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ON THE BOUNDARY BLOW-UP SOLUTIONS OF $p(x)$-LAPLACIAN EQUATIONS WITH GRADIENT TERMS

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Abstract. In this paper we investigate boundary blow-up solutions of the problem

$$\begin{cases} -\triangle_{p(x)}u + f(x,u) = \rho(x,u) + K(|x|)\,|\nabla u|^{\delta(|x|)} \text{ in } \Omega, \\ u(x) \to +\infty \text{ as } d(x,\ \partial\Omega) \to 0, \end{cases}$$

where $-\triangle_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\,\nabla u)$ is called $p(x)$-Laplacian. The existence of boundary blow-up solutions is proved and the singularity of boundary blow-up solution is also given for several cases including the case of $\rho(x,u)$ being a large perturbation (namely, $\frac{\rho(x,u(x))}{f(x,u(x))} \to 1$ as $x \to \partial\Omega$). In particular, we do not have the comparison principle.

1. INTRODUCTION

Let $\Omega = B(0,R) \subset \mathbb{R}^N (N \geq 2)$ be a bounded radial domain with $B(0,R) = \{x \in \mathbb{R}^N,\, |x| < R\}$. We consider boundary blow-up solutions of the variable exponent elliptic problem as follows:

$$\text{(P)} \quad \begin{cases} -\triangle_{p(x)}u + f(x,u) = \rho(x,u) + K(|x|)\,|\nabla u|^{\delta(|x|)} \text{ in } \Omega, \\ u(x) \to +\infty \text{ as } d(x,\ \partial\Omega) \to 0, \end{cases}$$

where $-\triangle_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\,\nabla u)$ with $\nabla u = (\partial_{x_1}u, \partial_{x_2}u, \cdots, \partial_{x_N}u)$ is so-called $p(x)-$Laplacian.

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The $p(x)-$Laplacian arises from the study of nonlinear elasticity, electrorheological fluids and image restoration etc. We refer readers to [2, 6, 42] and [51] for detailed application backgrounds. Clearly, if $p(x) \equiv p$ (a constant), (P) is a well known p-Laplacian elliptic problem; but for non-constant $p(x)$, $p(x)$-Laplacian problems are more complicated due to the non-homogeneity of $p(x)$-Laplacian. For example, if Ω is a smooth bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega) \backslash \{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)} \, dx}$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [13]), and it is also possible the first eigenvalue and eigenfunction of $p(x)$-Laplacian do not exist, even though the existence of the first eigenvalue and eigenfunction is very important in the study of elliptic problems related to p-Laplacian problems [5, 40].

There are many reference papers related to the the study of differential equations and variational problems with variable exponent, far from being complete, we refer readers to [2, 3, 10-15, 20-23, 25, 30-35, 43, 46-48, 52, 53] and references cited therein. For example, the regularity of weak solutions for differential equations with variable exponent was studied in [2] and [14], and existence of solutions for variable exponent problems was studied in a series of papers [12, 20, 30, 46, 47, 52]. In this paper, our aim is to study the existence of boundary blow-up solutions for problem (P) and the singularity of boundary blow-up solutions.

There are many papers on the boundary blow-up solutions of p-Laplacian problems [7-9, 16-18, 24, 26-29, 36-39, 41, 44, 45, 49, 50]. But the results on the boundary blow-up solutions with gradient terms are rare [1, 4, 19, 49]. In [47] and [48], the authors consider the existence and nonexistence of boundary blow-up solution for $-\triangle_{p(x)} u + f(x,u) = 0$. If $f(x,u) = (R - |x|)^{-\beta(|x|)} |u|^{q(|x|)-2} u$ is a typical form, then their main results mean that: (i) If $\frac{p(R)-\beta(R)}{q(R)-p(R)} > 0$, then it has radial boundary blow-up solutions; (ii) If $\frac{p(R)-\beta(R)}{q(R)-p(R)} < 0$, then it does not have radial boundary blow-up solution. When $p(x) \equiv p$ (a constant), $K(|x|) \equiv 0$ and $\rho(x,u) = \lambda |u|^{p-2} u$, many papers deal with the boundary blowup solutions of (P) (see [8, 16, 29, 44]), and generally speaking, the boundary blow-up solutions u satisfy $\frac{\lambda |u(x)|^{p-2} u(x)}{f(x,u(x))} \to 0$ as $x \to \partial\Omega$. In this paper, we will discuss the existence of boundary blow-up solutions of (P) for a general function $p(|x|)$ and $\rho(x,u) = b(\frac{1}{R-|x|})^{\alpha(|x|)} |u|^{\theta(|x|)-2} u$, where θ can be larger than p, and the case that $b(\frac{1}{R-|x|})^{\alpha(|x|)}$ tends to ∞ as $x \to \partial\Omega$ is admissible. Especially, when $\rho(x,u)$ is a large perturbation, we obtain the existence of boundary blow-up solutions, namely, $\frac{\rho(x,u(x))}{f(x,u(x))} \to 1$ as $x \to \partial\Omega$.

Denote

$$p^*(x) := \begin{cases} \dfrac{Np(x)}{N-p(x)}, \, p(x) < N, \\ \infty, \, p(x) \geq N. \end{cases}$$

Before stating our main results, we make the following assumptions throughout this paper with $\sigma \in [\frac{R}{2}, R)$ being a constant.

(H$_1$): $f(x,u) = a(\frac{1}{R-|x|})^{\beta(|x|)} |u|^{q(|x|)-2} u$, $\rho(x,u) = b(\frac{1}{R-|x|})^{\alpha(|x|)} |u|^{\theta(|x|)-2} u$, where $a, b \in \mathbb{R}^+$, $\beta, q, \alpha, \theta \in C^1[0,R]$, $\theta, q \geq 1$ and $\theta(r) < q(r) \leq p^*(x), \forall r \in [0,R]$.

(H$_2$): $p(x) \in C^1(\overline{\Omega})$ is a radial symmetric function, and satisfy $1 < p^- \leq p^+$, where $p^- = \inf\limits_{\Omega} p(x)$, $p^+ = \sup\limits_{\Omega} p(x)$, $p(0) > N$ and $p(r) < q(r), \forall r \in [\sigma, R]$.

(H$_3$): $K : \Omega \to \mathbb{R}$ is a continuous function and satisfies $|K(x)| \leq C_1(R-|x|)^\gamma$, as $|x| \to R^-$, and $|K(x)| \leq C_2 |x|^{\frac{\delta(x)(N-1)}{p(x)-1}}$ as $|x| \to 0$, where $\gamma \in \mathbb{R}$, $\delta \in C[0,R]$, $0 \leq \delta(x) \leq p(x)$ for $|x| \leq R$, and C_1, C_2 are generic positive constants.

We will discuss the existence of boundary blow-up solutions of (P) in the following three cases:

Case (I) $\alpha(R) - \beta(R) < s_1(q(R) - \theta(R))$, $\gamma > (s_1+1)(\delta(R)+1-p(R)) - 1$;

Case (II) $\alpha(R) - \beta(R) = s_1(q(R) - \theta(R))$, $\gamma > (s_1+1)(\delta(R)+1-p(R)) - 1$;

Case (III) $\alpha(R) - \beta(R) > s_1(q(R) - \theta(R))$, $\gamma > (s_2+1)(\delta(R)+1-p(R)) - 1$;

where $s_1 = \frac{p(R)-\beta(R)}{q(R)-p(R)}$, $s_2 = \frac{\alpha(R)-\beta(R)}{q(R)-\theta(R)}$.

Under the assumptions **(H$_1$)–(H$_3$)** as above, now our main results can be stated as follows:

Theorem 1.1. *Suppose case* (*I*) *holds, then* (*P*) *has a radial boundary blow-up solution* $u(\cdot)$ *with the singularity of* $C_0[d(x,\partial\Omega)]^{-s_1}$, *i.e.,*

$$\lim_{d(x,\partial\Omega)\to 0} \frac{u(x)}{C_0[d(x,\partial\Omega)]^{-s_1}} = 1,$$

where $s_1 = \frac{p(R)-\beta(R)}{q(R)-p(R)}$, $C_0 = [\frac{1}{a} s_1^{p(R)-1}(s_1+1)(p(R)-1)]^{\frac{1}{q(R)-p(R)}}$.

Theorem 1.2. *Suppose case* (*II*) *holds, then* (*P*) *has a radial boundary blow-up solution* $u(\cdot)$ *with the singularity of* $t_0 C_0[d(x,\partial\Omega)]^{-s_1}$, *i.e.,*

$$\lim_{d(x,\partial\Omega)\to 0} \frac{u(x)}{t_0 C_0[d(x,\partial\Omega)]^{-s_1}} = 1,$$

where s_1 *and* C_0 *are defined in Theorem 1.1, and* t_0 *is the unique positive solution of*

$$-aC_0^{q(R)-1} t^{p(R)-1} + aC_0^{q(R)-1} t^{q(R)-1} = bC_0^{\theta(R)-1} t^{\theta(R)-1}.$$

Theorem 1.3. *Suppose case* (*III*) *holds, then (P) has a radial boundary blow-up solution* $u(\cdot)$ *with the singularity of* $t_*[d(x,\partial\Omega)]^{-s_2}$, *i.e.,*

$$\lim_{d(x,\partial\Omega)\to 0}\frac{u(x)}{t_*[d(x,\partial\Omega)]^{-s_2}}=1,$$

where $s_2=\frac{\alpha(R)-\beta(R)}{q(R)-\theta(R)}$ *and* $t_*=(\frac{b}{a})^{\frac{1}{q(R)-\theta(R)}}$.

The main difficulties to prove above results are as follows. (i) The non-homogeneity of $p(x)$-Laplacian, (ii) The gradient term contained in the equation, and (iii) Lack of comparison principle.

This paper is organized as follows. First we do some preparations and prove some Lemmas which will be used to prove the theorems in Section 2. In Section 2, we present proofs of Theorems 1.1-1.3 stated as above.

2. Preliminaries

In order to deal with $p(x)$-Laplacian problem, we need introduce functional spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and properties of $p(x)$ -Laplacian which we will use later (see [11, 25]). Let

$$L^{p(\cdot)}(\Omega)=\left\{u\mid u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)}\,dx<\infty\right\},$$

with the norm

$$|u|_{p(\cdot)}=\inf\left\{\lambda>0\left|\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\le 1\right.\right\}.$$

The space $(L^{p(\cdot)}(\Omega),\ |\cdot|_{p(\cdot)})$ becomes a Banach space. We call it variable exponent Lebesgue space. Moreover, the space $(L^{p(\cdot)}(\Omega),\ |\cdot|_{p(\cdot)})$ is a separable and uniform convex Banach space (see [11], Theorem 1.10, Theorem 1.14).

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega)=\left\{u\in L^{p(\cdot)}(\Omega)\,\Big|\ |\nabla u|\in L^{p(\cdot)}(\Omega)\right\},$$

and it can be equipped with the norm

$$\|u\|=|u|_{p(\cdot)}+|\nabla u|_{p(\cdot)},\forall u\in W^{1,p(\cdot)}(\Omega).$$

$W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and uniform convex Banach spaces (see [11], Theorem 2.1).

Definition 2.1. $u\in W_{loc}^{1,p(\cdot)}(\Omega)$ is called a boundary blow-up solution of (P) if it satisfies

$$\int_Q|\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx+\int_Q f_\#(x,u,\nabla u)\varphi dx=0,\ \forall\varphi\in W_0^{1,p(\cdot)}(Q)\cap L^\infty(Q),$$

for any open domain $Q \Subset \Omega$, and $\max(k-u,0) \in W_0^{1,p(\cdot)}(\Omega)$ for every positive integer k, where

$$f_{\#}(x,u,\nabla u) = f(x,u) - \rho(x,u) - K(|x|)\,|\nabla u|^{\delta(|x|)}.$$

For any open domain $Q \Subset \Omega$, $\forall u \in W_{loc}^{1,p(\cdot)}(\Omega)$, we define

$$\Psi, L : W_{loc}^{1,p(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*,$$

as

$$\begin{aligned}\langle \Psi u, \varphi\rangle &= \int_Q (|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + f_{\#}(x,u,\nabla u)\varphi)dx, \forall \varphi \in W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q),\\ \langle Lu, \varphi\rangle &= \int_Q (|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + f(x,u)\varphi)dx, \forall \varphi \in W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q),\end{aligned}$$

then we have

Lemma 2.2. (see [[12], Theorem 3.1]). *For any open domain $Q \Subset \Omega$, if $q \le p^*$, $h \in W^{1,p(\cdot)}(Q)$ and $X = h + W_0^{1,p(\cdot)}(Q)$, then, $L : X \to (W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$ is strictly monotone.*

Let $g \in (W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$, if

$$\langle g, \varphi\rangle \ge 0,\ \forall \varphi \in W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q),\ \varphi \ge 0 \text{ a.e. in } \Omega,$$

then denote $g \ge 0$ in $(W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$; correspondingly, if $-g \ge 0$ in $(W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$, then denote $g \le 0$ in $(W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$.

Definition 2.3. Let $u \in W_{loc}^{1,p(\cdot)}(\Omega)$. If $\Psi u \ge 0$ $(\Psi u \le 0)$ in $(W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$, for any open domain $Q \Subset \Omega$, then u is called a weak super-solution (weak sub-solution) of equation (P).

Lemma 2.4. (see [[14], Theorem 1.1]). *Under the conditions of (H_1)-(H_3), if $u \in W^{1,p(\cdot)}(Q)$ is a bounded weak solution of $-\triangle_{p(\cdot)}u + f_{\#}(x,u,\nabla u) = 0$ in Q, $u = w_0$ on ∂Q , where $w_0 \in W^{1,p(\cdot)}(Q)$, $Q \subset\subset \Omega$; then $u \in C_{loc}^{1,\alpha}(Q)$, where $\alpha \in (0,1)$ is a constant.*

Here we note that if $u(x) = u(|x|) = u(r)$, a radial solution of (P), then (P) can be rewritten as follows:

$$(1)\quad \begin{aligned}&(r^{N-1}\,|u'|^{p(r)-2}\,u')' = r^{N-1}[f(r,u) - \rho(r,u) - K(r)\,|\nabla u|^{\delta(r)}], \quad r \in (0,R),\\ &u(0) = u_*, u'(0) = 0.\end{aligned}$$

In order to deal with the existence of solutions of (P), we need to do some preparation work. For any $(t,x) \in [0,R] \times \mathbb{R}^N$, denote $\varphi(t,x) = |x|^{p(t)-2}x$. It is well known that $\varphi(t,\cdot)$ is a homeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any fixed $t \in [0,R]$. For any $t \in [0,R]$, denote by $\varphi^{-1}(t,\cdot)$ the inverse operator of $\varphi(t,\cdot)$, then

$$\varphi^{-1}(t,x) = |x|^{\frac{2-p(t)}{p(t)-1}}x, \text{ for } x \in \mathbb{R}^N \backslash \{0\},\ \varphi^{-1}(t,0) = 0.$$

It is clear that $\varphi^{-1}(t,\cdot)$ is continuous and maps a bounded set into a bounded set.

Next we consider the existence of solutions for the following auxiliary weighted $p(r)$-Laplacian ordinary equation with right hand terms depending on the first order derivative

$$-(r^{N-1}\left|u'\right|^{p(r)-2}u')' + r^{N-1}f_*(r,u,r^{\frac{N-1}{p(r)-1}}u') = 0, r \in (0,R_\#), \tag{2}$$

where $R_\# \in (0,R)$, and with one of the following boundary value conditions:

$$u(0) = c,\ u(R_\#) = d. \tag{3}$$

$$\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}}u'(r) = 0,\ u(R_\#) = d. \tag{4}$$

$$\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}}u'(r) = 0,\ u'(R_\#) = d^*. \tag{5}$$

The function $f_* : [0,R_\#] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be Caratheodory, by this we mean:

(i) for almost every $t \in [0,R_\#]$ the function $f_*(t,\cdot,\cdot)$ is continuous;

(ii) for each $(x,y) \in \mathbb{R} \times \mathbb{R}$ the function $f_*(\cdot,x,y)$ is measurable on $[0,R_\#]$;

(iii) for each $\varsigma > 0$ there is a $\eta_\varsigma \in L^1([0,R_\#],\mathbb{R})$ such that, for almost every $t \in [0,R_\#]$ and every $(x,y) \in \mathbb{R} \times \mathbb{R}$ with $|x| \le \varsigma$, $|y| \le \varsigma$, one has

$$|f_*(t,x,y)| \le \eta_\varsigma(t).$$

Denote

$$C^1_\#[0,R_\#] = \{u \in C[0,R_\#] \mid u' \text{ is continuous in } (0,R_\#],\ \lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}}u'(r) \text{ exist}\},$$

$\|u\|_0 = \sup_{r\in(0,R_\#)} |u(r)|$ and $\|u\|_1 = \|u\|_0 + \|r^{\frac{N-1}{p(r)-1}}u'\|_0$. The spaces $C[0,R_\#]$ and $C^1_\#[0,R_\#]$ are equipped with the norm $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively.

Definition 2.5. Function $\phi \in C[0,R_\#]$ (resp. ψ) is called a sub-solution of (2) (resp. super-solution), if there exists $R^* \in (0,R_\#)$ such that $\phi \in C^1_\#[0,R^*]$

and $\phi \in C^1[R^*, R_\#]$ (resp. ψ), $r^{N-1}|\phi'|^{p(r)-2}\phi'(r)$ and $r^{N-1}|\psi'|^{p(r)-2}\psi'(r)$ are absolutely continuous on $(0, R^*)$ and $(R^*, R_\#)$ respectively (resp. ψ) and

$$-(r^{N-1}|\phi'|^{p(r)-2}\phi'(r))' + r^{N-1}f_*(r, \phi, r^{\frac{N-1}{p(r)-1}}\phi')$$
$$\leq 0, \ a.e. \text{ in } (0, R^*) \text{ and } (R^*, R_\#), \ (\geq)$$
$$\phi'(R^{*-}) \leq \phi'(R^{*+}). \quad (\geq)$$

u is a solution of (2) if and only if u is a sub-solution and a super-solution of (2).

Denote

$$Q_0 = \{(t, x) \mid t \in [0, R_\#], x \in [\phi(t), \psi(t)]\},$$
$$Q_1 = \{(t, x, y) \mid t \in [0, R_\#], x \in [\phi(t), \psi(t)], y \in \mathbb{R}\}.$$

We also assume that

(H$_*$) $|f_*(t, x, y)| \leq A_1(t, x)(|y|^{p(t)} + 1)$ for all $(t, x, y) \in Q_1$, where $A_1(t, x)$ is positive and continuous on Q_0.

In this section, we always assume that ϕ and ψ are a sub-solution and a super-solution of (2) respectively. In this section, the main goal is to give the following lemma 2.6-2.8.

Lemma 2.6. *If f_* is Caratheodory and satisfies (H$_*$), $\phi \leq \psi$ satisfies $\phi(0) \leq c \leq \psi(0)$, $\phi(R_\#) \leq d \leq \psi(R_\#)$, then (2) with (3) has a solution u satisfying $\phi \leq u \leq \psi$.*

Lemma 2.7. *If f_* is Caratheodory and satisfies (H$_*$), $\phi \leq \psi$ satisfies $\phi(R_\#) \leq d \leq \psi(R_\#)$ and*

$$\lim_{r \to 0^+} r^{\frac{N-1}{p(r)-1}}\phi'(r) \geq 0 \geq \lim_{r \to 0^+} r^{\frac{N-1}{p(r)-1}}\psi'(r),$$

then (2) with (4) has a solution u satisfying $\phi \leq u \leq \psi$.

Lemma 2.8. *If f_* is Caratheodory and satisfies (H$_*$), $\phi \leq \psi$ satisfies $\phi'(R_\#) \leq d^* \leq \psi'(R_\#)$ and*

$$\lim_{r \to 0^+} r^{\frac{N-1}{p(r)-1}}\phi'(r) \geq 0 \geq \lim_{r \to 0^+} r^{\frac{N-1}{p(r)-1}}\psi'(r),$$

then (2) with (5) has a solution u satisfying $\phi \leq u \leq \psi$.

Our main task in the rest of this section is to prove the Lemmas 2.6-2.8 stated as above. But we need do some preparation work before giving proofs. Now let's consider the problem

$$(r^{N-1}\varphi(r, u'(r)))' = r^{N-1}f_*(r), \tag{6}$$

with boundary value condition (3), where $f_* \in L^1$. If u is a solution of (6) with (3), by integrating (6) from 0 to r, we find that

$$r^{N-1}\varphi(r, u'(r)) = \varpi + F(r^{N-1}f_*)(r), \tag{7}$$

where
$$F(r^{N-1}f_*)(r)=\int_0^r t^{N-1}f_*(t)dt,\ \varpi=\lim_{r\to 0^+} r^{N-1}\varphi(r,u'(r)).$$

The boundary conditions imply that

$$\int_0^{R_\#}\varphi^{-1}[r,r^{1-N}(\varpi+F(r^{N-1}f_*)(r))]dr=d-c.$$

For fixed $h\in C[0,R_\#]$, we denote

$$\Lambda_h(\varpi)=\int_0^{R_\#}\varphi^{-1}[r,r^{1-N}(\varpi+h(r))]dr+c-d.$$

We have

Lemma 2.9. *The function Λ_h has the following properties*

(i) *For any fixed $h\in C[0,R_\#]$, the equation*

$$\Lambda_h(\varpi)=0 \tag{8}$$

has a unique solution $\widehat{\varpi}(h)\in\mathbb{R}$.

(ii) *The function $\widehat{\varpi}:C[0,R_\#]\to\mathbb{R}$ defined in (i), is continuous and maps bounded sets into bounded sets.*

Proof. (i) It is not difficult to check that for any fixed $h\in C[0,R_\#]$, $\Lambda_h(\cdot)$ is continuous and strictly increasing, therefore, if (8) has a solution, it must be unique.

By the assumption (H_2), $p(0)>N$, thus $r^{\frac{1-N}{p(r)-1}}\in L^1(0,R)$. Since $h\in C[0,R_\#]$, it is easy to see that

$$\lim_{\varpi\to+\infty}\Lambda_h(\varpi)=+\infty,\ \lim_{\varpi\to-\infty}\Lambda_h(\varpi)=-\infty.$$

It means the existence and boundedness of solutions of $\Lambda_h(\varpi)=0$.

In this way, we define a function $\widehat{\varpi}(h):C[0,R_\#]\to\mathbb{R}$, which satisfies

$$\int_0^{R_\#}\varphi^{-1}[r,r^{1-N}(\widehat{\varpi}(h)+h(r))]dr+c-d=0.$$

(ii) Similar to the proof of (i), $\widehat{\varpi}$ maps bounded sets into bounded sets. Next we show the continuity of $\widehat{\varpi}$. Let $\{u_n\}$ is a convergent sequence in $C[0,R_\#]$ and $u_n\to u$, as $n\to+\infty$. Then $\{\widehat{\varpi}(u_n)\}$ is a bounded sequence and therefore it contains a convergent subsequence $\{\widehat{\varpi}(u_{n_j})\}$. Let $\widehat{\varpi}(u_n)\to\varpi_0$ as $j\to+\infty$. Since

$$\int_0^{R_\#}\varphi^{-1}[r,r^{1-N}(\widehat{\varpi}(u_{n_j})+u_{n_j}(r))]dr+c-d=0,$$

letting $j \to +\infty$, we have

$$\int_0^{R_\#} \varphi^{-1}[r, r^{1-N}(\varpi_0 + u(r))]dr + c - d = 0,$$

from (i) we get $\varpi_0 = \widehat{\varpi}(u)$, it means $\widehat{\varpi}$ is continuous. This completes the proof.

Now we define $\varpi : L^1 \to \mathbb{R}$ by

$$\varpi(h) = \widehat{\varpi}(F(r^{N-1}h)).$$

Then it is clear that ϖ is a continuous function which maps bounded sets of L^1 into bounded sets of $\mathbb{R}$, and hence it is a compact continuous mapping.

Now we continue our argument previous to Lemma 2.9. By solving for u' in (7) and integrating we get

$$u(r) = u(0) + F\{\varphi^{-1}[r, r^{1-N}(\varpi(f_*) + F(r^{N-1}f_*)(r))]\}(r).$$

It is clear that $\varphi^{-1}(r, \cdot)$ is continuous and maps bounded sets into bounded sets. We define

$$M(h)(t) = F\{\varphi^{-1}[r, r^{1-N}(\varpi(h) + F(r^{N-1}h))]\}(t), \forall t \in [0, R_\#].$$

Let $N_{f_*}(u) : C^1_\#[0, R_\#] \to L^1$ be the Nemytskii operator associated to f_* defined by

$$N_{f_*}(u)(r) = f_*(r, u(r), r^{\frac{N-1}{p(r)-1}}u'(r)),\ a.e. \text{ on } [0, R_\#].$$

Then it is easy to see that

Lemma 2.10. *u is a solution of (2) with (3) if and only if u is a solution of the following abstract equation*

$$u = c + M(N_{f_*}(u)).$$

Next we present a lemma related to the operator M.

Lemma 2.11. *The operator M is continuous and maps equi-integrable sets in L^1 into relatively compact sets in $C^1_\#[0, R_\#]$.*

Proof. First we remark that $M(h)(t) \in C^1_\#[0, R_\#]$. Moreover, M is a continuous operator from L^1 to $C^1_\#[0, R_\#]$ due to the fact that

$$t^{\frac{N-1}{p(t)-1}}M(h)'(t) = \varphi^{-1}[t, (\varpi(h) + F(t^{N-1}h)], \forall t \in [0, R_\#].$$

Now suppose U is an equi-integrable set in L^1, then there exists $\eta(\cdot) \in L^1$, such that, for any $u \in U$

$$|u(t)| \le \eta(t) \text{ a.e. in } [0, R_\#].$$

Next we show that $\overline{M(U)} \subset C^1_\#[0, R_\#]$ is a compact set.

Let $\{u_n\}$ be a sequence in $M(U)$, then there exists a sequence $\{h_n\} \in U$ such that $u_n = M(h_n)$. For $t_1,\ t_2 \in [0, R_\#]$, we have

$$\left|F(r^{N-1}h_n)(t_1) - F(r^{N-1}h_n)(t_2)\right| \le R_\# \int_{t_1}^{t_2} |\ \eta(t)|\ dt.$$

Hence the sequence $\{F(r^{N-1}h_n)\}$ is uniformly bounded and equi-continuous, then there exists a subsequence of $\{F(r^{N-1}h_n)\}$ which is convergent in $C[0, R_\#]$, and for simplicity we still denote the subsequence by $\{F(r^{N-1}h_n)\}$. Since the operator ϖ is bounded and continuous, we can choose a convergent subsequence of $\{\varpi(h_n) + F(r^{N-1}h_n)\}$ in $C[0, R_\#]$ which we still denote by $\{\varpi(h_n) + F(r^{N-1}h_n)\}$. Then we have

$$t^{N-1}\varphi(t, (M(h_n))'(t) = \varpi(h_n) + F(r^{N-1}h_n)$$

is convergent in $C[0, R_\#]$. Note that

$$M(h_n)(t) = F\{r^{\frac{1-N}{p(r)-1}}\varphi^{-1}[r, (\varpi(h_n) + F(r^{N-1}h_n)]\}(t),\ \forall t \in [0, R_\#].$$

Due to the continuity of φ^{-1}, $M(h_n)$ is convergent in $C[0, R_\#]$. Thus we conclude that $\{u_n\}$ is convergent in $C^1_\#[0, R_\#]$. The proof is completed.

Lemma 2.12. *Let $\phi, \psi \in C[0, R_\#]$ be a sub-solution and a super-solution of (2), respectively; and satisfy $\phi(t) \le \psi(t)$ for any $t \in [0, R_\#]$. Then there exists a positive constant L (which depends on A_1, p) such that for any solution y of (2) with (3) and $\phi(t) \le y(t) \le \psi(t)$, we have $\|t^{\frac{N-1}{p(t)-1}}y'\|_0 \le L$.*

Proof. Denote

$$\begin{aligned}
\mu_0 &= 4\int_0^{R_\#} t^{N-1}[1 + A_1(t, y(t))]dt,\\
a_0 &= \max\{r^{\frac{N-1}{p(r)-1}} \mid r \in [0, R_\#]\},\\
\sigma &= \max\{\psi(s) - \phi(t) \mid t, s \in [0, R_\#]\},\\
\varkappa &= \max\{t^{\frac{(N-1)p(t)}{p(t)-1}} A_1(t, x) \mid (t, x) \in Q_0\}.
\end{aligned}$$

Then there exists a $t_0 \in (0, R_\#)$ such that

$$\left|t_0^{\frac{N-1}{p(t_0)-1}}y'(t_0)\right| \le a_0\left|y'(t_0)\right| \le a_0\frac{\sigma}{R_\#}.$$

Here we note that there exist positive numbers σ_1 and $N_1 \ge 1 + (a_0\frac{\sigma}{R_\#})^{p(t_0)-1}$ such that

$$\int_{\sigma_1}^{N_1} \frac{1}{w}dw > \varkappa\sigma + \mu_0.$$

Now suppose our conclusion is not true. Without loss of generality, we may assume that there exists $t_\# \in [0, R_\#]$ such that $t_\#{}^{N-1} |y'(t_\#)|^{p(t_\#)-2} y'(t_\#) \geq \sigma_1 + N_1$. Since $r^{N-1} |y'|^{p(r)-2} y'(r)$ is absolutely continuous, there exists $[t_1, t_2] \subset [0, R_\#]$ such that

$$r^{N-1} \left|y'\right|^{p(r)-2} y'(r) \geq 1 + (a_0 \frac{\sigma}{R_\#})^{p(t_0)-1} \text{ on } [t_1, t_2]$$

and either

$$t_1^{N-1} \left|y'\right|^{p(t_1)-2} y'(t_1) = \sigma_1,\ t_2^{N-1} \left|y'\right|^{p(t_2)-2} y'(t_2) = N_1$$

or

$$t_1^{N-1} \left|y'\right|^{p(t_1)-2} y'(t_1) = N_1,\ t_2^{N-1} \left|y'\right|^{p(t_2)-2} y'(t_2) = \sigma_1.$$

Without loss of generality, we assume the former case happens. Hence

$$\begin{aligned} \varkappa\sigma + \mu_0 &< \left| \int_{\sigma_1}^{N_1} \frac{1}{w} dw \right| \\ &= \left| \int_{t_1}^{t_2} \frac{(r^{N-1} |y'|^{p(r)-2} y')'}{r^{N-1} |y'|^{p(r)-2} y'} dr \right| \\ &= \int_{t_1}^{t_2} \left| \frac{r^{N-1} f_*(r, x, r^{\frac{N-1}{p(r)-1}} y')}{r^{N-1} |y'|^{p(r)-2} y'} \right| dr \\ &\leq \int_{t_1}^{t_2} r^{\frac{(N-1)p(r)}{p(r)-1}} A_1(r, y(r)) \left|y'\right| dr + \mu_0 \\ &\leq \varkappa\sigma + \mu_0, \end{aligned}$$

which is a contradiction. The proof is completed.

Next we consider an auxiliary problem of the form

$$\text{(SBVP)} \quad \begin{aligned} (r^{N-1} |u'|^{p(r)-2} u')' &= r^{N-1} f_*(r, R(r,u), R_1[r^{\frac{N-1}{p(r)-1}} u']) + r^{N-1} R_2(r,u) \\ &\stackrel{def}{=} r^{N-1} \widetilde{f}(r, u, r^{\frac{N-1}{p(r)-1}} u'),\ r \in (0, R_\#), \end{aligned}$$

where

$$R(t,u) = \begin{cases} \psi(t), & u(t) > \psi(t), \\ u\ , & \phi(t) \leq u(t) \leq \psi(t), \\ \phi, & u(t) < \phi(t), \end{cases}$$

$$R_1[y] = \begin{cases} L_1, & y > L_1, \\ y, & |y| \leq L_1, \\ -L_1, & y < -L_1, \end{cases}$$

with $L_1 = 1 + \max\{L, \sup\limits_{r \in (0,R_\#)} \left| r^{\frac{N-1}{p(r)-1}} \psi'(r) \right|, \sup\limits_{r \in (0,R_\#)} \left| r^{\frac{N-1}{p(r)-1}} \phi'(r) \right| \}$, and

$$R_2(t,u)=\begin{cases} e(t,u)\dfrac{u-\psi(t)}{2+u^2+\psi^2(t)}, & u(t)>\psi(t),\\ 0, & \phi(t)\le u(t)\le \psi(t),\\ e(t,u)\dfrac{u-\phi(t)}{2+u^2+\phi^2(t)}, & u(t)<\phi(t),\end{cases}$$

where $e(t,u)=1+A_1(t,R(t,u))$.

Lemma 2.13. *Let $\phi,\psi\in C[0,R_\#]$ be a sub-solution and a super-solution of (2), respectively; and satisfy $\phi(t)\le\psi(t)$ for any $t\in[0,R_\#]$, $\phi(0)\le c\le\psi(0)$, $\phi(R_\#)\le d\le\psi(R_\#)$. Let $u(t)$ be a solution of SBVP with (3), then $\phi(t)\le u(t)\le\psi(t)$ for any $t\in[0,R_\#]$.*

Proof. We only prove that $u(t)\le\psi(t)$ for any $t\in[0,R_\#]$. For the case of $\phi(t)\le u(t)$ for any $t\in[0,R_\#]$, the argument is similar and thus it is omitted.

First we note that u satisfies the boundary value condition $\phi(0)\le c=u(0)\le\psi(0)$, $\phi(R_\#)\le d=u(0)\le\psi(R_\#)$. Assume that $u(t)>\psi(t)$ for some $t\in(0,R_\#)$, then $u(t)-\psi(t)$ achieves its positive maximum at t_0, i.e., there exist a $t_0\in(0,R_\#)$ and a positive number δ such that $u(t_0)=\psi(t_0)+\delta$, $u(t)\le\psi(t)+\delta$ for any $t\in[0,R_\#]$.

At first, we may assume that $t_0=R^*$ (recall R^* is defined in the Definition 2.5). We will prove the result according to the following three cases:

Case (a) $\psi'(R^{*-})=\psi'(R^{*+})$;

Case (b) $u'(R^*)<\psi'(R^{*-})$;

Case (c) $u'(R^*)\ge\psi'(R^{*-})$ and $\psi'(R^{*-})\ne\psi'(R^{*+})$.

Suppose **Case (a)** holds. Then $\psi\in C^1_\#[0,R_\#]$. Hence

$$t_0^{\frac{N-1}{p(t_0)-1}}u'(t_0)=t_0^{\frac{N-1}{p(t_0)-1}}\psi'(t_0). \tag{9}$$

There exists a positive number η such that $u(t)>\psi(t)$ for any $t\in J:=(t_0-\eta,t_0+\eta)\subset[0,R_\#]$. From the definition of ψ,u and $\widetilde{f}$ we conclude that

$$\begin{aligned}(r^{N-1}\left|\psi'\right|^{p(r)-2}\psi')' &\le r^{N-1}f_*(r,\psi,r^{\frac{N-1}{p(r)-1}}\psi')=r^{N-1}\widetilde{f}_*(r,\psi,r^{\frac{N-1}{p(r)-1}}\psi')\\ &< r^{N-1}\widetilde{f}_*(r,u,r^{\frac{N-1}{p(r)-1}}u') \text{ on } [t_0-\eta_1,t_0+\eta_1],\end{aligned}$$

where $\eta_1\in(0,\eta)$ is small enough. For any $r\in(t_0,t_0+\eta_1]$, we have

$$\begin{aligned}&\int_{t_0}^{r}(t^{N-1}\left|\psi'\right|^{p(t)-2}\psi')'dt\\ &<\int_{t_0}^{r}t^{N-1}\widetilde{f}_*(t,u,t^{\frac{N-1}{p(t)-1}}u')dt=\int_{t_0}^{r}(t^{N-1}\left|u'\right|^{p(t)-2}u')'dt.\end{aligned} \tag{10}$$

From (9) and (10) we have

$$\left|\psi'\right|^{p(r)-2}\psi' < \left|u'\right|^{p(r)-2}u' \text{ on } (t_0, t_0+\eta_1],$$

it means that

$$(\psi+\delta)' < u' \text{ on } (t_0, t_0+\eta_1],$$

which is a contradiction to the definition of t_0, so $u(t) \leq \psi(t)$ for any $t \in [0, R_{\#}]$.

Suppose **Case (b)** holds. Since $u'(R^*) < \psi'(R^{*-})$, it is easy to see that $u'(t) < \psi'(t)$ in $(R^* - \iota, R^*)$ for some constant $\iota > 0$. Then $u(t) - \psi(t)$ is larger than $u(R^*) - \psi(R^*)$ in $(R^* - \iota, R^*)$, which is a contradiction.

Suppose **Case (c)** holds. We have $u'(R^*) \geq \psi'(R^{*-}) > \psi'(R^{*+})$, and we can see that $u'(t) > \psi'(t)$ in $(R^*, R^* + \iota)$ for some constant $\iota > 0$, then $u(t) - \psi(t)$ is larger than $u(R^*) - \psi(R^*)$ on $(R^*, R^* + \iota)$, which is a contradiction.

If $t_0 \neq R^*$, then $u(t) - \psi(t)$ is differentiable at t_0. Since $u(t) - \psi(t)$ achieves its positive maximum at t_0, (9) is valid. By repeating the proof of **Case (a)**, we can also get a contradiction, so $u(t) \leq \psi(t)$ for any $t \in [0, R_{\#}]$. The proof of Lemma 2.13 is completed.

Next we present proofs of Lemma 2.6, Lemma 2.7 and Lemma 2.8.

Proof of Lemma 2.6. At first we note that $\widetilde{f}(\cdot,\cdot,\cdot)$ satisfies (H_*). It follows from Lemma 2.13 that the solution u of SBVP with (3) satisfies $\phi(\cdot) \leq u(\cdot) \leq \psi(\cdot)$. From the proof of Lemma 2.12, we can see that the solution u of SBVP with (3) satisfies $\| t^{\frac{N-1}{p(t)-1}} u' \|_0 \leq L_1$, thus we only need to prove the existence of solutions for SBVP with (3). We remark here that u is a solution of SBVP with (3) if and only if u is a solution of

$$u = \Phi_{\widetilde{f}_*}(u) := c + M(N_{\widetilde{f}_*}(u)).$$

Denote

$$C^1_{c,d} = \{u \in C^1_{\#}[0, R_{\#}] \mid u(0) = c \text{ and } u(R_{\#}) = d\}.$$

Since $N_{\widetilde{f}_*}(u)$ maps $C^1_{c,d}$ into a set of equi-integrable functions in $L^1(0, R_{\#})$, and thus similar to the proof of Lemma 2.11, we can conclude that M maps sets of equi-integrable functions in $L^1(0, R_{\#})$ into relatively compact sets in $C^1_{c,d}$, then $\Phi_{\widetilde{f}_*}(u)$ is compact and continuous.

For any $u \in C^1_{c,d}$, we have $\Phi_{\widetilde{f}_*}(u) \in C^1_{c,d}$. Moreover, $\Phi_{\widetilde{f}_*}(C^1_{c,d})$ is bounded. Then it follows from the Schauder fixed point theorem that $\Phi_{\widetilde{f}_*}(u)$ has at least one fixed point u in $C^1_{c,d}$. Then u is a solution of SBVP with (3). Proof of Lemma 2.6 is completed.

Proof of Lemma 2.7. Let d be a fixed constant satisfying $\phi(R_{\#}) \leq d \leq \psi(R_{\#})$.

We claim that there are two sequences $\{u_n\}$ and $\{v_n\}$, all of them are solutions of (2) with the boundary value condition (3), and satisfy

$$\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} u_n'(r) > 0 > \lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} v_n'(r), \tag{11}$$

$$u_n(t) \le v_n(t),\ [u_{n+1}(t), v_{n+1}(t)] \subseteq [u_n(t), v_n(t)],\ t \in [0, R_\#], \tag{12}$$

$$u_n(R_\#) = d = v_n(R_\#), \tag{13}$$

and

$$v_{n+1}(0) - u_{n+1}(0) = \frac{v_n(0) - u_n(0)}{2}. \tag{14}$$

By Lemma 2.12, both sequences $\{u_n(t)\}$ and $\{v_n(t)\}$ are bounded in $C^1_\#[0, R_\#]$. Then it follows that $\{\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} u_n'(r)\}$ is a bounded set, and $\{\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} u_n'(r)\}$ has a convergent subsequence. Note that $\{u_n(t)\}$ are solutions of (2), and satisfy

$$r^{N-1}\varphi(r, u_n'(r)) = \varpi_n + F(r^{N-1}N_{f_*}(u_n))(r),$$

where

$$F(r^{N-1}N_{f_*}(u_n))(r) = \int_0^r r^{N-1}N_{f_*}(u_n)dt,\ \varpi_n = \lim_{r\to 0^+} r^{N-1}\varphi(r, u_n'(r)).$$

Similar to the proof of Lemma 2.12, $\{u_n(t)\}$ has a convergent subsequence $\{u_{n_i}(t)\}$ in $C^1_\#[0, R_\#]$, and then $\{\varpi_n\}$ is bounded. We may assume that $u_{n_i}(t) \to u(t)$ in $C^1_\#[0, R_\#]$ and $v_{n_j}(t) \to v(t)$ in $C^1_\#[0, R_\#]$. It is easy to see that $u(t) \le v(t)$ and both are solutions of (2) with the boundary value condition (4).

It only remain to prove the existence of $\{u_n\}$ and $\{v_n\}$, which are the solutions of (2) with the boundary value condition (3), and satisfy (11)-(14).

According to Lemma 2.6, equation (2) with boundary value condition

$$u_1(0) = \phi(0),\ u_1(R_\#) = d,$$

has a solution u_1 such that

$$\phi(t) \le u_1(t) \le \psi(t),\ t \in [0, R_\#].$$

Since $u_1(0) = \phi(0)$, we can see that

$$u_1(r_n) - \phi(r_n) = (u_1'(\xi_n) - \phi'(\xi_n))r_n \ge 0,\ \text{where } \xi_n \in (0, r_n).$$

Thus

$$u_1'(\xi_n) - \phi'(\xi_n) \ge 0,\ \text{where } \xi_n \to 0^+.$$

Since $\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} u_1'(r)$ and $\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} \phi'(r)$ exist, we have

$$\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} u_1'(r) \geq \lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} \phi'(r).$$

We may assume that $\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} u_1'(r) > 0$, or else we get a solution for (2) with the boundary value condition (4).

Similarly, equation (2) with the following boundary value condition

$$v_1(0) = \psi(0),\ v_1(R_\#) = d,$$

has a solution v_1 such that

$$u_1(t) \leq v_1(t) \leq \psi(t),\ t \in [0, R_\#],$$

which satisfies

$$\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} v_1'(r) \leq \lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} \psi'(r) \leq 0.$$

Obviously, $u_1(t)$ and $v_1(t)$ are a sub-solution and a super-solution of equation (2) with (4) respectively.

According to Lemma 2.6, equation (2) with the following boundary value condition

$$u(0) = \frac{u_1(0) + v_1(0)}{2},\ u(R_\#) = d,$$

has a solution y such that

$$u_1(t) \leq y(t) \leq v_1(t),\ t \in [0, R_\#].$$

We may assume that $\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} y(r) \neq 0$, or else we get a solution for (2) with the boundary value condition (4).

If $\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} y(r) > 0$, then denote $u_2(t) = y(t)$ and $v_2(t) = v_1(t)$; if $\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}}$ $y(r) < 0$, then denote $v_2(t) = y(t)$ and $u_2(t) = u_1(t)$. It is easy to see that $u_2(t)$ and $v_2(t)$ both are solutions of (2) and satisfy

$$\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} u_2'(r) > 0 > \lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} r v_2'(r),$$

$$u_2(t) \leq v_2(t),\ [u_2(t), v_2(t)] \subseteq [u_1(t), v_1(t)],\ \forall t \in [0, R_\#],$$

$$u_2(R_\#) = d = v_2(R_\#),$$

and

$$v_2(0) - u_2(0) = \frac{v_1(0) - u_1(0)}{2}.$$

By repeating the steps as above, we get the existence of $\{u_n\}$ and $\{v_n\}$, which are the solutions of (2) with the boundary value condition (3), and satisfy (11)-(14). Proof of Lemma 2.7 is completed.

Proof of Lemma 2.8 . The idea is similar to the proof of Lemma 2.7.

By Lemma 2.7, for any constant d satisfying $\phi(R_\#) \le d \le \psi(R_\#)$, (2) with the boundary value condition (4) has a solution u satisfying $\phi \le u \le \psi$. Let $d = \phi(R_\#)$, (2) with (4) has a solution $\underline{u}_1$ satisfying $\phi \le \underline{u}_1 \le \psi$. Obviously, $\underline{u}_1'(R_\#) \le \phi'(R_\#)$. Let $d = \psi(R_\#)$, (2) with (4) has a solution $\overline{u}_1$ satisfying $\underline{u}_1 \le \overline{u}_1 \le \psi$. Moreover, $\overline{u}_1'(R_\#) \ge \psi'(R_\#)$. Let $d = \frac{\phi(R_\#)+\psi(R_\#)}{2}$, then (2) with (4) has a solution y satisfy $\underline{u}_1 \le y \le \overline{u}_1$. If $y'(R_\#) = d^*$, then it is a solution of (2) with the boundary value condition (5). If $y'(R_\#) > d^*$, then denote $\underline{u}_2 = \underline{u}_1$, $\overline{u}_2 = y$. If $y'(R_\#) < d^*$, then denote $\underline{u}_2 = y$, $\overline{u}_2 = \overline{u}_1$. Similar to the proof of Lemma 2.7, we can get the existence of solutions of (2) with (5). Proof of Lemma 2.8 is completed.

Next we finish this section with the following lemma.

Lemma 2.14. *If $\psi(r)$ and $\phi(r)$ are a super-solution and a sub-solution of (1), respectively, and satisfy $\psi'(r) \to 0$ and $\phi'(r) \to 0$ as $r \to 0$, then $\psi(|x|)$ and $\phi(|x|)$ are a super-solution and a sub-solution of (P), respectively. Moreover, if u is a solution of (1) with $\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} u'(r) = 0$ and $u(r) \to \infty$ as $r \to R^-$, then it is a solution of (P).*

Proof. At first, we prove that $\phi(|x|)$ is a sub-solution of (P). Denote

$$\Omega_1 = \{x \in \Omega \mid |x| < R^*\}, \Omega_2 = \{x \in \Omega \mid |x| > R^*\}.$$

For nonnegative radial symmetric function $w = w(|x|) \in C_0^1(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega \{|\nabla\phi|^{p(x)-2}\nabla\phi\nabla w + f(x,\phi)w - \rho(x,\phi)w - K(|x|)\left|\phi'\right|^{\delta(|x|)} w\}dx \\
&= \int_{\Omega_1} \{|\nabla\phi|^{p(x)-2}\nabla\phi\nabla w + f(x,\phi)w - \rho(x,\phi)w - K(|x|)\left|\phi'\right|^{\delta(|x|)} w\}dx \\
&\quad + \int_{\Omega_2} \{|\nabla\phi|^{p(x)-2}\nabla\phi\nabla w + f(x,\phi)w - \rho(x,\phi)w - K(|x|)\left|\phi'\right|^{\delta(|x|)} w\}dx \\
&= J_1 + J_2.
\end{aligned}$$

Let $\xi_n \in C^1[0, R^*]$ satisfy $\xi_n(r) = 0$ for $r \in [\frac{1}{2n+1}R^*, R^*(1 - \frac{1}{2n+1})]$, and $\xi_n(r) = 1$ for $r \in [0, \frac{1}{4n+2}R^*] \cup [R^*(1 - \frac{1}{4n+2}), R^*]$. By the definition of ϕ, we have

$$\begin{aligned}
J_1 = &\lim_{n\to\infty} \int_{\Omega_1} \{|\nabla\phi|^{p(x)-2}\nabla\phi\nabla(1-\xi_n)w + f(x,\phi)(1-\xi_n)w \\
&-\rho(x,\phi)(1-\xi_n)w - K(|x|)\left|\phi'\right|^{\delta(|x|)}(1-\xi_n)w\}dx \\
&+\lim_{n\to\infty} \int_{\Omega_1} \{|\nabla\phi|^{p(x)-2}\nabla\phi\nabla(\xi_n w) + f(x,\phi)\xi_n w \\
&-\rho(x,\phi)\xi_n w - K(|x|)\left|\phi'\right|^{\delta(|x|)}\xi_n w\}dx
\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{n\to\infty}\int_{\Omega_1} |\nabla\phi|^{p(x)-2}\nabla\phi\nabla(\xi_n w)dx \\
&= \lim_{n\to\infty}\int_{\Omega_1} w|\nabla\phi|^{p(x)-2}\nabla\phi\nabla\xi_n dx + \lim_{n\to\infty}\int_{\Omega_1}\xi_n|\nabla\phi|^{p(x)-2}\nabla\phi\nabla w dx \\
&= \int_{\partial\Omega_1} |\phi'(R^{*-})|^{p(R^*)-2}\phi'(R^{*-})w(R^*)dS.
\end{aligned}
$$

and

$$
\begin{aligned}
J_2 &= \int_{\Omega_2}\{-\triangle_{p(x)}\phi + f(x,\phi) - \rho(x,\phi) - K(|x|)\left|\phi'\right|^{\delta(|x|)}\}wdx \\
&\quad + \int_{\partial\Omega_2}(|\nabla\phi|^{p-2}\nabla\phi\cdot n_2)wdS \\
&\le \int_{\partial\Omega_2}(|\nabla\phi|^{p(x)-2}\nabla\phi\cdot n_2)w(R^*)dS \\
&= -\int_{\partial\Omega_1}|\phi'|^{p(R^*)-2}\phi'(R^{*+})w(R^*)dS,
\end{aligned}
$$

where n_2 is the unit outer normal of $\partial\Omega_2$.

Then

$$
\begin{aligned}
&\int_{\Omega}\{|\nabla\phi|^{p(x)-2}\nabla\phi\nabla w + f(x,\phi)w - \rho(x,\phi)w - K(|x|)\left|\phi'\right|^{\delta(|x|)}w\}dx \\
&\le \int_{\partial\Omega_1}|\phi'(R^{*-})|^{p(R^*)-2}\phi'(R^{*-})w(R^*)dS \\
&\quad - \int_{\partial\Omega_1}|\phi'(R^{*+})|^{p(R^*)-2}\phi'(R^{*+})w(R^*)dS \\
&= \int_{\partial\Omega_1}[|\phi'(R^{*-})|^{p(R^*)-2}\phi'(R^{*-}) - |\phi'(R^{*+})|^{p(R^*)-2}\phi'(R^{*+})]w(R^*)dS \le 0.
\end{aligned}
$$

The last inequality follows from definition of ϕ. Thus ϕ is a sub-solution of (P). Similarly, we can prove that ψ is a super-solution of (P).

If u is a solution of (1) with $\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}}u'(r) = 0$, then we have

$$
\begin{aligned}
\left|\varphi(r,u_n'(r))\right| &= \left|\frac{1}{r^{N-1}}\int_0^r t^{N-1}N_{f_\#}(u_n)(t)dt\right| \\
&\le \frac{1}{r^{N-1}}\int_0^r t^{N-1}\left|N_{f_\#}(u_n)(t)\right|dt \\
&\le \int_0^r \left|N_{f_\#}(u_n)(t)\right|dt \to 0 \text{ as } r\to 0.
\end{aligned}
$$

Thus u is a solution of (P). Proof of Lemma 2.14 is completed.

3. Proofs of Theorems 1.1-1.3

In this section, we will discuss the existence of boundary blow-up solutions of (P) in Case (I)–Case (III) as stated in section 1 and then prove Theorems 1.1–1.3.

The method is the sub-super-solution method, that means that we will construct a super-solution g and a sub-solution v of (P) respectively, which satisfies $g \geq v$. Let $D_j = \{x \mid |x| < r_j^* := (1 - \frac{1}{j+1})R\}$ $(j = 1, 2, \cdots)$ be local domains of Ω. We will prove the existence of radial solution u_j of the following problem

$$\begin{cases} -\triangle_{p(x)} u_j + f(x, u_j) = \rho(x, u_j) + K(|x|) \left| u_j' \right|^{\delta(|x|)}, \text{ in } D_j, \\ u_j(x) = v(|x|, s_1, \epsilon), \text{ for } x \in \partial D_j, \end{cases}$$

which satisfy $g \geq u_j \geq v$. Similar to the proof of Theorem 3.3 of [47], the solution sequence $\{u_j\}$ on local domains has a subsequence converging to u which is a boundary blow-up solution of (P).

3.1. Case (I)

At first, we construct a super-solution of (P). Assume $q(r) - p(r) \geq \frac{1}{n_0}$ for $r \in [\sigma, R]$, where $n_0 > 3$ is an integer. Define a function $g(r, s, \epsilon)$ on $[0, R)$ as

$$g(r,s,\epsilon) =$$

$$\begin{cases} C(R-r)^{-s} + k, \ R_0 \leq r < R, \\ C(R-R_0)^{-s} + k - \int_r^{R_0} [Cs(R-R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}} [\frac{(R_0)^{N-1}}{t^{N-1}} \sin \varepsilon(t-\sigma)]^{\frac{1}{p(t)-1}} dt, \\ \qquad \sigma < r < R_0 , \\ C(R-R_0)^{-s} + k - \int_\sigma^{R_0} [Cs(R-R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}} [\frac{(R_0)^{N-1}}{t^{N-1}} \sin \varepsilon(t-\sigma)]^{\frac{1}{p(t)-1}} dt , \\ \qquad r \leq \sigma, \end{cases}$$

where s is a positive constant, $R_0 \in (\sigma, R)$ and $R - R_0$ is small enough, $\varepsilon = \frac{\pi}{2(R_0-\sigma)}$, $\epsilon \in (0, (4n_0^2)^{\frac{-1}{n_0-2}})$. k is a sufficiently large constant determined later in the proof of Lemma 3.1 and

$$C = C_\epsilon = C_\epsilon(s) = (1+\epsilon)[\frac{1}{a} s^{p(R)-1}(s+1)(p(R)-1)]^{\frac{1}{q(R)-p(R)}}. \tag{15}$$

Here we note that for any positive constant s, $g(\cdot, s, \epsilon) \in C^1[0, R)$.

Lemma 3.1. *For Case (I), $g(|x|, s_1, \epsilon)$ defined as above is a super-solution of (P).*

Proof. By Lemma 2.14, we only need to prove that $g(r, s_1, \epsilon)$ is a super-solution of (1). Since $g(\cdot, s, \epsilon) \in C^1[0, R)$, according to the definition of the super-solution of (1), we only need to prove

$$\begin{aligned}&(r^{N-1}\left|g'\right|^{p(r)-2}g')'\\ &\le r^{N-1}\left[f(r,g)-\rho(r,g)-|K(r)|\left|g'\right|^{\delta(r)}\right],\ \forall r\in[0,\sigma)\cup(\sigma,R_0)\cup(\sigma,R_0).\end{aligned}$$

Step 1. In the interval (σ, R_0).

When $r \in (R_0, R)$, we have

$$g' = Cs_1(R-r)^{-s_1-1},$$

and

$$(r^{N-1}\left|g'\right|^{p(r)-2}g')' = [r^{N-1}(Cs_1)^{p(r)-1}(R-r)^{-(s_1+1)(p(r)-1)}]'.$$

By computation, we have

$$\begin{aligned}&(r^{N-1}\left|g'\right|^{p(r)-2}g')'\\ &= r^{N-1}(Cs_1)^{p(r)-1}(s_1+1)(p(r)-1)(R-r)^{-(s_1+1)(p(r)-1)-1}\\ &+ r^{N-1}(Cs_1)^{p(r)-1}(R-r)^{-(s_1+1)(p(r)-1)}(-(s_1+1)p(r))'\ln(R-r)\\ &+ (r^{N-1}(Cs_1)^{p(r)-1})'(R-r)^{-(s_1+1)(p(r)-1)}\\ &= r^{N-1}(Cs_1)^{p(r)-1}(s_1+1)(p(r)-1)(R-r)^{-(s_1+1)(p(r)-1)-1}(1+h(r)),\end{aligned}\tag{16}$$

where

$$\begin{aligned}h(r) &= \frac{(-(s_1+1)p(r))'\ln(R-r)}{(s_1+1)(p(r)-1)}(R-r)\\ &\quad+\frac{(r^{N-1}(Cs_1)^{p(r)-1})'}{r^{N-1}(Cs_1)^{p(r)-1}(s_1+1)(p(r)-1)}(R-r)\\ &= \frac{n_0(-(s_1+1)p(r))'(R-r)^{\frac{1}{n_0}}\ln(R-r)^{\frac{1}{n_0}}}{(s_1+1)(p(r)-1)}(R-r)^{1-\frac{1}{n_0}}\\ &\quad+\frac{(r^{N-1}(Cs_1)^{p(r)-1})'(R-r)^{\frac{1}{n_0}}}{r^{N-1}(Cs_1)^{p(r)-1}(s_1+1)(p(r)-1)}(R-r)^{1-\frac{1}{n_0}}.\end{aligned}$$

It is easy to see that there exist positive constants $A, B \ge 1$ (A, B depend on C, R, p, q, n_0, s_1) such that

$$\begin{aligned}&\left|\frac{n_0(-(s_1+1)p(r))'(R-r)^{\frac{1}{n_0}}\ln(R-r)^{\frac{1}{n_0}}}{(s_1+1)(p(r)-1)}\right| \le A,\\ &\left|\frac{(r^{N-1}(Cs_1)^{p(r)-1})'}{r^{N-1}(Cs_1)^{p(r)-1}(s_1+1)(p(r)-1)}(R-r)^{\frac{1}{n_0}}\right| \le B,\end{aligned}$$

then we have

$$|h(r)| \leq (A+B)(R-r)^{1-\frac{1}{n_0}} \leq [(A+B+1)(R-R_0)^{\frac{1}{n_0}}]^{n_0-1}. \tag{17}$$

If $R-R_0>0$ is small enough, we get

$$(A+B+1)(R-R_0)^{\frac{1}{n_0}} \leq \frac{\epsilon}{2}. \tag{18}$$

Combining (17) and (18) together, we can find that when $R-R_0>0$ is small enough, then for any $r\in[R_0,R)$,

$$\begin{aligned} & (Cs_1)^{p(r)-1}(s_1+1)(p(r)-1)(1+h(r)) \\ \leq\ & (Cs_1)^{p(r)-1}(s_1+1)(p(r)-1)(1+\epsilon^{n_0-1}) \\ \leq\ & aC^{q(r)-1}(\frac{1}{1+\epsilon})^{\frac{1}{2n_0}}. \end{aligned} \tag{19}$$

Since $p(r)$, $q(r)$ and $\beta(r)$ are C^1 continuous, if $R-R_0>0$ is small enough, we obtain

$$(R-r)^{-(s_1+1)(p(r)-1)-1} \leq (R-r)^{-s_1(q(r)-1)-\beta(r)}(1+\epsilon)^{\frac{1}{4n_0}},\ \forall r\in[R_0,R). \tag{20}$$

Therefore, under the conditions of Case (I), we have

$$\frac{\rho(r,g)}{f(r,g)} \to 0 \text{ and } \frac{|K(r)|\,|g'|^{\delta(r)}}{f(r,g)} \to 0, \text{ as } r\to R^-. \tag{21}$$

From (16), (19), (20) and (21) it follows that when $R-R_0>0$ is small enough we can get

$$\begin{aligned} & (r^{N-1}|g'|^{p(r)-2}g')' \\ \leq\ & r^{N-1}(Cs_1)^{p(r)-1}(s_1+1)(p(r)-1)(R-r)^{-(s_1+1)(p(r)-1)-1}(1+\epsilon^{n_0-1}) \\ \leq\ & (1+\epsilon)^{\frac{-1}{4n_0}}r^{N-1}(R-r)^{-\beta(r)}a(C(R-r)^{-s_1})^{q(r)-1} \\ \leq\ & (1+\epsilon)^{\frac{-1}{4n_0}}r^{N-1}f(r,g) \\ \leq\ & r^{N-1}\left[f(r,g)-\rho(r,g)-|K(r)|\,|g'|^{\delta(r)}\right],\ \forall r\in(R_0,R). \end{aligned}$$

Thus, when $R-R_0>0$ is small enough, we have

$$(r^{N-1}|g'|^{p(r)-2}g')' \leq r^{N-1}\left[f(r,g)-\rho(r,g)-|K(r)|\,|g'|^{\delta(r)}\right],\ \forall r\in(R_0,R). \tag{22}$$

Step 2. On the interval $[0,\sigma)$ and (σ, R_0).

Note that $q(r)-1 > \theta(r)-1, \forall r \in [0,R]$. By computation, when k is large enough it follows that

$$\begin{aligned}
&r^{N-1}(R-r)^{-\beta(r)} a k^{q(r)-1} \\
\geq\ &(R_0)^{N-1}\Big[\varepsilon(Cs_1(R-R_0)^{-(s_1+1)})^{(p(R_0)-1)}\cos(\varepsilon(r-\sigma)) \\
&+ b(R-r)^{-\alpha(r)}(2k)^{\theta(r)-1} + |K(r)|\,|g'|^{\delta(r)}\Big] \\
\geq\ &(r^{N-1}|g'|^{p(r)-2}g')' + \rho(r,g) + |K(r)|\,|g'|^{\delta(r)}, \forall r \in (\sigma, R_0).
\end{aligned}$$

Therefore, when k is large enough, we have

$$(r^{N-1}|g'|^{p(r)-2}g')' \leq r^{N-1}\Big[f(r,g)-\rho(r,g)-|K(r)|\,|g'|^{\delta(r)}\Big],\ \forall r \in (\sigma, R_0), \tag{23}$$

and

$$(r^{N-1}|g'|^{p(r)-2}g')' = 0 \leq r^{N-1}\Big[f(r,g)-\rho(r,g)-|K(r)|\,|g'|^{\delta(r)}\Big],\quad 0 \leq r < \sigma. \tag{24}$$

Here we note that $g(|x|, s_1, \epsilon)$ is a C^1 function on $B(0,R)$. From (22), (23) and (24), we can see $g(|x|, s_1, \epsilon)$ is a super-solution of (P), when $R - R_0 > 0$ is small enough (R_0 is a constant depending on R, p, q, β, n_0, s_1) and k is large enough. This completes the proof.

Remark. It is easy to see that $g(r, s, \epsilon_*)$ is a super-solution of (2) for any $s \geq s_1$ and $\epsilon_* \geq \epsilon$, and then $g(|x|, s, \epsilon_*)$ is a super-solution of (P).

Next, we will construct a sub-solution of (P). Here we point out that there exists a very small positive number ε depending on R_0 such that

$$-f(r,\varepsilon) + \rho(r,\varepsilon) \leq 0, \forall r \in [0, R_0].$$

Obviously, for any $A \in [\varepsilon, g(R_0)]$, g is a super-solution of the following equation

$$\begin{cases} -(r^{N-1}|v'|^{p(r)-2}v')' \\ = r^{N-1}\Big[-f(r,v)+\rho(r,v)+K(r)\,|v'|^{\delta(r)}\Big], \forall r \in [0,R_0], \\ v'(0)=0, v(R_0)=A, \end{cases} \tag{25}$$

and ε is a sub-solution of (25).

By Lemma 2.7 and Lemma 2.14, (25) has a positive solution $\phi_A(r)$ satisfy $\varepsilon \leq \phi_A(r) \leq g(r,s,\epsilon)$. Define the function $v(r, s_1, \epsilon)$ on $[0,R)$ as

$$v(r,s,\epsilon) = \begin{cases} C^*(R-r)^{-s} - k^*,\ R_0 \leq r < R, \\ \phi_A(r), 0 \leq r < R_0, \end{cases}$$

where s is a positive constant, $R_0 \in (\sigma, R)$ and $R - R_0 > 0$ is small enough, ϵ is a small positive constant and

(26)
$$k^* = C^*(R-R_0)^{-s} - A,$$
$$C^* = C^*_\epsilon = C^*_\epsilon(s) = (1-\epsilon)[\frac{1}{a}s^{p(R)-1}(s+1)(p(R)-1)]^{\frac{1}{q(R)-p(R)}}.$$

Here we note that for any positive constant s, $v(\cdot, s, \epsilon) \in C[0, R)$, and $v'(r, s, \epsilon) \to 0$ as $r \to 0$.

Lemma 3.2. *Under the conditions of case* (I), *there exists a* $A \in [\varepsilon, C^*(R-R_0)^{-s_1}]$ *such that* $v(|x|, s_1, \epsilon)$ *is a sub-solution of* (P), *where* $s_1 = \frac{p(R)-\beta(R)}{q(R)-p(R)}$.

Proof. By the definition, $v(r, s_1, \epsilon)$ is a sub-solution of (25), and therefore $v(r, s_1, \epsilon)$ is a sub-solution of (P) on $[0, R_0]$. By noting that $v(r, s_1, \epsilon) \geq A > 0, \forall r \in [R_0, R)$. Since $A \in [\varepsilon, C^*(R-R_0)^{-s_1}]$, we have $k^* \geq 0$. Similar to the proof of Lemma 3.1, we can see that $v(r, s_1, \epsilon)$ satisfies

$$\begin{aligned}
&-(r^{N-1}\left|v'\right|^{p(r)-2}v')' + r^{N-1}\left[f(r,v) - \rho(r,v) - K(r)\left|v'\right|^{\delta(r)}\right] \\
&\leq -(r^{N-1}\left|v'\right|^{p(r)-2}v')' + r^{N-1}\left[f(r, C^*(R-r)^{-s_1}) - K(r)\left|v'\right|^{\delta(r)}\right] \\
&\leq 0, \forall r \in [R_0, R).
\end{aligned}$$

Thus v is a sub-solution of (P) on $[R_0, R)$. Denote

$$h_{k^*}(r) = C^*(R-r)^{-s_1} - k^*.$$

By Definition 2.5, v is a sub-solution of (P) provided

$$\phi'_A(R_0^-) - h'_{k^*}(R_0^+) \leq 0.$$

If $\phi'_A(R_0^-) > h'_{k^*}(R_0^+)$, we will prove that there exists a constant $A_1 \in [\varepsilon, A]$ such that $v(\cdot, s, \epsilon) \in C^1[0, R)$. It is sufficient to prove that exists a constant $A_1 \in [\varepsilon, A]$ such that $\phi'_{A_1}(R_0) = h'_{k^*}(R_0)$. Obviously,

$$\phi'_\varepsilon(R_0) \leq 0 < h'_{k^*}(R_0).$$

Let's consider the following equation

(27)
$$\begin{cases}
-(r^{N-1}\left|v'\right|^{p(r)-2}v')' \\
= r^{N-1}\left[-f(r,v) + \rho(r,v) + K(r)\left|v'\right|^{\delta(r)}\right], \forall r \in [0, R_0], \\
v'(0) = 0, v'(R_0) = h'_{k^*}(R_0).
\end{cases}$$

Clearly, ε is a sub-solution of (27), ϕ_A is a super-solution of (27), and $\varepsilon \leq \phi_A$. According to Lemma 2.8, there exist a solution y of (27) which satisfies $\varepsilon \leq y \leq \phi_A$.

Let $A_1 = y(R_0)$, then $\phi_{A_1}(\cdot) = y(\cdot)$ is a solution of (27). In the definition of v, let A_1 replace A, then v is a sub-solution of (2). By Lemma 2.14, it is a sub-solution of (P). The proof is completed.

Definition 3.3. If u is a boundary blow-up function and satisfies

$$\lim_{d(x,\partial\Omega)\to 0} \frac{u(x)}{\mu[d(x,\partial\Omega)]^{-s}} = 1,$$

where μ and s are positive constants, then we say that the singularity of u is $\mu[d(x,\partial\Omega)]^{-s}$, and the blowup rate of u is s.

Proof of Theorem 1.1.

Step 1. The existence of solution.

From Lemma 3.1-3.2 and Lemma 2.14 it follows that (P) has a super-solution $g(|x|, s_1, \epsilon)$ and a sub-solution $v(|x|, s_1, \epsilon)$, respectively. Moreover, we have $g(|x|, s_1, \epsilon) \geq v(|x|, s_1, \epsilon)$, for any $x \in \Omega$.

Let $D_j = \{x \mid |x| < r_j^* := (1 - \frac{1}{j+1})R\}$ $(j = 1, 2, \cdots)$. Let's consider the radial solutions of the following problem

$$\begin{cases} -\triangle_{p(x)} u_j + f(x, u_j) = \rho(x, u_j) + K(|x|) \left| u_j' \right|^{\delta(|x|)}, \text{ in } D_j, \\ u_j(x) = v(|x|, s_1, \epsilon), \text{ for } x \in \partial D_j. \end{cases} \tag{28}$$

It is easy to see that the solution of the following ODE is a radial solution of (28)

$$\begin{cases} (r^{N-1} \left| u_j' \right|^{p(r)-2} u_j')' = r^{N-1} \left[f(r, u_j) - \rho(r, u_j) - K(r) \left| u_j' \right|^{\delta(r)} \right] \\ u_j'(0) = 0, u_j(r_j^*) = v(r_j^*, s_1, \epsilon). \end{cases} \tag{29}$$

Next Let's consider

$$\begin{cases} (r^{N-1} \left| u_j' \right|^{p(r)-2} u_j')' = r^{N-1} \left[f(r, u_j) - \rho(r, u_j) - K(r) \left| u_j' \right|^{\delta(r)} \right] \\ \lim\limits_{r\to 0} r^{\frac{N-1}{p(r)-1}} u_j'(r) = 0, u_j(r_j^*) = v(r_j^*, s_1, \epsilon). \end{cases} \tag{30}$$

From Lemma 2.6, we can see that (30) has at least one solution u_j. By Lemma 2.14, $u_j'(0) = 0$. It means that every solution of (30) is a solution of (29), and it is a radial solution of (28). Similar to the proof of Theorem 3.3 of [47], $\{u_j\}$ has a subsequence converging to u which is a boundary blow-up solution of (P).

Step 2. The asymptotic behavior of solution.

We claim that there are a family sub-solution $v_*(r, s_1, \epsilon)$ and a family super-solution $g_*(r, s_1, \epsilon)$ satisfy

$$v(r, s_1, \frac{1}{2}) \leq v_*(r, s_1, \epsilon) \leq g_*(r, s_1, \epsilon) \leq g(r, s_1, 1), \epsilon \in (0, \frac{1}{3}). \tag{31}$$

In the definition of $g_*(r, s_1, \frac{1}{n+2})$ and $v_*(r, s_1, \frac{1}{n+2})$, let $\epsilon = \frac{1}{n+2}$, it follows from the former discussion that (P) has a solution u_n which is between $g_*(r, s_1, \frac{1}{n+2})$ and $v_*(r, s_1, \frac{1}{n+2})$. Since $g_*(r, s_1, \frac{1}{n+2})$ and $v_*(r, s_1, \frac{1}{n+2})$ are between $g_*(r, s_1, 1)$ and $v_*(r, s_1, \frac{1}{2})$, each solution u_n $(n = 1, 2, \cdots)$ is between $g_*(r, s_1, 1)$ and $v_*(r, s_1, \frac{1}{2})$. Similar to the former discussion, the sequence $\{u_n\}$ has a subsequence converging to u, which is a solution of (P). Obviously, u has the singularity of $\mu[d(x, \partial\Omega)]^{-s_1}$, where

$$s_1 = \frac{p(R) - \beta(R)}{q(R) - p(R)}, \mu = C_0 = [\frac{1}{a} s_1^{p(R)-1}(s_1 + 1)(p(R) - 1)]^{\frac{1}{q(R)-p(R)}}.$$

It only remain to prove the existence of a family sub-solution $v_*(r, s_1, \epsilon)$ and a family super-solution $g_*(r, s_1, \epsilon)$ which satisfy (31).

At first, we construct a family of $g_*(r, s_1, \epsilon)$ *which is between* $v(r, s_1, \frac{1}{2})$ *and* $g(r, s_1, 1)$.

By the definition of super-solution $g(r, s_1, \epsilon)$ and sub-solution $v(r, s_1, \epsilon)$, we have $g(r, s_1, 1) > v(r, s_1, \frac{1}{2}), \forall r \in [0, R)$. Now, let's consider

$$\text{(I)} \begin{cases} -(r^{N-1} |v'|^{p(r)-2} v')' = r^{N-1} \left[-f(r, v) + \rho(r, v) + K(r) |v'|^{\delta(r)} \right], \forall r \in [0, R_1], \\ v'(0) = 0, v(R_1) = A, \end{cases}$$

where

$$R_0 < R_1 < R, \text{ and } A_* := v(R_1, s_1, \frac{1}{2}) \leq A \leq A^* := g(R_1, s_1, 1).$$

Note that $g(r, s_1, 1)$ and $v(r, s_1, \frac{1}{2})$ are a super-solution and a sub-solution of (I), respectively. According to Lemma 2.6, for any $A \in [A_*, A^*]$, (I) has a solution $\phi_A(r)$ satisfying $v(r, s_1, \frac{1}{2}) \leq \phi_A(r) \leq g(r, s_1, 1)$. Let $\epsilon \in (0, \frac{1}{3}]$ be small enough. We define

$$g_*(r, s_1, \epsilon) = \begin{cases} C_\epsilon (R - r)^{-s_1} + k, \ R_1 \leq r < R, \\ \phi_{A^*}(r), 0 \leq r < R_1, \end{cases}$$

where C_ϵ is defined in (15), and

$$k := A^* - C_\epsilon (R - R_1)^{-s_1} > 0.$$

We can see that

$$\begin{aligned} g'_*(R_1^-, s_1, \epsilon) &= \phi'_{A^*}(R_1^-) \geq g'(R_1, s_1, 1) \\ &= s_1 C_1 (R - r)^{-s_1 - 1} > s_1 C_\epsilon (R - r)^{-s_1 - 1} = g'_*(R_1^+, s_1, \epsilon). \end{aligned}$$

Similar to the proof of Lemma 3.1 and Lemma 2.14, we can see that g_* is a super-solution of (P). Moreover, $v(r, s_1, \frac{1}{2}) \le g_*(r, s_1, \epsilon) \le g(r, s_1, 1)$.

At last, we construct a family of $v_*(r, s_1, \epsilon)$ *which satisfy* (*31*).

According to Lemma 2.6, (I) has a solution $\phi_{A_*}(r)$ satisfy $v(r, s_1, \frac{1}{2}) \le \phi_{A_*}(r) \le g_*(r, s_1, \epsilon)$. We define

$$v_*(r, s_1, \epsilon) = \begin{cases} C^*_\epsilon (R-r)^{-s_1} - k^*, \ R_1 \le r < R, \\ \phi_{A_*}(r), 0 \le r < R_1, \end{cases}$$

where C^*_ϵ is defined in (26), and

$$k^* = C^*_\epsilon (R-R_0)^{-s_1} - A_* > 0.$$

Note that

$$\begin{aligned} v'_*(R_1^-, s_1, \epsilon) &= \phi'_{A_*}(R_1^-) \le v'(R_1, s_1, \frac{1}{2}) \\ &= s_1 C^*_{\frac{1}{2}} (R-r)^{-s_1-1} < s_1 C^*_\epsilon (R-r)^{-s_1-1} = v'_*(R_1^+, s_1, \epsilon). \end{aligned}$$

Similar to the proof of Lemma 3.2 and Lemma 2.14, we can see that v_* is a sub-solution of (P). The proof is completed.

3.2. Case (II)

Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1, the main task is to construct a pair of sub-solution and super-solution of (P). Set

$$\begin{aligned} g_2 &= t_0 g(r, s_1, \epsilon), \forall r \in [0, R), \\ v_2(r, s_1, \epsilon) &= \begin{cases} t_0 C^*(R-r)^{-s_1} - k^*, \ R_0 \le r < R, \\ \phi_A(r), 0 \le r < R_0, \end{cases} \\ k^* &= t_0 C^*(R-R_0)^{-s_1} - A, \end{aligned}$$

where g is defined in **Case (I)**, $A \in [\varepsilon, t_0 C^*(R-R_0)^{-s_1}], \varepsilon > 0$ is small enough such that $f(r, \varepsilon) \le \rho(r, \varepsilon)$ for any $0 \le r \le R_0$, C^* is defined in (26), t_0 is the unique positive solution of

$$-aC_0^{q(R)-1} t^{p(R)-1} + aC_0^{q(R)-1} t^{q(R)-1} = bC_0^{\theta(R)-1} t^{\theta(R)-1},$$

where $C_0 = C_0^*$ is defined in (15).

Next, we will prove that g_2 and v_2 are a super-solution and a sub-solution of (P), respectively. The idea is similar to the proof of Theorem 1.1, and we need small adjustments.

Step 1. We will prove g_2 is a super-solution of (P).

The major difficulty is to prove

$$(r^{N-1}\left|g_2'\right|^{p(r)-2}g_2')' \leq r^{N-1}\left[f(r,g_2)-\rho(r,g_2)-|K(r)|\left|g_2'\right|^{\delta(r)}\right], \forall r\in(R_0,R).$$

Denote $\zeta=(s_1+1)(p(R)-1)+1$. Under the conditions of Case (II), we have

(32) $\beta(R)+s_1(q(R)-1)=\alpha(R)+s_1(\theta(R)-1)=\zeta$, and $(s_1+1)\delta(R)-\gamma<\zeta$.

Let C be defined in (15). Denote

$$\psi=t_0C(R-r)^{-s_1}.$$

Similar to the proof of Lemma 3.1, by computation, we have

(33)
$$\begin{aligned}&(r^{N-1}\left|\psi'\right|^{p(r)-2}\psi')'\\&=r^{N-1}(t_0Cs_1)^{p(r)-1}(s_1+1)(p(r)-1)(R-r)^{-(s_1+1)(p(r)-1)-1}(1+h(r)),\end{aligned}$$

where

$$h(r)=\frac{(-(s_1+1)p(r))'\ln(R-r)}{(s_1+1)(p(r)-1)}(R-r)+\frac{(r^{N-1}(t_0Cs_1)^{p(r)-1})'}{r^{N-1}(t_0Cs_1)^{p(r)-1}(s_1+1)(p(r)-1)}(R-r).$$

It is easy to see that $h(r)\to 0$ as $r\to R^-$.

Note that $(C_0s_1)^{p(R)-1}(s_1+1)(p(R)-1)=aC_0^{q(R)-1}$. Since $p(r)$, $q(r)$ and $\beta(r)$ are C^1 continuous, from (32) and the definition of C and $C=(1+\epsilon)C_0$ it follows that

$$(r^{N-1}\left|\psi'\right|^{p(r)-2}\psi')'\times r^{1-N}(R-r)^{\zeta}\to(t_0Cs_1)^{p(R)-1}(s_1+1)(p(R)-1)$$

(34a)
$$=a[t_0(1+\epsilon)]^{p(R)-1}C_0^{q(R)-1}, \text{ as } r\to R^-,$$

(34b)
$$r^{N-1}f(r,\psi)\times r^{1-N}(R-r)^{\zeta}\to a[t_0(1+\epsilon)C_0]^{q(R)-1}, \text{ as } r\to R^-,$$

(34c)
$$r^{N-1}\rho(r,\psi)\times r^{1-N}(R-r)^{\zeta}\to b[t_0(1+\epsilon)C_0]^{\theta(R)-1}, \text{ as } r\to R^-,$$

(34d)
$$r^{N-1}|K(r)|\left|\psi'\right|^{\delta(r)}\times r^{1-N}(R-r)^{\zeta}\to 0, \text{ as } r\to R^-.$$

Since t_0 is the unique positive solution of the following equation

$$-aC_0^{q(R)-1}t^{p(R)-1}+aC_0^{q(R)-1}t^{q(R)-1}=bC_0^{\theta(R)-1}t^{\theta(R)-1},$$

when $t>t_0$, it follows from $q(R)>\max\{p(R),\theta(R)\}$ that

$$-aC_0^{q(R)-1}t^{p(R)-1}+aC_0^{q(R)-1}t^{q(R)-1}>bC_0^{\theta(R)-1}t^{\theta(R)-1}.$$

Therefore

(35)
$$\begin{aligned}&-aC_0^{q(R)-1}[t_0(1+\epsilon)]^{p(R)-1}+aC_0^{q(R)-1}[t_0(1+\epsilon)]^{q(R)-1}\\&>bC_0^{\theta(R)-1}[t_0(1+\epsilon)]^{\theta(R)-1}.\end{aligned}$$

From (35) and (34a)-(34d), when $R - R_0 > 0$ is small enough, we can get

$$0 < (r^{N-1} \left|\psi'\right|^{p(r)-2} \psi')' \leq r^{N-1}[f(r,\psi) - \rho(r,\psi) - K(r) \left|\psi'\right|^{\delta(r)}], \forall r \in [R_0, R).$$

By (35), we have $f(r,\psi) > \rho(r,\psi) > 0, \forall r \in [R_0, R)$. It is easy to see that $f(r,\psi+k) - \rho(r,\psi+k)$ is increasing with respect to k. By noting that $g_2 = \psi + k$ for $r \in [R_0, R)$, when $R - R_0 > 0$ is small enough, it is easy to check

$$(36)\quad (r^{N-1} \left|g_2'\right|^{p(r)-2} g_2')' \leq r^{N-1}\left[f(r,g_2) - \rho(r,g_2) - |K(r)|\left|g_2'\right|^{\delta(r)}\right], \forall r \in (R_0, R).$$

Note that $q(r) - 1 > \theta(r) - 1, \forall r \in [0, R]$. By computation, when k is large enough it is easy to check

$$(37)\quad (r^{N-1} \left|g_2'\right|^{p(r)-2} g_2')' \leq r^{N-1}\left[f(r,g_2) - \rho(r,g_2) - |K(r)|\left|g_2'\right|^{\delta(r)}\right], \forall r \in (\sigma, R_0),$$

and

$$(38)\quad (r^{N-1}\left|g_2'\right|^{p(r)-2} g_2')' = 0 \leq r^{N-1}\left[f(r,g_2) - \rho(r,g_2) - |K(r)|\left|g_2'\right|^{\delta(r)}\right], 0 \leq r < \sigma.$$

It follows from (36)-(38) that g_2 is a super-solution of (1) with $\lim_{r\to 0^+} r^{\frac{N-1}{p(r)-1}} g_2'(r) = 0$ and $g_2(r) \to \infty$ as $r \to R^-$, then it is a super-solution of (P).

Step 2. We will prove v_2 is a sub-solution of (P).

The major difficulty is to prove

$$(r^{N-1} \left|v_2'\right|^{p(r)-2} v_2')' + r^{N-1}K(r) \left|v_2'\right|^{\delta(r)} \geq r^{N-1}[f(r,v_2) - \rho(r,v_2)], \forall r \in [R_0, R).$$

Obviously

$$-aC_0^{q(R)-1}[t_0(1-\epsilon)]^{p(R)-1} + aC_0^{q(R)-1}[t_0(1-\epsilon)]^{q(R)-1} < bC_0^{\theta(R)-1}[t_0(1-\epsilon)]^{\theta(R)-1}.$$

Thus when $R - R_0 > 0$ is small enough, we can get

$$(39)\quad (r^{N-1} \left|\phi'\right|^{p(r)-2} \phi')' \geq r^{N-1}[f(r,\phi) - \rho(r,\phi) - K(r) \left|\phi'\right|^{\delta(r)}], \forall r \in [R_0, R).$$

When $R - R_0 > 0$ is small enough, we have

$$(40)\qquad (r^{N-1} \left|\phi'\right|^{p(r)-2} \phi')' + r^{N-1}K(r) \left|\phi'\right|^{\delta(r)} > 0, \forall r \in [R_0, R).$$

From (39), we have

$$(41)\quad (r^{N-1}\left|\phi'\right|^{p(r)-2} \phi')' + r^{N-1}K(r) \left|\phi'\right|^{\delta(r)} \geq r^{N-1}[f(r,\phi) - \rho(r,\phi)], \forall r \in [R_0, R).$$

Note that $0 \leq k^* < t_0 C^*(R - R_0)^{-s_1} = \min_{r\in[R_0,R)} \phi(r)$. We claim that

(42)
$$\begin{aligned}&(r^{N-1}\left|\phi'\right|^{p(r)-2}\phi')' + r^{N-1}K(r)\left|\phi'\right|^{\delta(r)}\\ &\geq r^{N-1}[f(r,\phi-k^*)-\rho(r,\phi-k^*)], \forall r\in[R_0,R).\end{aligned}$$

If $r \in [R_0, R)$ satisfies $f(r,\phi-k^*)-\rho(r,\phi-k^*)\leq 0$, it follows from (40) that

$$(r^{N-1}\left|\phi'\right|^{p(r)-2}\phi')' + r^{N-1}K(r)\left|\phi'\right|^{\delta(r)} > 0 \geq r^{N-1}[f(r,\phi-k^*)-\rho(r,\phi-k^*)].$$

If $r \in [R_0, R)$ satisfies $f(r,\phi-k^*)-\rho(r,\phi-k^*) > 0$, it is easy to check

$$f(r,\phi-k^*)-\rho(r,\phi-k^*) \leq f(r,\phi)-\rho(r,\phi).$$

Thus (42) is valid. Since $v_2(r,s_1,\epsilon)=\phi-k^*$ for any $r\in[R_0,R)$, then

$$(r^{N-1}\left|v_2'\right|^{p(r)-2}v_2{'})' + r^{N-1}K(r)\left|v_2{'}\right|^{\delta(r)} \geq r^{N-1}[f(r,v_2)-\rho(r,v_2)], \forall r\in[R_0,R).$$

Similar to the proof of Lemma 3.2, there exists a $A\in[\varepsilon, t_0C^*(R-R_0)^{-s_1}]$ such that $v_2(|x|, s_1, \epsilon)$ is a sub-solution of (P).

Step 3 The existence and asymptotic behavior of solution of (P).

Also similar to the proof of Theorem 1.1, we get existence of solution u satisfying

$$\lim_{d(x,\partial\Omega)\to 0}\frac{u(x)}{t_0\mu[d(x,\partial\Omega)]^{-s_1}} = 1,$$

The proof is completed.

3.3. Case (III)

Proof of Theorem 1.3. The idea is similar to the proof of Theorem 1.1-1.2, and we need small adjustments.

Under the conditions of Case (III), we have

(43)
$$\begin{aligned}&(s_2+1)\delta(R)-\gamma\\ &< (s_2+1)(p(R)-1)+1 < \beta(R)+s_2(q(R)-1) = s_2(\theta(R)-1)+\alpha.\end{aligned}$$

Note that $at_*^{q(R)-1} = bt_*^{\theta(R)-1}$. Similar to the proof of Theorem 1.2, the terms $f(x,t_*(R-|x|)^{-s_2})$ and $\rho(x,t_*(R-|x|)^{-s_2})$ have the same blowup singularity, and the blowup rate is larger than $-\triangle_{p(x)}t_*(R-|x|)^{-s_2}$ and $K(x)\left|\nabla t_*(R-|x|)^{-s_2}\right|^{\delta(x)}$, i.e.,

(44)
$$\frac{f(x,t_*(R-|x|)^{-s_2})}{\rho(x,t_*(R-|x|)^{-s_2})} \to 1 \text{ as } |x|\to R^-,$$

and

(45)
$$\frac{-\triangle_{p(x)}t_*(R-|x|)^{-s_2}}{f(x,t_*(R-|x|)^{-s_2})} \to 0 \text{ as } |x|\to R^-,$$

(46)
$$\frac{K(x)\left|\nabla t_*(R-|x|)^{-s_2}\right|^{\delta(x)}}{-\triangle_{p(x)}t_*(R-|x|)^{-s_2}} \to 0 \text{ as } |x|\to R^-.$$

Denote

$$\psi(r) = (1+\epsilon)t_*(R-r)^{-s_2}, \forall r \in [0, R).$$

Define

$$g_3(r, s_2, \epsilon) = \begin{cases} \psi(r) + k, \ R_0 \le r < R, \\ \psi(R_0) + k - \int_r^{R_0} [\psi'(R_0)]^{\frac{p(R_0)-1}{p(t)-1}} [\frac{(R_0)^{N-1}}{t^{N-1}} \sin \varepsilon(t-\sigma)]^{\frac{1}{p(t)-1}} dt, \\ \qquad \sigma < r < R_0 \ , \\ \psi(R_0) + k - \int_r^{R_0} [\psi'(R_0)]^{\frac{p(R_0)-1}{p(t)-1}} [\frac{(R_0)^{N-1}}{t^{N-1}} \sin \varepsilon(t-\sigma)]^{\frac{1}{p(t)-1}} dt \ , \\ \qquad r \le \sigma, \end{cases}$$

$$v_3(r, s_2, \epsilon) = \begin{cases} (1-\epsilon)t_*(R-r)^{-s_2} - k^*, \ R_0 \le r < R, \\ \phi_A(r), 0 \le r < R_0, \end{cases}$$

$$k^* = (1-\epsilon)t_*(R-r)^{-s_2} - A,$$

where $k > 0$ is large enough, $A \in [\varepsilon, (1-\epsilon)t_*(R-r)^{-s_2}]$, $\varepsilon > 0$ is small enough such that $f(r, \varepsilon) \le \rho(r, \varepsilon)$ for any $0 \le r \le R_0$.

We will prove that g_3 and v_3 are a super-solution and a sub-solution of (P), respectively.

Step 1. We will prove that g_3 is a super-solution and a sub-solution of (P).

Since g_3 is C^1, when k is large enough, we only to prove

$$(r^{N-1} |g_3'|^{p(r)-2} g_3')' \le r^{N-1} \left[f(r, g_3) - \rho(r, g_3) - |K(r)| \left| g_3' \right|^{\delta(r)} \right], \ \forall r \in [0, \sigma) \cup (\sigma, R_0) \cup (R_0, R).$$

It follows from (44) and condition (H_1) that

$$\frac{f(x, \psi)}{\rho(x, \psi)} \to (1+\epsilon)^{q(R)-\theta(R)} > 1 \text{ as } |x| \to R^-.$$

Combining the above inequality, (45) and (46) together, when $R - R_0 > 0$ is small enough, we have

$$(r^{N-1} |\psi'|^{p(r)-2} \psi')' \le r^{N-1}[f(r, \psi) - \rho(r, \psi) - K(r) |\psi'|^{\delta(r)}], \forall r \in [R_0, R).$$

Similar to the proof of Theorem 1.2 and Lemma 3.1, when k is large enough, we can see that g_3 is a super-solution of (P).

Step 2. We will prove that v_3 is a sub-solution of (P).

We claim that the following inequality is valid when $R - R_0 > 0$ is small enough

$$(47) \quad (r^{N-1} |v_3'|^{p(r)-2} v_3')' \ge r^{N-1}[f(r, v_3) - \rho(r, v_3) - K(r) |v_3'|^{\delta(r)}], \forall r \in [R_0, R).$$

Similar to the proof of Lemma 3.2, there exists a $A \in [\varepsilon, (1-\epsilon)t_*(R-r)^{-s_2}]$ such that $v_3(|x|, s_2, \epsilon)$ is a sub-solution of (P). It only remain to prove (47).

Denote

$$\phi = (1-\epsilon)t_*(R-r)^{-s_2}.$$

It follows from (43) and the condition **(H$_1$)** that

$$\frac{f(x,\phi)}{\rho(x,\phi)} \to (1-\epsilon)^{q(R)-\theta(R)} < 1 \text{ as } |x| \to R^-.$$

Combining the above inequality, (45) and (46) together, when $R - R_0 > 0$ is small enough, we have

$$(r^{N-1}|\phi'|^{p(r)-2}\phi')' \geq r^{N-1}[f(r,\phi) - \rho(r,\phi) - K(r)|\phi'|^{\delta(r)}], \forall r \in [R_0, R).$$

It follows from (46) that

$$(r^{N-1}|\phi'|^{p(r)-2}\phi')' - r^{N-1}K(r)|\phi'|^{\delta(r)} \geq \frac{1}{2}(r^{N-1}|\phi'|^{p(r)-2}\phi')' > 0 \text{ as } r \to R^-.$$

Note that $k^* \in [0, (1-\epsilon)t_*(R-R_0)^{-s_2} - \varepsilon]$. Similar to the proof of Theorem 1.2, when $R - R_0 > 0$ is small enough, we have

$$(r^{N-1}|\phi'|^{p(r)-2}\phi')' + r^{N-1}K(r)|\phi'|^{\delta(r)} \geq r^{N-1}[f(r,\phi-k^*) - \rho(r,\phi-k^*)], \forall r \in [R_0, R).$$

Note that $v_3 = \phi - k_*$ on $[R_0, R)$. Thus (47) is valid.

Step 3. The existence and asymptotic behavior of solution of (P).

Also similar to the proof of Theorem 1.1, we get the existence of solution u satisfying

$$\lim_{d(x,\partial\Omega)\to 0} \frac{u(x)}{t_*[d(x,\partial\Omega)]^{-s_2}} = 1.$$

The proof is completed.

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