). *Let $f : X \to X$ be a proper map, where X is a locally compact separable metric space. Let d be the metric given by the restriction to X of some metric $\widetilde{d}$ on $\widetilde{X}$, the one-point compactification of X. Then d is an admissible metric and for any $\varepsilon > 0$ there exists an admissible cover of X, such that the diameter of this cover is less than ε.*

Proof of theorem 4.2. It is easy to see that

$$\overline{\lim_{|\mathcal{U}|\to 0}} h_Z(f,\mathcal{U}) \le \sup_{\mathcal{U}} h_Z(f,\mathcal{U}) = h_Z(f).$$

We are going to show that $\underline{\lim}_{|\mathcal{U}|\to 0} h_Z(f,\mathcal{U}) \ge h_Z(f)$. By Lemma 4.1, for any $\varepsilon > 0$ there exists an admissible cover $\mathcal{V}$ such that $|\mathcal{V}| < \varepsilon$ and $\sup_{\mathcal{U}} h_Z(f,\mathcal{U}) - \varepsilon \le h_Z(f,\mathcal{V})$. Let δ be a Lebesgue number of $\mathcal{V}$ and $\mathcal{W}$ be an admissible cover of X with $|\mathcal{W}| < \delta$. Then each element of $\mathcal{W}$ is contained in some element of $\mathcal{V}$. Hence we get that $M(Z,\alpha,\mathcal{V},N) \le M(Z,\alpha,\mathcal{W},N)$. Then $h_Z(f,\mathcal{V}) \le h_Z(f,\mathcal{W})$. Moreover, $\sup_{\mathcal{U}} h_Z(f,\mathcal{U}) - \varepsilon \le h_Z(f,\mathcal{V}) \le h_Z(f,\mathcal{W})$. Letting $|\mathcal{W}| \to 0$, we have $\sup_{\mathcal{U}} h_Z(f,\mathcal{U}) - \varepsilon \le \underline{\lim}_{|\mathcal{W}|\to 0} h_Z(f,\mathcal{W})$. By the arbitrarily of ε we have $\sup_{\mathcal{U}} h_Z(f,\mathcal{U}) \le \underline{\lim}_{|\mathcal{W}|\to 0} h_Z(f,\mathcal{W})$. Hence $\lim_{|\mathcal{U}|\to 0} h_Z(f,\mathcal{U}) = h_Z(f)$. The existence of two other limits can be proved in a similar fashion. ■

Remark 4.2. *It is easy to see that the topological entropy is a generalization of the PP entropy.*

Theorem 4.3. *Let X be a locally compact separable metric space and $f : X \to X$ be a proper map. Let $\widetilde{X}$ be the one-point compactification of X and $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ be the extension of f. Then for any $Z \subset X$, we have that*

$$h_Z(f) = h_Z^{PP}(\widetilde{f}), \underline{Ch}_Z(f) = \underline{Ch}_Z^{PP}(\widetilde{f}), \overline{Ch}_Z(f) = \overline{Ch}_Z^{PP}(\widetilde{f}).$$

Where $h_Z^{PP}(\widetilde{f})$, $\underline{Ch}_Z^{PP}(\widetilde{f})$, *and* $\overline{Ch}_Z^{PP}(\widetilde{f})$ *denote the PP entropy and lower and upper capacity PP entropy of* $\widetilde{f}$ *on set* Z *respectively.*

Proof. Let d be the metric given by the restriction to X of some metric $\widetilde{d}$ on $\widetilde{X}$ and $\mathcal{U} = \{U_0, U_1, \cdots, U_{k-1}\}$ be an admissible cover of X. Let $\widetilde{U}_i = \{y : \exists x \in U_i, \widetilde{d}(x,y) < |U_i|\}, 0 \le i \le k-1$, then $\widetilde{\mathcal{U}} = \{\widetilde{U}_0, \cdots, \widetilde{U}_{k-1}\}$ be an open cover of $\widetilde{X}$ and $|\mathcal{U}| \to 0$ implies $|\widetilde{\mathcal{U}}| \to 0$. Let $\mathcal{G} \subset \bigcup_{n \ge N} S_n(\mathcal{U})$ cover a set $Z \subset X$. For any $\mathbf{U} = \{U_{i_0}, U_{i_1}, \cdots, U_{i_{n-1}}\} \in \mathcal{G}$, define $\widetilde{\mathbf{U}} = \{\widetilde{U}_{i_0}, \widetilde{U}_{i_1}, \cdots, \widetilde{U}_{i_{n-1}}\} \in S_n(\widetilde{\mathcal{U}})$ and denote by $\widetilde{\mathcal{G}}$ the collection of all these strings, then $\widetilde{\mathcal{G}}$ covers $Z \subset \widetilde{X}$. Moreover, we have that

$$M(Z,\alpha,\mathcal{U},N) = \inf_{\mathcal{G}}\{\sum_{\mathbf{U}\in\mathcal{G}} \exp(-\alpha m(\mathbf{U}))\}$$

$$\ge \inf_{\widetilde{\mathcal{G}}}\{\sum_{\widetilde{\mathbf{U}}\in\widetilde{\mathcal{G}}} \exp(-\alpha m(\widetilde{\mathbf{U}}))\} := \widetilde{M}(Z,\alpha,\widetilde{\mathcal{U}},N).$$

Letting $N \to \infty$ we have

$$m(Z,\alpha,\mathcal{U}) \ge \widetilde{m}(Z,\alpha,\widetilde{\mathcal{U}})$$

and then $h_Z(f,\mathcal{U}) \ge h_Z^{PP}(\widetilde{f},\tilde{\mathcal{U}})$. Applying Theorem 4.2 and letting $|\mathcal{U}| \to 0$, it follows that$h_Z(f) \ge h_Z^{PP}(\widetilde{f})$.

we are going to show that $h_Z(f) \le h_Z^{PP}(\widetilde{f})$. If $\widetilde{\mathcal{U}}_{\frac{\varepsilon}{2}} = \{\widetilde{B}(\widetilde{x}_0, \frac{\varepsilon}{2}), \cdots, \widetilde{B}(\widetilde{x}_{k-1}, \frac{\varepsilon}{2})\}$ is a cover of $\widetilde{X}$, for every $\varepsilon \in (a,b)$, where $0 < a < b$. By the density of X in $\widetilde{X}$, it follows that there exist $\{x_0, \cdots, x_{k-1}\} \subset X$, such that $\widetilde{d}(x_i, \widetilde{x}_i) < \frac{\varepsilon}{2}, 0 \le i \le k-1$. If $x \in X$, we have that $\widetilde{d}(x, \widetilde{x}_i) < \frac{\varepsilon}{2}$, for some $\widetilde{x}_i \in \{\widetilde{x}_0, \cdots, \widetilde{x}_{k-1}\}$. Hence it follows that $d(x, x_i) \le \widetilde{d}(x, \widetilde{x}_i) + \widetilde{d}(x_i, \widetilde{x}_i) < \varepsilon$, showing that $\{B(x_0, \varepsilon), \cdots, B(x_{k-1}, \varepsilon)\}$ is a cover of X. Applying Lemma 4.1, we have that d is an admissible metric, then there exists $\delta \in (a,b)$ such that $\mathcal{U}_\delta := \{B(x_0, \delta), \cdots, B(x_{k-1}, \delta)\}$ is an admissible cover of X. For $a < \varepsilon_1 < \delta < b$, we have that

$$\widetilde{M}(Z,\alpha,\widetilde{\mathcal{U}}_{\frac{\varepsilon_1}{2}},N) = \inf_{\widetilde{\mathcal{G}}}\{\sum_{\widetilde{\mathbf{U}}\in\widetilde{\mathcal{G}}} \exp(-\alpha m(\widetilde{\mathbf{U}}))\}$$

$$\ge \inf_{\mathcal{G}}\{\sum_{\mathbf{U}\in\mathcal{G}} \exp(-\alpha m(\mathbf{U}))\} = M(Z,\alpha,\mathcal{U},N),$$

where the first infimum is taken over all finite or countable collections of strings $\widetilde{\mathcal{G}} \subset S(\widetilde{\mathcal{U}}_{\frac{\varepsilon_1}{2}})$ and the second infimum is taken over all finite or countable collections of strings $\mathcal{G} \subset S(\mathcal{U}_\delta)$. Letting $N \to \infty$, we have

$$\widetilde{m}(Z,\alpha,\widetilde{\mathcal{U}}_{\frac{\varepsilon_1}{2}}) \ge m(Z,\alpha,\mathcal{U}_\delta).$$

Moreover, $h_Z^{PP}(\widetilde{f},\widetilde{\mathcal{U}}_{\frac{\varepsilon_1}{2}}) \geq h_Z(f,\mathcal{U}_\delta)$. Letting $b \to 0$, then $\varepsilon_1 \to 0$ and $\delta \to 0$. Applying Theorem 4.2, it follows that $h_Z^{PP}(\widetilde{f}) \geq h_Z(f)$. Then $h_Z(f) = h_Z^{PP}(\widetilde{f})$. The other equalities can be proved in a similar fashion. ■

Remark 4.3. In [13], Patrão proved $h^P(f) = h^{AKM}(\widetilde{f})$, this is a special case of Theorem 4.3.

Example 4.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism then $h_Z(f) = 0, \forall Z \subset \mathbb{R}$.

Corollary 4.1. *Let X be a locally compact separable metric space and $f : X \to X$ be a proper map. Then we have that*

$$h_Z(f^n) = nh_Z(f), \underline{Ch}_Z(f^n) = n\underline{Ch}_Z(f), \overline{Ch}_Z(f^n) = n\overline{Ch}_Z(f), \forall Z \subset X, \forall n \in \mathbb{N}.$$

Proof. Since $h_Z^{PP}(\widetilde{f}^n) = nh_Z^{PP}(\widetilde{f}), \forall Z \subset \widetilde{X}, \forall n \in \mathbb{N}$[5]. Applying Theorem 4.3, we have $h_Z(f) = h_Z^{PP}(\widetilde{f}), h_Z(f^n) = h_Z^{PP}(\widetilde{f}^n)$. Hence

$$h_Z(f^n) = nh_Z(f), \forall Z \subset X, \forall n \in \mathbb{N}.$$

The other equalities can be proved in a similar fashion. ■

Corollary 4.2. *Let X be a locally compact separable metric space and $f : X \to X$ be a proper map. Let $g : X \to [-\infty, +\infty]$ be a function and $\widetilde{g} : \widetilde{X} \to [-\infty, +\infty]$ any extension of g. If the set functions G and $\widetilde{G}$ is defined as $G(Z) = h_Z(f)$ and $\widetilde{G}(\widetilde{Z}) = h_Z^{PP}(\widetilde{f})$ respectively, then (X, f) and $(\widetilde{X}, \widetilde{f})$ have the same multifractal spectrum specified by (g, G) and $(\widetilde{G}, \widetilde{f})$ respectively.*

Proof. Let $\alpha \in [-\infty, +\infty]$, then

$$K_\alpha^{\widetilde{g}} = \{x \in \widetilde{X} : \widetilde{g}(x) = \alpha\} = \{x \in X : g(x) = \alpha\} = K_\alpha^g,$$

or

$$K_\alpha^{\widetilde{g}} = \{x \in \widetilde{X} : \widetilde{g}(x) = \alpha\} = \{x \in X : g(x) = \alpha\} \cup \{\infty\} = K_\alpha^g \cup \{\infty\}.$$

Combining the fact that $h_{\{\infty\}}^{PP}(\tilde{f}) = 0$, Theorem 4.3 and Proposition 3.1 gives us our desired result. ■

Corollary 4.3. *Let X be a locally compact separable metric space, $f : X \to X$ be a proper map with the specification property respect to the metric that is the restriction of some metric on $\widetilde{X}$, $\varphi \in C(X, \mathbb{R})$ be bounded and can be continuously extended to $\widetilde{X}$ and satisfies $\inf_{\mu \in M(X,f)} \int \varphi d\mu < \sup_{\mu \in M(X,f)} \int \varphi d\mu$. Let*

$$Q_\varphi := \{x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=o}^{n-1} \varphi(f^i(x)) \quad \text{dose} \quad \text{not} \quad \text{exist.}\}$$

Then

$$h_{Q_\varphi}(f) = h_X(f) = h^P(f).$$

Lemma 4.2. *Let X be a locally compact separable metric space, $f : X \to X$ be a proper map with the specification property respect to the metric that is the restriction of some metric on $\widetilde{X}$. If f satisfies the specification property, then $\widetilde{f}$ satisfies the specification property too.*

Proof. Let $\widetilde{d}$ denote the metric on $\widetilde{X}$ and d the restriction of $\widetilde{d}$ to X. For any $\varepsilon > 0$ there exists an integer $m = m(\varepsilon)$ such that for arbitrary finite intervals $I_j = [a_j, b_j] \subset \mathbb{N}, j = 1, \cdots, k$, such that

$$dist(I_i, I_j) \geq m(\varepsilon), i \neq j,$$

and any $x_1, \cdots, x_k \in X = \widetilde{X} \setminus \{\infty\}$ there exists a point $x \in X$ such that

$$\widetilde{d}(f^{p+a_j}(x), f^p(x_j)) < \varepsilon$$

for all $p = 0, \cdots, b_j - a_j$ and every $j = 1, \cdots, k$.

If $x_1, \cdots, x_k \subset \widetilde{X}$ and $x_i = \infty, 1 \leq i \leq k$. Putting $\widetilde{m}(\varepsilon) = m(\frac{\varepsilon}{2})$ such that for arbitrary finite intervals $I_j = [a_j, b_j] \subset \mathbb{N}, j = 1, \cdots, k$, such that

$$dist(I_i, I_j) \geq \widetilde{m}(\varepsilon), i \neq j.$$

By the density of X in $\widetilde{X}$ and the uniform continuity of $\widetilde{f}, \widetilde{f}^2, \cdots, \widetilde{f}^{b_i - a_i}$, it follows that there exists $y_i \in X$, such that

$$\widetilde{d}(x_i, y_i) < \frac{\varepsilon}{2}, \widetilde{d}(\widetilde{f}(x_i), \widetilde{f}(y_i)) < \frac{\varepsilon}{2}, \cdots, \widetilde{d}(\widetilde{f}^{b_i - a_i}(x_i), \widetilde{f}^{b_i - a_i}(y_i)) < \frac{\varepsilon}{2}.$$

For $x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_k \in X$ there exists a point $z \in X$ such that

$$\widetilde{d}(\widetilde{f}^{p+a_j}(z), \widetilde{f}^p(x_j)) < \frac{\varepsilon}{2}, p = 0, \cdots, b_j - a_j, j = 1, \cdots, i-1, i+1, \cdots, k,$$

and

$$\widetilde{d}(\widetilde{f}^{p+a_i}(z), \widetilde{f}^p(y_i)) < \frac{\varepsilon}{2}, p = 0, \cdots, b_i - a_i.$$

Then $\widetilde{d}(\widetilde{f}^{p+a_i}(z), \widetilde{f}^p(x_i)) \leq \widetilde{d}(\widetilde{f}^{p+a_i}(z), \widetilde{f}^p(y_i)) + \widetilde{d}(\widetilde{f}^p(y_i), \widetilde{f}^p(x_i)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, p = 0, \cdots, b_i - a_i$. Hence we get that $\widetilde{f}$ satisfies the specification property. ∎

The Proof of Corollary 4.3. Let $\widetilde{\varphi} \in C(\widetilde{X}, \mathbb{R})$ be a extension of φ, then $Q_{\widetilde{\varphi}} = Q_\varphi$ or $Q_{\widetilde{\varphi}} = Q_\varphi \cup \{\infty\}$. Combining the fact that $h^{PP}_{\{\infty\}}(\widetilde{f}) = 0$, Theorem 4.3, Proposition 3.1, Lemma 4.2 and Theorem B gives us our desired result. ∎

Remark 4.4. If X is a compact metric space, under the same conditions as that of Corollary 4.3, Chen, Küpper and Shu[7] proved Theorem B. Then Corollary 4.3 is a extension of Theorem B.

Example 4.2. Let X be a locally compact separable metric space and $f : X \to X$ be a proper map. Let $\mu \in M(X, f)$ and $\xi = \{A_1, A_2, \cdots, A_k\}$ be a finite measurable partition of X. For every $n \in \mathbb{N}$, we write $\xi^n = \{A_{i_0} \cap f^{-1}A_{i_1} \cap \cdots \cap f^{-n}A_{i_n} : A_{i_j} \in \xi\}$ and denoted by $\xi^n(x)$ the element of the partition ξ^n that contains the point x. Consider the subset $Y \subset X$ consisting of all points $x \in X$ for which the limit

$$h_\mu(f, \xi, x) := \lim_{n\to\infty} -\frac{1}{n} \log \mu(\xi^n(x))$$

exists. The number $h_\mu(f, \xi, x)$ is called the $\mu-$local entropy of f at x(respect to ξ). By the Shannon-McMillan-Breman theorem, $\mu(X \setminus Y) = 0$. We define the function g on Y by $g(x) = h_\mu(f, \xi, x)$. Defining $\widetilde{\mu}(\widetilde{A}) = \mu(\widetilde{A} \cap X)$, it is immediate that $\widetilde{\mu}$ be a $\widetilde{f}-$invariant probability measures on $\widetilde{X}$, since X and $\{\infty\}$ are $\widetilde{f}-$invariant sets. Let $\widetilde{\xi} = \{A_1 \cup \{\infty\}, A_2, \cdots, A_k\}$, then $\widetilde{\xi}$ is a finite measurable partition of $\widetilde{X}$ respect to $\widetilde{\mu}$. Let $\widetilde{Y} \subset \widetilde{X}$ consisting of all points $x \in \widetilde{X}$ for which $h_{\widetilde{\mu}}(\widetilde{f}, \widetilde{\xi}, x)$ exists. We define the function $\widetilde{g}$ on $\widetilde{Y}$ by $\widetilde{g}(x) = h_{\widetilde{\mu}}(\widetilde{f}, \widetilde{\xi}, x)$. If the set functions G and $\widetilde{G}$ is defined as $G(Z) = h_Z(f)$ and $\widetilde{G}(\widetilde{Z}) = h_Z^{PP}(\widetilde{f})$ respectively, then (X, f) and $(\widetilde{X}, \widetilde{f})$ have the same multifractal spectrum specified by (g, G) and $(\widetilde{g}, \widetilde{G})$ respectively.

5. Some Variational Principles

In this section we present three variational principles involving entropies.

Theorem 5.1 *Let $f : X \to X$ be a proper map, where X is a locally compact separable metric space. Then it follows that*

$$\overline{Ch}_Z(f) = \min_d h_d^{BD}(f, Z), \quad \forall Z \subset X.$$

Where the minimum is attained whenever d is the metric given by the restriction to X of some metric $\widetilde{d}$ on $\widetilde{X}$ the one-point compacification of X and $h_d^{BD}(f, Z)$ denotes the BD entropy.

Lemma 5.1. *Let (X, d) be a metric space, then every admissible cover of X has a Lebesgue number.*

Proof. If (X, d) is compact, the existence of a Lebesgue number is a classical result. Thus assume that (X, d) is not compact. If $\mathcal{U} = \{A_1, A_2, \cdots, A_n\}$ is an admissible cover of X, there exists at least one $A_k \in \mathcal{U}$ with compact complement in X. Assume $\mathcal{U}$ has not Lebesgue numbers, i.e., for every $i \in \mathbb{N}$, there exists a set

$E_i \subset X$, such that $diam(E_i) < \frac{1}{i}$ and E_i can not contain in any element of $\mathcal{U}$. Then there exists $x_i \in E_i$ and $x_i \in X \backslash A_k$. Since $X \backslash A_k$ is compact, then there exists a convergent subsequence $\{x_{i_j}\}$ of $\{x_i\}$. If $\lim\limits_{j \to +\infty} x_{i_j} = x_0$, then $x_0 \in X \backslash A_k$. By $\mathcal{U}$ is an admissible cover of X, then there exists $A_m \in \mathcal{U}$, such that $x_0 \in A_m$. Since A_m is an open set, then there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset A_m$.

On the other hand, there exists a $N \in \mathbb{N}$ such that if $j > N$ then $x_{i_j} \in B(x_0, \frac{\varepsilon}{2}) \subset A_m$. Letting $j > N + \frac{2}{\varepsilon}$, then for any $x \in E_{i_j}$, we have that

$$d(x, x_0) \leq d(x, x_{i_j}) + d(x_{i_j}, x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $E_{i_j} \subset B(x_0, \varepsilon) \subset A_m$, then we get a contradiction. ∎

Remark 5.1. By Lemma 5.1, the (2) of the definition of admissible metric[13] can be deleted. In [18], the authors proved that every co-compact open cover of a metric space has a lebesgue number.

The proof of theorem 5.1. For every metric d induced the topology of X. Let $\mathcal{U}$ be an admissible cover of X. Applying lemma 5.1, $\mathcal{U}$ has a Lebesgue number denoted by δ. Let S be a $(n, \varepsilon)-$spanning set of Z, such that $\varepsilon < \delta$, then for every $x \in S$, there exists a string $\mathbf{U} \in S_n(\mathcal{U})$ such that $X(\mathbf{U}) \supset B_n(x, \varepsilon)$, where $B_n(x, \varepsilon)$ denote the Bowen ball. Hence we have that $\Lambda(Z, \mathcal{U}, n) \leq S_n(Z, \varepsilon)$, where $S_n(Z, \varepsilon)$ denote the smallest cardinality of all $(n, \varepsilon)-$spanning sets of Z. So $\overline{Ch}_Z(f, \mathcal{U}) = \overline{\lim\limits_{n \to +\infty}} \frac{1}{n} \log \Lambda(Z, \mathcal{U}, n) \leq \overline{\lim\limits_{n \to +\infty}} \frac{1}{n} \log S_n(Z, \varepsilon)$, Letting $\varepsilon \to 0$, we have that $\overline{Ch}_Z(f, \mathcal{U}) \leq h_d^{BD}(f, Z)$. Moreover, we have that $\overline{Ch}_Z(f) = \sup\limits_{\mathcal{U}} \overline{Ch}_Z(f, \mathcal{U}) \leq h_d^{BD}(f, Z)$. By the arbitrarily of the metric d, we have that $\overline{Ch}_Z(f) \leq \inf\limits_{d} h_d^{BD}(f, Z)$.

On the other hand, if the metric d' is given by the restriction to X of the $\widetilde{d}$ on $\widetilde{X}$. By the lemma 4.1, d' is an admissible metric, then for any $\varepsilon > 0$, there exists an admissible cover of X such that the diameter of the cover is less than ε. Let $\mathcal{U}$ be an admissible cover with $|\mathcal{U}| < +\infty$. Assume $\mathcal{G} \in S_n(\mathcal{U})$ covers Z. Fixing $x_{\mathbf{U}} \in X(\mathbf{U}), \forall \mathbf{U} \in \mathcal{G}$, then $S := \{x_{\mathbf{U}} : \mathbf{U} \in \mathcal{G}\}$ be a $(n, |\mathcal{U}|)-$spanning set of Z. Moreover, we have that $S_n(Z, |\mathcal{U}|) \leq \Lambda(Z, \mathcal{U}, n)$. Hence

$$\overline{\lim_{n \to +\infty}} \frac{1}{n} \log S_n(Z, |\mathcal{U}|) \leq \overline{\lim_{n \to +\infty}} \frac{1}{n} \log \Lambda(Z, \mathcal{U}, n) = \overline{Ch}_Z(f, \mathcal{U}).$$

Applying Theorem 4.2 and letting $|\mathcal{U}| \longrightarrow 0$, we have that $h_{d'}^{BD}(f, Z) \leq \overline{Ch}_Z(f)$. Then $\overline{Ch}_Z(f) = \min\limits_{d} h_d^{BD}(f, Z)$, and $\overline{Ch}_Z(f) = h_{d'}^{BD}(f, Z)$. ∎

Remark 5.2. In [13], Patrão proved $h^P(f) = \min\limits_{d} h_d^{BD}(f, X)$. This is a special case of Theorem 5.1.

Theorem 5.2. (Inverse variational principle). *Let X be a locally compact separable metric space, $f : X \to X$ be a proper map and $\mu \in M(X, f)$ then*

$$h_\mu(f) = \inf\{h_Z(f) : Z \subset X, \mu(Z) = 1\}.$$

where $M(X, f)$ denotes the set of all $f-$invariant probability measures on X and $h_\mu(f)$ the measure-theoretic entropy of f with respect to $\mu \in M(X, f)$.

Proof. If $\mu \in M(X, f)$, defining $\widetilde{\mu}(\widetilde{A}) = \mu(\widetilde{A} \cap X)$. It is immediate that $\widetilde{\mu} \in M(\widetilde{X}, \widetilde{f})$, since X and $\{\infty\}$ are $\widetilde{f}-$invariant sets. It is also immediate that $h_\mu(f) = h_{\widetilde{\mu}}(\widetilde{f})$. By Bowen and Pesin's result([5,15]), i.e., $h_{\widetilde{\mu}}(\widetilde{f}) = \inf\{h^{PP}_{\widetilde{Z}}(\widetilde{f}) : \widetilde{Z} \subset \widetilde{X}, \widetilde{\mu}(\widetilde{Z}) = 1\}$. If $Z \subset X$ and $\mu(Z) = 1$, then $\widetilde{\mu}(Z) = 1$. If $\widetilde{Z} \subset \widetilde{X}, \infty \in \widetilde{Z}$ and $\widetilde{\mu}(\widetilde{Z}) = 1$, then $\mu(\widetilde{Z} \cap X) = 1$ and $\widetilde{\mu}(\{\infty\}) = 0$. Applying theorem 4.3, we have that

$$\begin{aligned}
&\inf\{h^{PP}_{\widetilde{Z}}(\widetilde{f}) : \widetilde{Z} \subset \widetilde{X}, \widetilde{\mu}(\widetilde{Z}) = 1\}\\
&= \inf\{h^{PP}_{\widetilde{Z}}(\widetilde{f}) : \widetilde{Z} = Z \cup \{\infty\} \quad \text{or} \quad \widetilde{Z} = Z \subset X, \mu(Z) = 1\}\\
&= \inf\{h_Z(f) : Z \subset X, \mu(Z) = 1\}.
\end{aligned}$$

Hence, $h_\mu(f) = \inf\{h_Z(f) : Z \subset X, \mu(Z) = 1\}$. ■

Remark 5.3. Theorem 5.2 is an extension of the classical result of Bowen and Pesin ([5, 15]).

Relative to the classical variational principle[17], we call Theorem 5.2 the inverse variational principle. Recently, there are some extensions of the classical variational principle, such as [9] and [13]. Under the same conditions as that of Theorem 5.2, Patrão[13] proved a variational principle, i.e., $h^P(f) = \sup\{h_\mu(f) : \mu \in M(X, f)\}$. For the compact system, Feng and Huang[9] defined the measure theoretical lower and upper entropies and obtained some valuable variational principles for topological entropies of subsets. We can consider the extensions of these results for some proper maps. We will do this work in another paper.

Theorem 5.3. *Let X be a locally compact separable metric space, $f : X \to X$ be a proper map with the specification property respect to the metric that is the restriction of some metric on $\widetilde{X}$ to X. Let $\varphi \in C(X, \mathbb{R})$ be bounded and can be continuously extended to $\widetilde{X}$. Let $g(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=o}^{n-1} \varphi(f^i(x)) = \alpha$. If there exists some $\alpha \in \mathbb{R}$ such that $K^g_\alpha \neq \emptyset$, then*

$$h_{K^g_\alpha}(f) = \sup\{h_\mu(f) : \mu \in M(X, f), \int \varphi d\mu = \alpha\}.$$

Proof. Combining Lemma 4.2, Corollary 4.2 and Theorem A, we get the result immediately. ■

Remark 5.4. If X is a compact metric space, under the same conditions as that of Theorem 5.3, Takens and Verbitskiy[16] proved Theorem A. Then Theorem 5.3 is a extension of Theorem A.

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