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TWO OPTIMAL INEQUALITIES FOR ANTI-HOLOMORPHIC SUBMANIFOLDS AND THEIR APPLICATIONS

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Abstract. The CR δ-invariant for CR-submanifolds was introduced by B.-Y. Chen in a recent article [13]. In this paper, we prove two new optimal inequalities for anti-holomorphic submanifolds in complex space forms involving the CR δ-invariant. Moreover, we obtain some classification results for certain anti-holomorphic submanifolds in complex space forms which satisfy the equality case of either inequality.

1. Introduction

Let $\tilde{M}$ be a Kähler manifold with complex structure J and let N be a Riemannian manifold isometrically immersed in $\tilde{M}$. For each point $x \in N$, we denote by $\mathcal{D}_x$ the maximal complex subspace $T_xN \cap J(T_xN)$ of the tangent space T_xN of N. If the dimension of $\mathcal{D}_x$ is the same for all $x \in N$, then $\{\mathcal{D}_x, x \in N\}$ defines a holomorphic distribution $\mathcal{D}$ on N. A subspace $\mathcal{V}$ of T_xN, $x \in N$, is called *totally real* if $J(\mathcal{V})$ is a subspace of the normal space $T_x^{\perp}N$ at x. A submanifold N of a Kähler manifold is called a *totally real submanifold* if each tangent space of N is totally real.

A submanifold N of a Kähler manifold $\tilde{M}$ is called a CR-*submanifold* if there exists a totally real distribution $\mathcal{D}^{\perp}$ on N whose orthogonal complement is the holomorphic distribution $\mathcal{D}$ (cf. [1, 8, 12]), i.e.,

$$TN = \mathcal{D} \oplus \mathcal{D}^{\perp}, \ \ J\mathcal{D}_x^{\perp} \subset T_x^{\perp}N, \ \ x \in N.$$

Throughout this paper, we denote by h the complex rank of the holomorphic distribution $\mathcal{D}$ and by p the (real) rank of the totally real distribution $\mathcal{D}^{\perp}$ for a CR-submanifold. A

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warped product submanifold $N^T \times_f N^\perp$ with warping function f in a Kähler manifold $\tilde{M}$ is called a *CR-warped product* if N^T is a holomorphic submanifold and $N^\perp$ is a totally real submanifold of $\tilde{M}$.

It is well-known that the totally real distribution $\mathcal{D}^\perp$ of every CR-submanifold of a Kähler manifold is an integrable distribution (cf. [8, 10, 12]).

In order to provide some answers to an open question concerning minimal immersions proposed by S. S. Chern in the 1960s and to provide some applications of the well-known Nash embedding theorem, the second author introduced in early 1990s the notion of δ-invariants (see [6, 12, 14, 20] for details). For a CR-submanifold N of a Kähler manifold, he introduced in [13] a δ-invariant $\delta(\mathcal{D})$, called the *CR δ-invariant*, defined by

$$\delta(\mathcal{D})(x) = \tau(x) - \tau(\mathcal{D}_x), \tag{1.1}$$

where τ is the scalar curvature of N and $\tau(\mathcal{D})$ is the scalar curvature of the holomorphic distribution $\mathcal{D}$ of N (see [12] for details). In [13], the second author established a sharp inequality involving the CR δ-invariant $\delta(\mathcal{D})$ for anti-holomorphic warped product submanifolds in complex space forms.

In this paper, we prove two new optimal inequalities involving the CR δ-invariant for arbitrary anti-holomorphic submanifolds in complex space forms. Moreover, we obtain some classification results for anti-holomorphic submanifolds in complex space forms which satisfy the equality case of either inequality.

2. Preliminaries

2.1. Basic definitions and formulas

Let N be a Riemannian n-manifold equipped with an inner product $\langle\ ,\ \rangle$. Denote by ∇ the Levi-Civita connection of N.

Assume that N is isometrically immersed in a Kähler manifold $\tilde{M}$. Then the formulas of Gauss and Weingarten are given respectively by (cf. [5, 12])

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for vector fields X and Y tangent to N and ξ normal to N, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$, σ is the second fundamental form, D is the normal connection, and A is the shape operator of N.

The second fundamental form σ and the shape operator A are related by

$$\langle A_\xi X, Y \rangle = \langle \sigma(X, Y), \xi \rangle, \tag{2.3}$$

where $\langle\ ,\ \rangle$ is the inner product on N as well as on $\tilde{M}$. The mean curvature vector of N is defined by

$$\overrightarrow{H} = \left(\frac{1}{n}\right) \operatorname{trace}\sigma, \quad n = \dim N. \tag{2.4}$$

The squared mean curvature H^2 is given by $H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle$.

The *equation of Gauss* is

$$\begin{aligned} R(X,Y;Z,W) = {} & \tilde{R}(X,Y;Z,W) + \langle \sigma(X,W), \sigma(Y,Z)\rangle \\ & - \langle \sigma(X,Z), \sigma(Y,W)\rangle \end{aligned} \tag{2.5}$$

for vectors X, Y, Z, W tangent to N, where R and $\tilde{R}$ denote the Riemann curvature tensors of N and $\tilde{M}$, respectively.

For the second fundamental form σ, we define its covariant derivative $\bar{\nabla}\sigma$ with respect to the connection on $TN \oplus T^{\perp}N$ by

$$(\bar{\nabla}_X \sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.6}$$

The *equation of Codazzi* is

$$(\tilde{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X \sigma)(Y,Z) - (\bar{\nabla}_Y \sigma)(X,Z), \tag{2.7}$$

for vectors X, Y, Z tangent to N, where $(\tilde{R}(X,Y)Z)^{\perp}$ denotes the normal component of $\tilde{R}(X,Y)Z$.

2.2. Real hypersurfaces

A real hypersurface N of a Kähler manifold $\tilde{M}$ is called a *Hopf hypersurface* if $J\xi$ is a principal curvature vector, i.e., an eigenvector of the shape operator A_ξ, where ξ is a unit normal vector of N. Obviously, every Hopf hypersurface is mixed totally geodesic.

A real hypersurface N of a Kähler manifold $\tilde{M}$ with $\dim_{\mathbf{C}} \tilde{M} = m$ is called *totally real m-ruled* if for each point $x \in N$ there exists an m-dimensional totally real totally geodesic submanifold V_x^m of N through x.

2.3. Complex space forms

A Kähler manifold of constant holomorphic sectional curvature is called a *complex space form*. Throughout this paper, we denote a complete, simply-connected (complex) m-dimensional complex space form of constant holomorphic sectional curvature $4c$ by $\tilde{M}^m(4c)$.

It is well-known that $\tilde{M}^m(4c)$ is holomorphically isometric to the complex projective m-space $CP^m(4c)$, the complex Euclidean m-space $\mathbf{C}^m$, or the complex hyperbolic m-space $CH^m(4c)$ according to $c > 0$, $c = 0$, or $c < 0$, respectively.

The curvature tensor $\tilde{R}$ of a complex space form $\tilde{M}^m(4c)$ satisfies

$$\begin{aligned}\tilde{R}(U,V,W) = c\{&\langle V,W\rangle U - \langle X,W\rangle V + \langle JV,W\rangle JU \\ &- \langle JU,W\rangle JV + 2\langle U,JV\rangle JW\}.\end{aligned} \tag{2.8}$$

2.4. Anti-holomorphic submanifolds and CR-submanifolds

A CR-submanifold N of a Kähler manifold $\tilde{M}$ is called *anti-holomorphic* if we have

$$J\mathcal{D}_x^{\perp} = T_x^{\perp}N, \ \ x \in N.$$

A CR-submanifold is called *mixed totally geodesic* if its second fundamental form σ satisfies $\sigma(X,Z) = 0$ for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

A mixed totally geodesic CR-submanifold is called *mixed foliate* if its holomorphic distribution $\mathcal{D}$ is also integrable. Moreover, a CR-submanifold N is called a *CR-product* if it is a Riemannian product of a holomorphic submanifold N^T and a totally real submanifold $N^{\perp}$ of $\tilde{M}$.

Obviously, real hypersurfaces of a Kähler manifold are exactly anti-holomorphic submanifolds with $p = \operatorname{rank}\mathcal{D}^{\perp}$=1.

2.5. H-umbilical Lagrangian submanifolds

An anti-holomorphic submanifold of a Kähler manifold is called *Lagrangian* if $\mathcal{D} = \{0\}$, i.e.,

$$J(T_xN) = T_x^{\perp}N, \ \ x \in N.$$

A Lagrangian submanifold is said to be *H-umbilical* if its second fundamental form satisfies the following simple form (cf. [7]):

$$\begin{aligned}&\sigma(e_1,e_1) = \varphi Je_1, \ \ \sigma(e_1,e_j) = \psi Je_j, \\ &\sigma(e_2,e_2) = \cdots = \sigma(e_n,e_n) = \psi Je_1, \\ &\sigma(e_j,e_k) = 0, \ \ j \neq k, \quad j,k = 2,\ldots,n,\end{aligned} \tag{2.9}$$

for some suitable functions φ and ψ with respect to some suitable orthonormal local frame field $\{e_1,\ldots,e_n\}$.

Since there do not exist umbilical Lagrangian submanifold in Kähler manifolds, H-umbilical Lagrangian submanifolds are the simplest Lagrangian submanifolds next to totally geodesic one (cf. [7]).

3. Some Basic Lemmas for CR-Submanifolds

We need the following two lemmas from [1, 8] for later use.

Lemma 3.1. *Let N be a CR-submanifold of a Kähler manifold $\tilde{M}$. Then we have:*

(1) *the totally real distribution* $\mathcal{D}^{\perp}$ *is an integrable distribution,*
(2) $\langle \sigma(U, JX), JZ\rangle = \langle \nabla_U X, Z\rangle$,
(3) $A_{JZ}W = A_{JW}Z$,

for vector field U *tangent to* N, X, Y *in* $\mathcal{D}$, *and* Z, W *in* $\mathcal{D}^{\perp}$.

Lemma 3.2. *Let* N *be a* CR*-submanifold of a Kähler manifold* $\tilde{M}$. *Then we have:*

(1) *the holomorphic distribution* $\mathcal{D}$ *is integrable if and only if*

$$\langle \sigma(X, JY), JZ\rangle = \langle \sigma(JX, Y), JZ\rangle \tag{3.1}$$

holds for any $X, Y \in \mathcal{D}$ *and* $Z \in \mathcal{D}^{\perp}$,

(2) *the leaves of the totally real distribution* $\mathcal{D}^{\perp}$ *are totally geodesic in* N *if and only if*

$$\langle \sigma(X, Z), JW\rangle = 0 \tag{3.2}$$

holds for any $X \in \mathcal{D}$ *and* $Z, W \in \mathcal{D}^{\perp}$.

We also recall the following result for later use.

Lemma 3.3. *A complex space form* $\tilde{M}^m(4c)$ *with* $c \neq 0$ *admits no mixed foliate proper* CR*-submanifolds.*

Lemma 3.3 is due to [2] for $c > 0$ and due to [16] for $c < 0$.

For mixed foliate CR-submanifolds in a complex Euclidean space, we have the following result from [8].

Lemma 3.4. *Let* N *be a* CR*-submanifold of* $\mathbf{C}^m$. *Then* N *is mixed foliate if and only if* N *is a* CR*-product.*

We also need the following result from [8, Theorem 4.6].

Lemma 3.5. *Every* CR*-product in a complex Euclidean* m*-space* $\mathbf{C}^m$ *is a direct product of a holomorphic submanifold of a linear complex subspace and a totally real submanifold of another linear complex subspace.*

4. An Inequality for Anti-holomorphic Submanifolds with $p \geq 2$

Let N be a CR-submanifold of a Kähler manifold. Denote by $\mathcal{D}$ and $\mathcal{D}^{\perp}$ the holomorphic distribution and the totally real distribution of N as before. For a CR submanifold N, let us choose a local orthonormal frame $\{e_1, \ldots, e_{2h+p}\}$ on N in such way that $e_1, \ldots, e_h, e_{h+1}, \ldots, e_{2h}$ are in $\mathcal{D}$ and $e_{2h+1}, \ldots, e_{2h+p}$ are in $\mathcal{D}^\perp$, where $e_{h+1} = Je_1, \ldots, e_{2h} = Je_h$.

The CR *δ-invariant*, denoted by $\delta(\mathcal{D})$, for a CR-submanifold N with $p = \operatorname{rank} \mathcal{D}^\perp \geq 1$ is defined by (see [13] for details)

$$\delta(\mathcal{D})(x) = \tau(x) - \tau(\mathcal{D}_x), \tag{4.1}$$

where τ and $\tau(\mathcal{D})$ denote the scalar curvature of N and the scalar curvature of the holomorphic distribution $\mathcal{D} \subset TN$, respectively.

Through out this paper, we shall use the following convention on the range of indices *unless mentioned otherwise*:

$$i, j, k = 1, \ldots, 2h; \;\; \alpha, \beta, \gamma = 1, \ldots, h,$$
$$r, s, t = 2h+1, \ldots, 2h+p; \;\; A, B, C = 1, \ldots, 2h+p.$$

For a CR-submanifold N we define the two *partial mean curvature vectors* $\overrightarrow{H}_{\mathcal{D}}$ and $\overrightarrow{H}_{\mathcal{D}^\perp}$ of N by

$$\overrightarrow{H}_{\mathcal{D}} = \frac{1}{2h} \sum_{i=1}^{2h} \sigma(e_i, e_i), \qquad \overrightarrow{H}_{\mathcal{D}^\perp} = \frac{1}{p} \sum_{r=2h+1}^{2h+p} \sigma(e_r, e_r). \tag{4.2}$$

An anti-holomorphic submanifold N of a Kähler manifold $\tilde{M}$ is called *minimal* (resp., *$\mathcal{D}$-minimal* or *$\mathcal{D}^\perp$-minimal*) if $H = 0$ holds identical (resp., $\overrightarrow{H}_{\mathcal{D}} = 0$ or $\overrightarrow{H}_{\mathcal{D}^\perp} = 0$ hold identically).

We define the coefficients of the second fundamental form by

$$\sigma^r_{AB} = \langle \sigma(e_A, e_B), Je_r \rangle$$

for $A, B = 1, \ldots, 2h+p$ and $r = 1, \ldots, p$.

For anti-holomorphic submanifolds with $p = \operatorname{rank} \mathcal{D}^\perp \geq 2$, we have the following optimal inequality.

Theorem 4.1. *Let N be an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \operatorname{rank}_{\mathbf{C}} \mathcal{D} \geq 1$ and $p = \operatorname{rank} \mathcal{D}^\perp \geq 2$. Then we have*

$$\delta(\mathcal{D}) \leq \frac{(p-1)(2h+p)^2}{2(p+2)} H^2 + \frac{p}{2}(4h+p-1)c. \tag{4.3}$$

The equality sign of (4.3) *holds identically if and only if the following three conditions are satisfied:*

(a) *N is $\mathcal{D}$-minimal, i.e., $\overrightarrow{H}_{\mathcal{D}} = 0$,*

(b) *N is mixed totally geodesic, and*

(c) *there exist an orthonormal frame* $\{e_{2h+1},\dots,e_n\}$ *of* $\mathcal{D}^\perp$ *such that the second fundamental* σ *of* N *satisfies*

$$\begin{cases}\sigma^r_{rr}=3\sigma^r_{ss}, & \text{for } 2h+1\le r\ne s\le 2h+p,\\ \sigma^r_{st}=0, & \text{for distinct } r,s,t\in\{2h+1,\dots,2h+p\}.\end{cases}\tag{4.4}$$

Proof. Let N be an anti-holomorphic submanifold in a complex space form $\tilde{M}^{h+p}(4c)$. Let us choose an orthonormal frame $\{e_1,\dots,e_{2h+p}\}$ on N as above.

It follows from the equation of Gauss and the definition of CR δ-invariant that $\delta(\mathcal{D})$ satisfies

$$\begin{aligned}\delta(\mathcal{D})=&\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}K(e_i,e_r)+\sum_{2h+1\le r\ne s\le 2h+p}\frac{1}{2}K(e_r,e_s)\\ =&\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}\langle\sigma(e_i,e_i),\sigma(e_r,e_r)\rangle+\sum_{r,s=2h+1}^{2h+p}\frac{1}{2}\langle\sigma(e_r,e_r),\sigma(e_s,e_s)\rangle\\ &-\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2-\sum_{r,s=2h+1}^{2h+p}\frac{1}{2}||\sigma(e_r,e_s)||^2\\ &+\frac{p}{2}(4h+p-1)c.\end{aligned}\tag{4.5}$$

On the other hand, we have

$$\begin{aligned}&\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}\langle\sigma(e_i,e_i),\sigma(e_r,e_r)\rangle+\sum_{r,s=2h+1}^{2h+p}\frac{1}{2}\langle\sigma(e_r,e_r),\sigma(e_s,e_s)\rangle\\ &\quad-\sum_{r,s=2h+1}^{2h+p}\frac{1}{2}||\sigma(e_r,e_s)||^2\\ &=\frac{(2h+p)^2}{2}H^2-2h^2|\vec{H}_{\mathcal{D}}|^2-\frac{1}{2}||\sigma_{\mathcal{D}^\perp}||^2,\end{aligned}\tag{4.6}$$

where $||\sigma_{\mathcal{D}^\perp}||^2$ is defined by

$$||\sigma_\perp||^2=\sum_{r,s=2h+1}^{2h+p}||\sigma(e_r,e_s)||^2.\tag{4.7}$$

By combining (4.5) and (4.6) we find

$$\begin{aligned}\delta(\mathcal{D})=&\frac{(2h+p)^2}{2}H^2+\frac{p}{2}(4h+p-1)c-2h^2|\vec{H}_{\mathcal{D}}|^2\\ &-\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2-\frac{1}{2}||\sigma_{\mathcal{D}^\perp}||^2.\end{aligned}\tag{4.8}$$

It follows from statement (2) of Lemma 3.1 the coefficients of the second fundamental form satisfy

$$\sigma^r_{st} = \sigma^s_{rt} = \sigma^t_{rs}. \tag{4.9}$$

We find from (4.2), (4.7) and (4.9) that

$$\begin{aligned}(p+2)||\sigma_{\mathcal{D}^\perp}||^2 - 3p^2|H_{\mathcal{D}^\perp}|^2 &= (p-1)\sum_{r=2h+1}^{2h+p}\left(\sum_{s=2h+1}^{2h+p}\sigma^r_{ss}\right)^2 \\ &\quad + \sum_{2h+1\le r\ne s\le 2h+p} 3(p+1)(\sigma^r_{ss})^2 + \sum_{2h+1\le r<s<t\le 2h+p} 6(p+2)(\sigma^r_{st})^2 \\ &\quad + \sum_{r=2h+1}^{2h+p}\ \sum_{2h+1\le s<t\le 2h+p} 2(p+2)\sigma^r_{ss}\sigma^r_{tt} \\ &= \sum_{r=2h+1}^{2h+p}(p-1)(\sigma^r_{rr})^2 + \sum_{2h+1\le r\ne s\le 2h+p} 3(p+1)(\sigma^r_{ss})^2 \\ &\quad + \sum_{2h+1\le r<s<t\le 2h+p} 6(p+2)(\sigma^r_{st})^2 - \sum_{r=2h+1}^{2h+p}\ \sum_{2h+1\le s<t\le 2h+p} 6\sigma^r_{ss}\sigma^r_{tt} \\ &= \sum_{2h+1\le r<s<t\le 2h+p} 6(p+2)(\sigma^r_{st})^2 + \sum_{2h+1\le s\ne r\le 2h+p}(\sigma^r_{rr} - 3\sigma^r_{ss})^2 \\ &\qquad + \sum_{r\ne s,t}\ \sum_{2h+1\le s<t\le 2h+p} 3(\sigma^r_{ss} - \sigma^r_{tt})^2 \\ &\ge 0.\end{aligned} \tag{4.10}$$

Thus we get

$$||\sigma_{\mathcal{D}^\perp}||^2 \ge \frac{3p^2}{p+2}|H_{\mathcal{D}^\perp}|^2, \tag{4.11}$$

with equality holding if and only if

$$\begin{aligned}&\sigma^r_{rr} = 3\sigma^r_{ss}, && \text{for } 2h+1 \le r \ne s \le 2h+p, \\ &\sigma^r_{st} = 0, && \text{for distinct } r,s,t \in \{2h+1,\ldots,2h+p\}.\end{aligned} \tag{4.12}$$

Now, by combining (4.8) and (4.11), we obtain

$$\begin{aligned}&\frac{(2h+p)^2}{2}H^2 + \frac{p}{2}(4h+p-1)c - \delta(\mathcal{D}) \\ &\quad \ge 2h^2|\overrightarrow{H}_{\mathcal{D}}|^2 + \sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2 + \frac{3p^2}{2(p+2)}|H_{\mathcal{D}^\perp}|^2\end{aligned} \tag{4.13}$$

$$
\begin{aligned}
&= \frac{3}{2(p+2)}\left\{(2h+p)^2H^2 - 4h^2|\overrightarrow{H}_{\mathcal{D}}|^2 - 2\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2\right\} \\
&+2h^2|\overrightarrow{H}_{\mathcal{D}}|^2 + \sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2 \\
&= \frac{3(2h+p)^2}{2(p+2)}H^2 + \frac{2h^2(p-1)}{p+2}|\overrightarrow{H}_{\mathcal{D}}|^2 + \frac{p-1}{p+2}\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2 \\
&\geq \frac{3(2h+p)^2}{2(p+2)}H^2.
\end{aligned}
$$

It is obvious that the equality of the last inequality in (4.13) holds if and only if N is $\mathcal{D}$-minimal and mixed totally geodesic. Consequently, we may obtain inequality (4.3) from (4.13).

It is straightforward to verify that the equality sign of (4.3) holds identically if and only if conditions (a), (b) and (c) of Theorem 4.1 are satisfied. ■

5. Anti-holomorphic Submanifolds with $p \geq 2$ satisfying Equality

First, we give the following example satisfying the equality case of (4.3).

Example 5.1. Let $w : S^p(1) \to \mathbf{C}^p$, $p \geq 2$, be the map of the unit p-sphere $S^p(1)$ into $\mathbf{C}^p$ defined by

$$w(y_0, y_1, \ldots, y_p) = \frac{1+\mathrm{i}y_0}{1+y_0^2}(y_1,\ldots,y_p), \quad y_0^2+y_1^2+\ldots+y_p^2 = 1.$$

The map w is a (non-isometric) Lagrangian immersion with one self-intersection point. This immersion is called the *Whitney p-sphere.* It is well-known that Whitney spheres are the only H-umbilical Lagrangian submanifolds of the complex Euclidean spaces satisfying $\alpha = 3\beta \neq 0$ in (2.8) (see for instance, [4, 12]).

Consider the product immersion:

$$\phi : \mathbf{C}^h \times S^p(1) \to \mathbf{C}^h \oplus \mathbf{C}^p = \mathbf{C}^{h+p}$$

defined by

$$\phi(z,x) = (z, w(x)), \ \ \forall z \in \mathbf{C}^h, \ \ \forall x \in S^p(1). \tag{5.1}$$

It is straight-forward to verify that ϕ is an anti-holomorphic isometric immersion which satisfies the equality sign of (4.3) identically.

In this section we provide the following two classification theorems for anti-holomorphic submanifolds satisfying the equality case of (4.3) identically.

Theorem 5.1. *Let N be an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \text{rank}_{\mathbf{C}}\,\mathcal{D} \geq 1$ and $p = \text{rank}\,\mathcal{D}^{\perp} \geq 2$. If N satisfies the equality case of* (4.3) *identically and if the holomorphic distribution $\mathcal{D}$ is integrable, then $c = 0$ so that $\tilde{M}^{h+p}(4c) = \mathbf{C}^{h+p}$. Moreover, either*

(i) *N is a totally geodesic anti-holomorphic submanifold of $\mathbf{C}^{h+p}$ or,*

(ii) *up to dilations and rigid motions of $\mathbf{C}^{h+p}$, N is given by an open portion of the following product immersion:*

$$\phi : \mathbf{C}^h \times S^p(1) \to \mathbf{C}^{h+p}; \quad (z, x) \mapsto (z, w(x)), \quad z \in \mathbf{C}^h, \ x \in S^p(1),$$

where $w : S^p(1) \to \mathbf{C}^p$ is the Whitney p-sphere defined in Example 5.1.

Proof. Assume that N is an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \text{rank}_{\mathbf{C}}\,\mathcal{D} \geq 1$ and $p = \text{rank}\,\mathcal{D}^{\perp} \geq 2$. If N satisfies the equality case of (4.3) and if the holomorphic distribution $\mathcal{D}$ is integrable, then it follows from Theorem 4.1 that N is mixed foliate. Hence Lemma 3.3 implies that $c = 0$. Therefore, according to Lemma 3.4, N is a CR-product. Hence, N is locally a CR-product given by

$$\mathbf{C}^h \times N^{\perp} \subset \mathbf{C}^h \times \mathbf{C}^p,$$

where $\mathbf{C}^h$ is a complex Euclidean h-subspace and $N^{\perp}$ is a Lagrangian submanifold of $\mathbf{C}^p$. Consequently, condition (c) of Theorem 4.1 implies that $N^{\perp}$ is a Lagrangian H-umbilical submanifold in $\mathbf{C}^p$ whose second fundamental form satisfying

$$\begin{aligned}
&\sigma(e_{2h+1}, e_{2h+1}) = 3\lambda J e_{2h+1}, \quad \sigma(e_{2h+1}, e_s) = \lambda J e_s, \\
&\sigma(e_{2h+2}, e_{2h+2}) = \cdots = \sigma(e_{2h+p}, e_{2h+p}) = \lambda J e_{2h+1}, \\
&\sigma(e_r, e_s) = 0, \quad 2h+2 \leq r \neq s \leq 2h+p,
\end{aligned} \tag{5.2}$$

for some suitable function λ with respect to some suitable orthonormal local frame field $\{e_{2h+1}, \ldots, e_{2h+p}\}$ of $TN^{\perp}$.

If $\lambda = 0$, then $N^{\perp}$ is an open portion of a totally geodesic totally real p-plane in $\mathbf{C}^p$. Hence, in this case N is a totally geodesic anti-holomorphic submanifold.

If $\lambda \neq 0$, it follows from (5.2) that, up to dilations and rigid motions, $N^{\perp}$ is an open part of the Whitney p-sphere in $\mathbf{C}^p$ (cf. [4, 12]). Therefore, up to dilations and rigid motions of $\mathbf{C}^{h+p}$ the anti-holomorphic submanifold is locally given by the product immersion:

$$\phi : \mathbf{C}^h \times S^p(1) \to \mathbf{C}^{h+p}; \qquad (z, x) \mapsto (z, w(x)), \tag{5.3}$$

for $z \in \mathbf{C}^h$ and $x \in S^p(1)$, where $w : S^p(1) \to \mathbf{C}^p$ is the Whitney p-sphere.

The converse is easy to verify. ∎

Theorem 5.2. *Let N be an anti-holomorphic submanifold in a complex space form $\tilde{M}^{1+p}(4c)$ with $h = \operatorname{rank}_{\mathbf{C}} \mathcal{D} = 1$ and $p = \operatorname{rank} \mathcal{D}^{\perp} \geq 2$. Then we have*

$$\delta(\mathcal{D}) \leq \frac{(p-1)(p+2)^2}{2(p+2)} H^2 + \frac{p}{2}(p+3)c. \tag{5.4}$$

The equality case of (5.4) *holds identically if and only if $c = 0$ and either*

(i) *N is a totally geodesic anti-holomorphic submanifold of $\mathbf{C}^{h+p}$ or,*

(ii) *up to dilations and rigid motions, N is given by an open portion of the following product immersion:*

$$\phi : \mathbf{C} \times S^p(1) \to \mathbf{C}^{1+p}; \quad (z, x) \mapsto (z, w(x)), \ z \in \mathbf{C}, \ x \in S^p(1),$$

where $w : S^p(1) \to \mathbf{C}^p$ is the Whitney p-sphere.

Proof. Let N be an anti-holomorphic submanifold in a complex space form $\tilde{M}^{1+p}(4c)$. Then we have inequality (5.4) from inequality (4.3).

Assume that N satisfies the equality case of (5.4) identically. Then Theorem 4.1 implies that N satisfies conditions (a), (b) and (c) of Theorem 4.1.

By condition (a), N is $\mathcal{D}$-minimal. Thus we find

$$\sigma(Je_1, Je_1) = -\sigma(e_1, e_1) \tag{5.5}$$

for any unit vector $e_1 \in \mathcal{D}$. It is direct to verify from (5.5) and polarization that the second fundamental form satisfies the following condition:

$$\sigma(X, JY) = \sigma(JX, Y), \ \ \forall X, Y \in \mathcal{D}.$$

Therefore, according to Lemma 3.2(1), we may conclude that $\mathcal{D}$ is integrable. Consequently, we obtain Theorem 5.2 from Theorem 5.1. ∎

6. An Optimal Inequality for Real Hypersurfaces

Clearly, anti-holomorphic submanifolds with $\operatorname{rank} \mathcal{D}^{\perp} = 1$ are nothing but real hypersurfaces. The Ricci tensor Ric of real hypersurfaces in complex space forms have been studied in [11, 17, 19] among others.

In the following, a Hopf hypersurface N is called *special* if $J\xi$ is an eigenvector of A_ξ with eigenvalue 0, i.e., $A_\xi(J\xi) = 0$, where ξ is a unit normal vector field.

For real hypersurfaces, we have the following.

Theorem 6.1. *If N is a real hypersurface of a complex space form $\tilde{M}^{h+1}(4c)$, then the Ricci tensor Ric of N satisfies*

$$Ric(J\xi, J\xi) \leq \frac{(2h+1)^2}{2} H^2 + 2hc. \tag{6.1}$$

where ξ is a unit normal vector field of N in $\tilde{M}^{h+1}(4c)$.

The equality sign of (6.1) *holds identically if and only if N is a minimal special Hopf hypersurface.*

Proof. Let N be a real hypersurface of a complex space form $\tilde{M}^{h+1}(4c)$. Then it follows from the definition of $\delta(\mathcal{D})$ that

$$\delta(\mathcal{D}) = Ric(J\xi, J\xi). \tag{6.2}$$

Let us choose an orthonormal frame $\{e_1, \ldots, e_h, e_{h+1} = Je_1, \ldots, e_{2h} = Je_h\}$ for the holomorphic distribution $\mathcal{D}$ and let e_{2h+1} be a unit vector field in $\mathcal{D}^{\perp}$.

We put

$$\sigma_{a,b} = \langle \sigma(e_a, e_b), Je_{2h+1} \rangle, \quad a, b = 1, \ldots, 2h+1. \tag{6.3}$$

Let us define the connection forms by

$$\begin{aligned} \nabla_X e_i &= \sum_{j=1}^{2h} \omega_i^j(X) e_j + \omega_i^{2h+1}(X) e_{2h+1}, \\ \nabla_X e_{2h+1} &= \sum_{j=1}^{2h} \omega_{2h+1}^j(X) e_j, \end{aligned} \tag{6.4}$$

for $i = 1, \ldots, 2h$. It follows from (4.1) and the equation of Gauss that

$$\delta(\mathcal{D}) = \sum_{i=1}^{2h} \sigma_{i,i}\sigma_{2h+1,2h+1} - \sum_{i=1}^{2h} (\sigma_{i,2h+1})^2 + 2hc. \tag{6.5}$$

On the other hand, we have

$$\sum_{i=1}^{2h} \sigma_{i,i}\sigma_{2h+1,2h+1} = \frac{(2h+1)^2}{2} H^2 - \frac{1}{2}(\sigma_{2h+1,2h+1})^2 - 2h^2 |\overrightarrow{H}_{\mathcal{D}}|^2. \tag{6.6}$$

By combining (6.6) and (6.6) we obtain

$$\begin{aligned} \delta(\mathcal{D}) = & \frac{(2h+1)^2}{2} H^2 + 2hc - 2h^2 |\overrightarrow{H}_{\mathcal{D}}|^2 - \frac{1}{2}(\sigma_{2h+1,2h+1})^2 \\ & - \sum_{i=1}^{2h} (\sigma_{i,2h+1})^2 \\ \leq & \frac{(2h+1)^2}{2} H^2 + 2hc. \end{aligned} \tag{6.7}$$

It follows from (6.7) and Lemma 3.2(2) that the equality sign of inequality (6.1) holds identically if and only if the following two statements hold:

(i) N is a special Hopf hypersurface and

(ii) N is $\mathcal{D}$-minimal in $\tilde{M}^{h+1}(4c)$.

Obviously, conditions (i) and (ii) imply that N is a minimal real hypersurface of $\tilde{M}^{h+1}(4c)$.

The converse is easy to verify. ∎

The following corollary follows easily from Theorem 6.1.

Corollary 6.1. *Let N be a real hypersurface of a complex space form $\tilde{M}^{h+1}(4c)$. If N satisfies the equality case of* (6.1) *identically, then the holomorphic distribution of N is non-integrable, unless $c = 0$ and N is totally geodesic.*

Proof. Under the hypothesis, if N satisfies the equality case of (6.1) identically and if the holomorphic distribution $\mathcal{D}$ is integrable, then Theorem 6.1 implies that N is mixed foliate. So, it follows from Lemma 3.3 and Lemma 3.4 that $c = 0$ and N is a CR-product of a complex h-subspace in $\mathbf{C}^h$ and an open portion of line in $\mathbf{C}$. Consequently, N must be totally geodesic. ∎

7. Some Applications of Theorem 6.1

We need the following lemma.

Lemma 7.1. *Let N be a special Hopf hypersurface of a complex space form $\tilde{M}^{h+1}(4c)$. Then there exist an orthonormal frame $\{e_1, \ldots, e_h, e_{h+1} = Je_1, \ldots, e_{2h} = Je_h\}$ of $\mathcal{D}$ and an integer $k \leq h$ such that*

$$\begin{aligned} &\sigma(e_\alpha, e_\beta) = \lambda_\alpha \delta_{\alpha\beta}\xi, \;\; \sigma(e_{h+\alpha}, e_{h+\beta}) = \mu_\alpha \delta_{\alpha\beta}\xi, \;\; 1 \leq \alpha, \beta \leq k; \\ &\sigma(e_a, e_b) = 0, \;\; otherwise, \end{aligned} \tag{7.1}$$

with $\lambda_\alpha \mu_\alpha = c$, where $\kappa_1, \ldots, \kappa_k$ are nonzero functions.

Proof. Let N be a special Hopf hypersurface of $\tilde{M}^{h+1}(4c)$ and let e_{2h+1} be a unit vector field in $\mathcal{D}^\perp$. Then $\xi = Je_{2h+1}$ is a unit normal vector field. Thus we have

$$\sigma(U, e_{2h+1}) = 0, \;\; U \in TN. \tag{7.2}$$

For an eigenvector X of A_ξ with eigenvalue $\kappa \neq 0$, we may choose an orthonormal frame $\{e_1, \ldots, e_h, e_{h+1} = Je_1, \ldots, e_{2h} = Je_h\}$ with $e_1 = X$. Hence we find

$$A_\xi(e_1) = \kappa e_1. \tag{7.3}$$

From (6.3), (6.4) and Lemma 3.1(2) we derive that

$$\begin{aligned} &\omega_\alpha^{2h+1}(e_\beta) = \sigma_{\alpha+h,\beta}, \;\; \omega_{h+\alpha}^{2h+1}(e_\beta) = -\sigma_{\alpha,\beta}, \\ &\omega_\alpha^{2h+1}(e_{h+\beta}) = \sigma_{h+\alpha,h+\beta}, \;\; \omega_{h+\alpha}^{2h+1}(e_{h+\beta}) = -\sigma_{\alpha,h+\beta}. \end{aligned} \tag{7.4}$$

It follows from (2.8) that

$$(\tilde{R}(e_\alpha, e_{h+\beta})e_{2h+1})^\perp = -2c\delta_{\alpha\beta}Je_{2h+1}. \tag{7.5}$$

On the other hand, we find from (7.2), (7.4) and the equation of Codazzi that

$$\begin{aligned}(\tilde{R}(e_\alpha, e_{h+\beta})e_{2h+1})^\perp &= (\bar{\nabla}_{e_\alpha}\sigma)(e_{h+\beta}, e_{2h+1}) - (\bar{\nabla}_{e_{h+\beta}}\sigma)(e_\alpha, e_{2h+1}) \\ &= 2\sum_{\gamma=1}^{h}(\sigma_{\alpha,h+\gamma}\sigma_{\gamma,h+\beta} - \sigma_{\alpha,\gamma}\sigma_{h+\beta,h+\gamma})Je_{2h+1}.\end{aligned} \tag{7.6}$$

By combining (7.5) and (7.6) we find

$$\sum_{\gamma=1}^{h}(\sigma_{\alpha,\gamma}\sigma_{h+\beta,h+\gamma} - \sigma_{\alpha,h+\gamma}\sigma_{\gamma,h+\beta}) = c\delta_{\alpha\beta}, \quad 1 \le \alpha, \beta \le h. \tag{7.7}$$

Also, it follows from $(\tilde{R}(e_\beta, e_\alpha)e_{2h+1})^\perp = \sigma(e_\alpha, \nabla_{e_\beta}e_{2h+1}) - \sigma(e_\beta, \nabla_{e_\alpha}e_{2h+1})$ that

$$\sum_{\gamma=1}^{h}(\sigma_{\alpha,h+\gamma}\sigma_{\beta,\gamma} - \sigma_{\alpha,\gamma}\sigma_{\beta,h+\gamma}) = 0. \tag{7.8}$$

Condition (7.3) gives

$$\sigma_{11} = \kappa \neq 0, \quad \sigma_{1a} = 0, \quad otherwise. \tag{7.9}$$

Now, by combining (7.7), (7.8) and (7.9) we obtain

$$\kappa\sigma_{1^*1^*} = c \ \text{ and } \ \sigma_{1^*a} = 0, \ \text{ for } \ a = 1, \ldots, h, 2^*, \ldots, h^*,$$

which implies that $JX = e_{1^*}$ is an eigenvector of A_ξ with eigenvalue c/κ. By applying this fact, we conclude the lemma. ∎

Remark 7.1. Lemma 7.1 is due to [18] and [3] for $c > 0$ and $c < 0$, respectively,

Lemma 7.1 implies the following two lemmas.

Lemma 7.2. *If N is a special Hopf hypersurface of $\mathbf{C}^{h+1}$, then there exists an orthonormal frame $\{e_1, \ldots, e_h, e_{h+1} = Je_1, \ldots, e_{2h} = Je_h\}$ of $\mathcal{D}$ and an integer $k \le h$ such that*

$$\sigma(e_\alpha, e_\beta) = \lambda_\alpha\delta_{\alpha\beta}\xi, \quad \sigma(e_a, e_b) = 0, \quad otherwise, \tag{7.10}$$

for $1 \le \alpha, \beta \le k$, where $\lambda_1, \ldots, \lambda_k$ are nonzero functions.

Proof. Under the hypothesis, Lemma 7.1 implies that there is an orthonormal frame $\{e_1, \ldots, e_h, e_{h+1} = Je_1, \ldots, e_{2h} = Je_h\}$ of $\mathcal{D}$ such that

$$\begin{aligned} &\sigma(e_\alpha, e_\beta) = \lambda_\alpha \delta_{\alpha\beta}\xi, \ \ \sigma(e_{h+\gamma}, e_{h+\eta}) = \mu_\gamma \delta_{\gamma\eta}\xi, \\ &\sigma(e_a, e_b) = 0, \ \ otherwise, \\ &1 \leq \alpha, \beta \leq n_1; \ \ n_1 + 1 \leq \gamma, \eta \leq n_1 + n_2, \end{aligned} \tag{7.11}$$

where n_1, n_2 are integers and $\lambda_1, \ldots, \lambda_{n_1+n_2}$ are functions. Thus, after replacing

$$e_{n_1+1}, \ldots, e_{n_1+n_2}, Je_{n_1+1}, \ldots, Je_{n_1+n_2}\, \mu_{n_1+1}, \ldots, \mu_{n_1+n_2}$$

by $Je_{n_1+1}, \ldots, Je_{n_1+n_2}, -e_{n_1+1}, \ldots, -e_{n_1+n_2}, \lambda_{n_1+1}, \ldots, \lambda_{n_1+n_2}$, respectively, we obtain (7.10). ■

Lemma 7.3. *Let N be a special Hopf hypersurface of $CP^{h+1}(4)$ (resp., $CH^{h+1}(-4)$). Then there exists an orthonormal frame $\{e_1, \ldots, e_h, e_{h+1} = Je_1, \ldots, e_{2h} = Je_h\}$ of the holomorphic distribution $\mathcal{D}$ such that*

$$\sigma(e_\alpha, e_\beta) = \lambda_\alpha \delta_{\alpha\beta}\xi, \ \ \sigma(e_{h+\alpha}, e_{h+\beta}) = \frac{\delta_{\alpha\beta}}{\lambda_\alpha}\xi, \ \ \sigma(e_a, e_b) = 0 \ otherwise,$$

$$(resp., \ \sigma(e_\alpha, e_\beta) = \lambda_\alpha \delta_{\alpha\beta}\xi, \ \sigma(e_{h+\alpha}, e_{h+\beta}) = -\frac{\delta_{\alpha\beta}}{\lambda_\alpha}\xi, \ \sigma(e_a, e_b) = 0, \ otherwise),$$

for $1 \leq \alpha, \beta \leq h$, where $\lambda_1, \ldots, \lambda_h$ are nowhere zero functions.

By applying Theorem 6.1 and Lemma 7.1, we have the following.

Theorem 7.1. *If N is a real hypersurface of $\tilde{M}^2(4c)$, then we have*

$$Ric(J\xi, J\xi) \leq \frac{9}{2}H^2 + 2c. \tag{7.12}$$

The equality sign of (7.12) *holds identically if and only if $c = 0$ and N is totally geodesic.*

Proof. Let N be a real hypersurface of a complex space form $\tilde{M}^2(4c)$. Then we obtain (7.12) from (6.1). Assume that N satisfies the equality case of (7.12) identically. Then Theorem 6.1 implies that N is a minimal special Hopf hypersurface. Therefore, by Lemma 7.1 there exists a unit vector field e_1 in $\mathcal{D}$ such that

$$\begin{aligned} &\sigma(e_1, e_1) = \lambda Je_3, \ \sigma(e_2, e_2) = -\lambda Je_3, \\ &\sigma(e_1, e_2) = \sigma(e_2, e_3) = 0, \ a = 1, 2, 3 \end{aligned} \tag{7.13}$$

for some function λ. It follows from (7.13) and Lemma 3.1(2) that

$$\omega_1^3(e_1) = \omega_2^3(e_2) = 0, \ \ \omega_3^2(e_1) = \omega_3^1(e_2) = \lambda. \tag{7.14}$$

On the other hand, we find from $(\tilde{R}(e_1, e_2)e_3)^\perp = (\bar{\nabla}_{e_1}\sigma)(e_2, e_3) - (\bar{\nabla}_{e_2}\sigma)(e_1, e_3)$, (2.8) and (7.4) that $-2c = \lambda(\omega_3^1(e_2) + \omega_3^2(e_1))$. Combining this with (7.14) gives

$$c = -\lambda^2 \leq 0. \tag{7.15}$$

If $c = 0$, (7.15) implies that $\lambda = 0$. Thus N is a totally geodesic hypersurface.

If $c < 0$, it follows from (7.15) that λ is a nonzero constant. Thus, N is a minimal Hopf hypersurface of $CH^2(-\lambda^2)$ with three constant principal curvatures $0, \lambda, -\lambda$. But this is impossible according to Theorem 1 of [3].

The converse is easy to verify. ∎

For real hypersurfaces in $\mathbf{C}^3$, we have the following.

Theorem 7.2. *Let N be a real hypersurface of $\mathbf{C}^3$. We have*

$$Ric(J\xi, J\xi) \leq \frac{25}{2}H^2. \tag{7.16}$$

If the equality case of (7.16) *holds identically, then N is a totally real 3-ruled minimal submanifold of $\mathbf{C}^3$.*

Proof. Let N be a real hypersurface of $\mathbf{C}^3$. Then we find inequality (7.16) from (6.1) of Theorem 6.1.

Assume that N satisfies the equality case of (7.16) identically. Then it follows from Theorem 6.1 and Lemma 7.2 that there exists an orthonormal local frame $\{e_1, e_2, e_3 = Je_1, e_4 = Je_2, e_5\}$ on N such that

$$\begin{aligned} &\sigma(e_1, e_1) = \lambda\xi, \ \ \sigma(e_2, e_2) = -\lambda\xi, \\ &\sigma(e_a, e_b) = 0 \ \ otherwise, \end{aligned} \tag{7.17}$$

for some function λ.

Let us put $W = \{x \in N : \lambda(x) \neq 0\}$, which is an open subset of W.

Case (a). $W = \emptyset$. In this case, N is a totally geodesic hypersurface. In particular, N is a totally real 3-ruled minimal submanifold of $\mathbf{C}^3$.

Case (b). $W \neq \emptyset$. If we put $\mathcal{D}_1 = \text{Span}\{e_1, e_2\}$ and $\mathcal{D}_2 = \text{Span}\{e_3, e_4, e_5\}$, then we find from (7.17) that

$$\sigma(\mathcal{D}_2, TN) = \{0\}, \ \ i.e., \ A_\xi V = 0, \ \ \forall V \in \mathcal{D}_2. \tag{7.18}$$

Thus, after applying (7.18) and the Codazzi equation

$$(\bar{\nabla}_U\sigma)(V, X) = (\bar{\nabla}_V\sigma)(U, X), \ \ U, V \in \mathcal{D}_2, \ \ X \in \mathcal{D}_1,$$

we obtain $\sigma([U,V],X)=0$. Therefore, it follows from (7.17) and $\lambda\neq 0$ that $\mathcal{D}_2$ is an integrable distribution.

Also, from (7.17) we derive that

$$\begin{aligned}\lambda\langle\nabla_U V,e_1\rangle &= \langle\nabla_U V,A_\xi e_1\rangle\\ &= -\langle V,(\nabla_U A_\xi)e_1\rangle-\langle V,A_\xi(\nabla_U e_1)\rangle\\ &= -\langle V,(\nabla_{e_1}A_\xi)U\rangle-\langle A_\xi V,\nabla_U e_1\rangle\\ &= -\langle V,\nabla_{e_1}(A_\xi U)\rangle+\langle V,A_\xi(\nabla_{e_1}X)\rangle\\ &= 0\end{aligned}$$

for U,V in $\mathcal{D}_2$. Hence we find $\langle\nabla_U V,e_1\rangle=0$. Similarly, we have $\langle\nabla_U V,e_2\rangle=0$. After combining these with (7.18), we conclude that each leave of $\mathcal{D}_2$ is a totally real totally geodesic submanifold of $\mathbf{C}^3$. Consequently, each connected component of W is a totally real 3-ruled minimal submanifold of $\mathbf{C}^3$. If W is dense in N, we have the same conclusion by continuity.

If W is not dense in N, then the interior of each connected component of $N-W$ is a totally geodesic real hypersurface of $\mathbf{C}^3$, which is obviously totally real 3-ruled. Consequently, by continuity the whose N is a totally real 3-ruled minimal submanifold. ∎

For real hypersurfaces in $CP^3(4)$, we have

Proposition 7.1. *If N is a real hypersurface of $CP^3(4)$, then we have*

$$Ric(J\xi,J\xi)\leq\frac{25}{2}H^2+4. \tag{7.19}$$

The equality sign of (7.19) *holds identically if and only if locally there exists an orthonormal frame $\{e_1,e_2,e_3=Je_1,e_4=Je_2,e_5\}$ such that*

$$\begin{aligned}&\sigma(e_1,e_1)=\lambda\xi,\ \sigma(e_2,e_2)=-\lambda\xi,\\ &\sigma(e_3,e_3)=\frac{1}{\lambda}\xi,\ \sigma(e_4,e_4)=-\frac{1}{\lambda}\xi,\\ &\sigma(e_a,e_b)=0\ \ otherwise,\end{aligned} \tag{7.20}$$

where λ is a nowhere zero function.

Proof. Follows from Theorem 6.1 and Lemma 7.3. ∎

Similarly, we also have the following result for real hypersurfaces in $CH^3(-4)$ by Theorem 6.1 and Lemma 7.3.

Proposition 7.2. *If N is a real hypersurface of $CH^3(-4)$, then we have*

$$Ric(J\xi,J\xi)\leq\frac{25}{2}H^2-4. \tag{7.21}$$

The equality sign of (7.21) *holds identically if and only if locally there exists an orthonormal frame* $\{e_1, e_2, e_3 = Je_1, e_4 = Je_2, e_5\}$ *on* N *such that*

$$\begin{aligned} &\sigma(e_1, e_1) = \lambda\xi,\ \sigma(e_2, e_2) = -\lambda\xi, \\ &\sigma(e_3, e_3) = -\frac{1}{\lambda}\xi,\ \sigma(e_4, e_4) = \frac{1}{\lambda}\xi, \\ &\sigma(e_a, e_b) = 0\ \ otherwise, \end{aligned} \tag{7.22}$$

where λ *is a nowhere zero function.*

Proposition 7.1 and Proposition 7.2 imply immediately the following.

Corollary 7.2. *Every real hypersurface of* $CP^3(4)$ $($*resp., of* $CH^3(-4))$ *satisfying the equality case of* (7.19) $($*resp., the equality case of* (7.21)$)$ *is* $\delta(2,2)$*-ideal in the sense of* [9, 12].

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